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The “Riemann hypothesis” for the Hawkins random Sieve


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Hawkins ([1, 2]) and Wunderlich ([6, 7]) have studied the sequence \( \{X_n : n \in \mathbb{N}\} \) of ‘random primes’ caught in the Hawkins Sieve which operates inductively as follows.

Let

\[ A_1 = \{2, 3, 4, 5, 6, \ldots\}. \]

**Stage 1.** Declare \( X_1 = \min A_1 \). From the set \( A_1 \setminus \{X_1\} \), each number in turn (and each independently of the others) is deleted with probability \( X_1^{-1} \). The set of elements of \( A_1 \setminus \{X_1\} \) which remain is denoted by \( A_2 \).

**Stage n.** Declare \( X_n = \min A_n \). From the set \( A_n \setminus \{X_n\} \), each number in turn (and each independently of the others) is deleted with probability \( X_n^{-1} \). The set of elements of \( A_n \setminus \{X_n\} \) which remain is denoted by \( A_{n+1} \).

Define

\[ Y_n = \prod_{k \leq n} (1 - X_k^{-1})^{-1}. \]

**Notation and conventions.** For \( x > 1 \), we write \( \text{li} (x) \) for the value of the logarithmic integral at \( x \):

\[
\text{li} (x) = \lim_{\delta \downarrow 0} \left( \int_0^{1-\delta} + \int_{1+\delta}^x \right) \frac{dz}{\log z} \sim \frac{x}{\log x}.
\]

Qualifying ‘with probability one’ phrases will be suppressed. An equation involving the symbol ‘\( \varepsilon \)’ is to be understood as true for every positive \( \varepsilon \).

**2**

Recall that the ‘real’ Riemann Hypothesis about the zeros of the Riemann zeta-function is equivalent to the statement:

\[
(1) \quad \text{li} (p_n) = n + O(n^{\varepsilon + \epsilon})
\]

(where \( p_n \) denotes the \( n \)th prime) and that equation (1) is the result which Riemann really wished to prove. See for example Ingham [3].
THEOREM: The following 'Riemann Hypothesis for the Hawkins Sieve' holds:

\[ L \lim (X_n L^{-1}) = n + O(n^{1+\varepsilon}) \]

where \( L \) denotes the random limit

\[ L = \lim_{n \to \infty} X_n \exp (-Y_n). \]

This theorem was motivated by the exactly analogous 'diffusion' result in Williams [5]. The proof now given mirrors that in [5].

3

The process \( \{(X_n, Y_n) : n \in N\} \) is Markovian with

\[ P[X_{n+1} - X_n = j|\mathscr{F}_n] = Y_n^{-1}(1 - Y_n^{-1})^{j-1} \quad (j \in N), \]

\[ Y_{n+1} = Y_n(1 - X_{n+1}^{-1})^{-1} = Y_n(1 + Z_{n+1}^{-1}) \]

and \( X_1 = Y_1 = 2 \). We have written (for \( n \in N \)):

\[ Z_n = X_n - 1, \quad \mathscr{F}_n = \sigma\{(X_k, Y_k) : k \leq n\}, \]

the latter equation signifying that \( \mathscr{F}_n \) is the smallest \( \sigma \)-algebra with respect to which \( X_k \) and \( Y_k \) are measurable for every \( k \leq n \). Introduce

\[ U_{n+1} = (Z_{n+1} - Z_n)Y_n^{-1} \quad (n \in N). \]

Elementary properties of geometric distributions now make things particularly neat. For \( x > 0 \) and \( n \in N \),

\[ P[U_{n+1} > x|\mathscr{F}_n] \leq (1 - Y_n^{-1})^{-1}(1 - Y_n^{-1})^{xY_n} \leq 2e^{-x}. \]

(Recall that \( Y_n \geq 2 \) for every \( n \).) By the Borel-Cantelli Lemma,

\[ U_{n+1} = O(\log n) = O(n^{\varepsilon}). \]

Because

\[ E[(U_{n+1} - 1)|\mathscr{F}_n] = 0, \]

\( \{(U_{n+1} - 1) : n \in N\} \) is a family of orthogonal random variables. Since also

\[ E[(U_{n+1} - 1)^2|\mathscr{F}_n] = 1 - Y_n^{-1} \leq 1, \]

Theorem 33B(ii) of Loève [4] provides the estimate:

\[ \sum_{k \leq n} (U_{k+1} - 1) = O(n^{1+\varepsilon}). \]

In particular,
The remainder of the proof is divided into three stages.

**PROPOSITION 1:** $Y_n \uparrow \infty$ and $Z_n \sim nY_n$.

**PROOF:** If $Y_n \uparrow Y < \infty$, then we could conclude from (4) that $\sum Z_n^{-1} < \infty$ and from (5) and (8) that (in contradiction) $n^{-1}Z_n \rightarrow Y$.

Thus $Y_n \uparrow \infty$ and

\[
Z_{n+1} Y_{n+1}^{-1} - Z_n Y_n^{-1} = U_{n+1} - Y_{n+1}^{-1} = U_{n+1} + o(1)
\]

so that (from (8)) $Z_n \sim nY_n$.

**PROPOSITION 2:** If $H_n = \log Z_n - Y_n (n \in \mathbb{N})$, then

\[
H_n = C + O(n^{-\frac{1}{2} + \epsilon})
\]

for some (random) $C$.

**PROOF:** Since $x(1+x)^{-1} \leq \log (1+x) \leq x$ for $x \geq 0$,

\[
H_{n+1} - H_n = \log (1 + \alpha_n U_{n+1}) - \alpha_n(1 + \alpha_n U_{n+1})^{-1} = \beta_n(U_{n+1} - 1) + O(\alpha_n^2 U_{n+1}^2)
\]

where

$\alpha_n = Y_n Z_n^{-1} = O(n^{-1})$ and $\beta_n = Y_n Z_{n+1}^{-1} = O(n^{-1})$.

Proposition 2 now follows by partial summation using (6), (7) and the further estimate:

$\beta_{n-1} - \beta_n = \beta_{n-1} \beta_n U_{n+1} - \beta_{n-1} Z_{n+1}^{-1} = O(n^{-2 + \epsilon})$.

**Note.** From equation (3), in which of course $L = \exp (C)$, and Proposition 1, it follows that

\[
Y_n \sim \log n, \quad X_n \sim n \log n.
\]

In other words, ‘Mertens’ Theorem’ and the ‘Prime Number Theorem’ hold. (One could not expect the $e^\gamma$ factor which is the rather tantalising feature of the real Mertens Theorem.) Wunderlich obtained both results at (11) by a more complicated method.
On summing equation (9) over \( n \) and utilising the exponentiated form:

\[
Z_n = L \exp \left( Y_n(1 + O(n^{-\frac{1}{2} + \varepsilon})) \right)
\]

of equation (10), we obtain

\[
LY_n^{-1} \exp (Y_n) = n - \sum_{k \leq n} Y_k^{-1} + O(n^{\frac{1}{2} + \varepsilon}).
\]

Extend the random function \( Y \) from \( \{1, 2, 3, \cdots\} \) to \( (1, \infty) \) by linear interpolation. Then it is easily checked that

\[
LY_t^{-1} \exp (Y_t) = \int_1^t (1 - Y_s^{-1})ds + f(t)
\]

where \( f(t) = O(t^{\frac{1}{2} + \varepsilon}) \). But now we may compute

\[
[L \operatorname{li} \left( \exp (Y_t) \right)]_1^t = \int_1^t (1 - Y_s^{-1})^{-1}d[LY_s^{-1} \exp (Y_s)]
\]

\[
= t - 1 + \int_1^t f'(s)[1 - Y_s^{-1}]^{-1}ds.
\]

Integration by parts using \( Y'_s = O(s^{-1}) \) establishes the following strong form of 'Mertens' Theorem':

\[
L \operatorname{li} \left( \exp (Y_t) \right) = t + O(t^{\frac{1}{2} + \varepsilon}).
\]

Equation (2) now follows on combining equations (12) and (13).

REFERENCES


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