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Factoriality of a ring of holomorphic functions


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A non negative Cousin II distribution $\mathcal{D} = \{U_i, f_i\}_{i \in I}$ on a complex (reduced) space $X$ is the datum of an open covering $\{U_i\}_{i \in I}$ of $X$ and, for every $i \in I$, of a holomorphic function $f_i : U_i \to \mathbb{C}$ with the property that for $U_i \cap U_j \neq \emptyset$ one has, on $U_i \cap U_j$, $f_i = \eta_{ij} f_j$ where $\eta_{ij}$ is a holomorphic function on $U_i \cap U_j$ and $\eta_{ij}$ is nowhere zero. A solution of $\mathcal{D}$ is a holomorphic function $f : X \to \mathbb{C}$ such that $f = \eta_i f_i$ on every $U_i$ when $\eta_i : U_i \to \mathbb{C}$ is a nowhere zero, holomorphic function. One says that $X$ is a Cousin-II space when every non negative Cousin-II distribution on $X$ has a solution; if $K \subset X$ we say that $K$ is a Cousin-II set when $K$ has, in $X$, an open neighborhood base $\mathcal{U}$ and every $U \in \mathcal{U}$ is a Cousin-II space.

The object of this note is to prove the following

**Theorem:** Let $X$ be a complex manifold; let $K$ be a semianalytic connected compact Cousin-II subset of $X$; then the ring of the holomorphic functions on $K$ is a unique factorization domain.

Observe that, in the preceding conditions, $K$ is not necessarily a Stein set.

In the proof we shall use some results of J. Frisch [1].

The proof is based on two lemmas. We begin with some notations and definitions.

Let $X$ be a complex space; the structure sheaf of $X$ is denoted by $\mathcal{O}_X$ or $\mathcal{O}$. Let $Y$ be a (closed) analytic subset of $X$; we say that a function $f \in H^0(X, \mathcal{O})$ is associated, in $X$, to $Y$ if for every $x \in X$ the germ $f_x$ generates the ideal of $Y$ in $x$, $I_Y(x)$. If $g \in H^0(X, \mathcal{O})$ we set $Z(X, g) := \{x \in X | g(x) = 0\}$; we write also $Z(g)$ to denote $Z(X, g)$.

**Lemma 1:** Let $X$ be a complex space, $f \in H^0(X, \mathcal{O})$; let $\mathcal{C} = \{C_i\}_{i \in I}$ ($I = \{1, \cdots, m\}$ $m \leq \infty$) be the family of the analytic irreducible components of $Z(X, f)$ and suppose that $\mathcal{C}$ does not contain any irreducible

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component of \( X \). For every \( i \in \{1, \ldots, d\} \subset I \) let a function \( g_i \) be given associated, in \( X \), to \( C_i \).

Then \( g_i \) is irreducible in the ring \( H^0(X, \mathcal{O}) \) for \( 1 \leq i \leq d \). Furthermore there are \( d \) natural numbers \( \lambda_1, \ldots, \lambda_d \) such that the following factorization holds

\[
f = k_d(g_1)^{\lambda_1} \cdots (g_d)^{\lambda_d}
\]

and \( k_d(x) \neq 0 \) for every \( x \in X - \bigcup_{j \geq d+1} C_j \).

**Proof:** Every \( g_i \) is irreducible in \( H^0(X, \mathcal{O}) \). Indeed, if \( g_i = ab \) with \( a, b \in H^0(X, \mathcal{O}) \) then we may suppose without loss of generality \( Z(X, a) = C_i \), because \( C_i \) is irreducible. Let \( z \in C_i \); we call \( \mathcal{M}_z \) the maximal ideal of \( \mathcal{O}_z \); there is a suitable natural number \( s \) such that \( (g_i)_z \in \mathcal{M}_z^{-1} \), \( (g_i)_z \notin \mathcal{M}_z^s \); by the definition of \( g_i \) there exists a germ \( c_z \) for which

\[
a_z = (g_i)_z c_z;
\]

it follows that \( (g_i)_z = (g_i)_z c_z b_z \) and hence \( b_z \notin \mathcal{M}_z \); therefore \( b \) is a unit in \( H^0(X, \mathcal{O}) \).

We will prove the assumptions by induction on \( d \). Let \( d = 1 \). Since \( f \) vanishes on \( C_1 \), it follows from the definition of \( g_1 \) that there exists \( k \in H^0(X, \mathcal{O}) \) such that

\[
f = kg_1.
\]

We observe that if \( k \) vanishes in a point \( z' \in C_1 - \bigcup_{j \geq 2} C_j \), then \( k \) vanishes on the whole of \( C_1 \). Indeed, if \( U \) is a neighborhood of \( z' \) in \( X \) with \( U \cap (\bigcup_{j \geq 2} C_j) = \emptyset \), then we have \( U \cap Z(k) = U \cap Z(k) \cap C_1 \); hence from the equality of analytic set germs

\[
Z(k)_{z'} = (Z(k) \cap C_1)_{z'}
\]

it follows \( \text{codim}_C(Z(k) \cap C_1)_{z'} = 1 = \text{codim}_C(C_1)_{z'} \); as \( C_1 \) is irreducible, it follows \( Z(k) \cap C_1 = C_1 \).

Then there is \( k' \in H^0(X, \mathcal{O}) \) with \( k = k'g_1 \), \( f = (k')g_1^2 \). If \( k' \) vanishes in \( z'' \in C_1 - \bigcup_{j \geq 2} C_j \), then \( k'|_{C_1} = 0 \); but since \( f \) has a zero of finite order in every point of \( C_1 \), a number \( \lambda_1 \geq 1 \) must exist such that

\[
f = k_1(g_1)^{\lambda_1}
\]

with \( k_1 \in H^0(X, \mathcal{O}) \) and \( k_1(z'') \neq 0 \); it follows that \( k_1(x) \neq 0 \) for every \( x \in C_1 - \bigcup_{j \geq 2} C_j \), hence \( k_1(x) \neq 0 \) for \( x \in X - \bigcup_{j \geq 2} C_j \).

Suppose that the thesis holds for \( d - 1 \). We have

\[
f = k_{d-1}(g_1)^{\lambda_1} \cdots (g_d-1)^{\lambda_{d-1}}
\]

with \( k_{d-1}(x) \neq 0 \) for every \( x \in X - \bigcup_{j \geq d} C_j \).

One has \( k_{d-1}(x) = 0 \) for every \( x \in C_d \); therefore there is \( h \in H^0(X, \mathcal{O}) \) such that \( k_{d-1} = hg_d \), \( f = h(g_1)^{\lambda_1} \cdots (g_{d-1})^{\lambda_{d-1}}g_d \). If \( h(w) = 0 \) at a point
w ∈ C_d − ∪_{j≥d+1} C_j, then h vanishes on C_d; indeed, Z(h)_w = (Z(h) ∩ C_d)_w.
As in the case d = 1, we deduce that there exists λ_d ≥ 1 with

\[ f = k_d(g_1)^{λ_1} \cdots (g_{d-1})^{λ_{d-1}}(g_d)^{λ_d} \]

where k_d ∈ H^0(X, O) and k_d(x) ≠ 0 for every x ∈ C_d − ∪_{j≥d+1} C_j, as required.

Let f be a holomorphic function on a complex space X; let K ⊆ X;
f_K denotes the 'holomorphic function' induced by f on K; i.e. f_K is the
class of the functions g which are holomorphic in a neighborhood of K
and such that g|U_f = f|U in some neighborhood U = U_g of K in X.
The ring of all the holomorphic functions on K is denoted by H(K).

We say that the property (C) holds on X when, for every holomorphic
function f on X, every irreducible analytic component D of Z(X, f) has,
in X, an associated function; it is said that the property (C) holds on K
when K has an open neighborhood base B such that the property (C)
holds on every B ∈ B.

**LEMMA 2:** Let X be a complex space; assume that f ∈ H^0(X, O) does not
vanish identically on any irreducible analytic component of X. Let K be
a compact semianalytic subset of K and let the property (C) hold on K.

Then f_K is uniquely expressible as finite product of irreducible elements
of H(K), except for units and for the order of the factors.

**PROOF:**

(I) Existence of the factorization. Let ℱ = {U_i}_{i∈I} be an open neigh-
borhood base of K in X such that on every U ∈ ℱ the property (C) holds.

Let ℱ_1 be the family of the irreducible components of Z(U_1, f|U_1) which
contain points of K. As K is compact, ℱ_1 has only a finite number of
elements. Lemma 1 proves that, without loss of generality, we may suppose that f|U_1 is irreducible in H^0(U_1, O); indeed, there is a factoriza-
tion f = u_1 \cdots u_r, where u_1, \cdots, u_r are irreducible elements of H^0(U_1, O)
and u ∈ H^0(U_1, O) induces on K a function u_K that is a unit in H(K).

Assume that there is no factorization of f_K as required by the thesis.

Under this hypothesis we shall prove, in the following step (a), that
there is a suitable sequence \{U_h, x_h\}_{h≥1} with U_h ∈ ℱ and x_h ∈ U_h satisfying
certain properties specified below and, later in (b), we shall prove that
the existence of this sequence, implies a contradiction.

(a) We define a sequence \{U_h, x_h\}_{h≥1} such that U_h ∈ ℱ and x_h ∈ U_h ∩ K
for every h ≥ 1; furthermore we impose that the four following properties
hold for every h ≥ 1:

1. there are m_h ≥ 1, functions g_1^{(h)}, \cdots, g_{m_h}^{(h)} holomorphic on U_h and
irreducible in H^0(U_h, O) such that on U_h
\[ f = k^{(h)}(g^{(h)}_1)^{\lambda_{h,1}} \cdots (g^{(h)}_{m_h})^{\lambda_{h,m}} \]

where \( \lambda_{h,1}, \ldots, \lambda_{h,m} \) are non zero natural numbers and \( k^{(h)} \in H^0(U_h, \mathcal{O}) \) is such that \( k^{(h)}(x) \neq 0 \) for every \( x \in K \).

(2) If \( \mathcal{C}_h \) is the family of the irreducible components \( C \) of \( Z(U_h, f|_{U_h}) \) such that \( C \cap K \neq \emptyset \), then \( \mathcal{C}_h = \{ Z(U_h, g^{(h)}_1), \ldots, Z(U_h, g^{(h)}_{m_h}) \} \) and \( \mathcal{C}_h \) has exactly \( m_h \) elements;

(3) \( m_h \geq h \);

(4) There are \( h \) different elements \( C_1, \ldots, C_h \) of \( \mathcal{C}_h \) with \( x_1 \in C_1, \ldots, x_h \in C_h \).

To define \( (U_1, x_1) \) it suffices to call \( x_1 \) an arbitrary point of \( Z(U_1, f|_{U_1}) \cap K \).

Now, we assume \( (U_1, x_1), \ldots, (U_{h-1}, x_{h-1}) \) already defined and we define \( (U_h, x_h) \).

First of all we choose \( U_h \). Since \( f_K \) has no finite decomposition as product of irreducible elements of \( H(K) \), there is, at least, a function \( g^{(h-1)}_1 \), say \( g^{(h-1)}_1 \), which satisfies the following properties. On an open set \( V \in \mathcal{U}, V \subset U_{h-1} \) two holomorphic functions \( a, b \) are given such that

\[ g^{(h-1)}_1 = ab, Z(V, a) \cap K \neq \emptyset, Z(V, b) \cap K \neq \emptyset; \]

furthermore there is a point \( x \in K \) such that \( a(x) \neq 0, b(x) = 0 \). We set \( U_h := V \). Certainly \( \mathcal{C}_h \) is a finite set; let \( m_h \) be the cardinality of \( \mathcal{C}_h \); for every \( G_j \in \mathcal{C}_h \) \( (j = 1, \ldots, m_h) \), we call \( g^{(h)}_j \) a holomorphic function on \( U_h \), associated to \( G_j \). Property (2) is true. Lemma 1 implies that also property (1) holds. For every \( j \in \{1, \ldots, m_{h-1}\} \) we write briefly \( C_j \) to indicate \( Z(U_{h-1}, g^{(h-1)}_j) \); let \( D^*_j, \ldots, D^*_{ij} \) be the different irreducible components of \( C_j \cap U_h \), such that \( D^*_j \cap K \neq \emptyset \) for every \( i = 1, \ldots, i_j \). Certainly \( i_j \geq 1 \) for every such \( j \); furthermore the definition of \( U_h \) implies \( i_1 > 1 \).

The analytic sets \( D^*_j \) are 1-codimensional in \( U_h \) (we consider the complex dimension); hence they are irreducible components of \( Z(U_h, f|_{U_h}) \) i.e. they are elements of \( \mathcal{C}_h \). If \( D^*_j = D^*_t \) then \( (i, j) = (s, t) \); indeed, there exist points which are both in \( C_i \) and in \( C_t \) and which are regular in \( Z(U_{h-1}, f|_{U_{h-1}}) \). It follows \( m_h \geq m_{h-1} + 1 \), that is condition (3) is fulfilled.

Let \( C_1, \ldots, C_{h-1} \) be different elements of \( \mathcal{C}_{h-1} \) such that

\[ x_1 \in C_1, \ldots, x_{h-1} \in C_{h-1}; \]

for each \( j = 1, \ldots, h-1 \) the analytic subset \( C_j \cap U_h \) of \( U_h \) contains at least one element \( D_j \) of \( \mathcal{C}_h \) such that \( x_j \in D_j \); it was previously observed that \( D_1, \ldots, D_{h-1} \) are different one from the other. Let \( D \in \mathcal{C}_h \) such that \( D \neq D_j \) for every \( j \in \{1, \ldots, h-1\} \); pick a point \( x_h \in D \cap K \).

With this definition \( \{(U_1, x_1), \ldots, (U_h, x_h)\} \) satisfies (1), (2), (3) and also (4).

In step (b) we shall find a contradiction.

(b) Let \( x \) be a cluster point of the sequence \( \{x_h\}_{h \geq 1} \); since \( K \) is compact,
$x \in K$. We call $\mathcal{M}_x$ the maximal ideal of $\mathcal{O}_x$; $f_x \in \mathcal{M}_x$; let $s \in \mathbb{N}$ be such that $f_x \in \mathcal{M}_x^{-1}$, $f_x \notin \mathcal{M}_x^s$.

Let $\mathcal{F}$ be the coherent ideal sheaf on $U_1$ associated to the analytic subset $Z(U_1, f|_{U_1})$ of $U_1$ and let $\mathcal{F} = \mathcal{O}/\mathcal{F}$.

With regard to $\mathcal{F}$ there is an open neighborhood base $\mathcal{B}$ of $x$ in $K$ such that every $V \in \mathcal{B}$ has an open neighborhood base $\mathcal{X}_V$ in $X$ so that every $W \in \mathcal{X}_V$ is an $\mathcal{F}$-privileged neighborhood of $x$. The existence of $\mathcal{B}$ and of the families $\mathcal{X}_V$ is proved in ([1], théor. (1.4), (1.5)).

Let $V \in \mathcal{B}$; $V$ contains $s$ terms $x_{i_1}, \ldots, x_{i_s}$ with $i_1 < \cdots < i_s$.

Let us consider $U_{i_t}$; because of property (4), there are $s$ irreducible components $C_1, \ldots, C_s$ of $Z(U_{i_t}, f|_{U_{i_t}})$ different one from the other such that $x_{i_t} \in C_j$ for every $j = 1, \ldots, s$. Let $S$ be the union of the irreducible components of $Z(U_{i_t}, f|_{U_{i_t}})$ which contain $x$; because of (2) for every of these components $C$ there is a suitable $t$, $1 \leq t \leq s$, such that $C = Z(U_{i_t}, g_t^{(i_t)})$. But the functions $g_t^{(i_t)}$ vanishing in $x$ may not be more than $s - 1$, because of (1). Hence we may suppose, without loss of generality, that $C_1$ is not contained in $S$. Let $W \in \mathcal{X}_V$ be an $\mathcal{F}$-privileged neighborhood of $V$ in $X$ such that $W \subset U_{i_t}$ and let $y \in W \cap C_1$, $y \notin S$. Property (C) implies that a function $F \in H^0(U_{i_t}, \mathcal{O})$ exists that vanishes in the points of $S$ and only in these points. The function $F$ induces a section $F' \in H^0(W, \mathcal{F})$; $F'$ is not the zero section of $\mathcal{F}$ on $W$ because $F(y) \neq 0$; but $F' \in \mathcal{F}_x$ is the zero germ. This is a contradiction.

(II) Uniqueness of the decomposition in irreducible factors. Let

$$f_K = \delta \alpha_1^{m_1} \cdots \alpha_t^{m_t},$$

$$f_K = \lambda \beta_1^{p_1} \cdots \beta_s^{p_s},$$

where $\delta, \lambda$ are units in $H(K)$ and $\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_s$ are irreducible elements of $H(K)$. Furthermore assume that, for $i \neq j$ and $1 \leq i, j \leq t$, $\alpha_i$ and $\alpha_j$, are not associated elements in $H(K)$ and that, for $i \neq j$ and $1 \leq i, j \leq s$, $\beta_i$ and $\beta_j$ are not associated elements in $H(K)$.

In a suitable neighborhood $U \in \mathcal{U}$ there are holomorphic functions $d, a_1, \ldots, a_t, l, b_1, \ldots, b_s$ such that they induce respectively $\delta, \alpha_1, \ldots, \alpha_t, \lambda, \beta_1, \ldots, \beta_s$ and on $U$

$$f = da_1^{m_1} \cdots a_t^{m_t} = lb_1^{p_1} \cdots b_s^{p_s}.$$

Each $a_i$, for $i = 1, \ldots, t$, defines an irreducible component $V_i$ of $Z(U, a_i)$ such that $V_i \cap K \neq \emptyset$. Such a $V_i$ is unique; indeed, if $Z_i \neq V_i$ is an irreducible component of $Z(U, a_i)$ and $Z_i \cap K \neq \emptyset$, then, by lemma 1 and by recalling that property (C) holds on $U$, we can write $a_i = vgh$ where $v, g, h$ are holomorphic functions on $U$ and $g, h$ are respectively associated in $U$ to $V_i$ and to $Z_i$; it follows that $x_i = vKgKkK$; this is a contradiction,
because \( g_K \) and \( h_K \) are not units in \( H(K) \). Let \( i, j \in \{1, \ldots, t\}, \ i \neq j \), then \( V_i \neq V_j \). Assume, in fact, \( V_i = V_j \); let \( g_i \) be a holomorphic function associated in \( U \) to \( V_j \); since \( g_i \) divides \( a_i \) and \( a_j \) in \( H(U, \mathcal{O}) \), it follows that \( \alpha_i, \alpha_j \) are associated elements in \( H(K) \). Every \( V_i \) is an irreducible component of \( Z = Z(U, f_i|_U) \); indeed, if \( x \in V_i \), then \( \dim_C (V_i)_x = \dim_C X_x - 1 = \dim_C Z_x \); vice versa for every irreducible component \( C \) of \( Z \) such that \( C \cap K \neq \emptyset \), there is one \( i \in \{1, \ldots, t\} \) such that \( C = V_i \). Every \( b_j \) defines, for \( j = 1, \ldots, s \), one and only one irreducible component \( W_j \) of \( Z(U, b_j) \) such that \( W_j \cap K \neq \emptyset \); by what we have just proved it follows that \( t = s \) and, eventually changing the indicization of \( b_1, \ldots, b_s \), \( V_i = W_i \) for every \( i \in \{1, \ldots, t\} \). Then we can deduce that for each \( i \) the elements \( \alpha_i, \beta_i \) are associated in the ring \( H(K) \).

Pick, now, \( i \in \{1, \ldots, t\} \) let \( x \in V_i \) be a regular point of \( Z \) with \( d(x) \neq 0 \), \( l(x) \neq 0 \) and let \( g_i \) be a function associated to \( V_i \) in \( U \). Recall that \( V_i \) is an irreducible component of \( Z \); by lemma 1, \( a_i = s_i g_i^r_i \) and \( s_i(x) \neq 0 \); by the irreducibility of \( \alpha_i \) it follows \( r = 1 \). Furthermore \( a_j(x) \neq 0, b_j(x) \neq 0 \) for \( j \neq i \). Then on \( U \) we can write

\[
    f = f_1 g_i^{m_i} = f_2 g_i^{p_i}
\]

where \( f_1, f_2 \in H^0 (U, \mathcal{O}), \ f_1(x) \neq 0, \ f_2(x) \neq 0 \); it follows that \( m_i = p_i \). The lemma is proved.

Note that property (C) holds on each Cousin-II space \( U \) in which the local rings \( \mathcal{O}_{U, y} \) are U.F.D. (= unique factorization domains) for every \( y \in U \). Thus, we have proved that if \( K \) is a semianalytic, connected, compact Cousin-II subset of a complex space \( X \) and if \( K \) has an open neighborhood \( U \) in \( X \) so that the rings \( \mathcal{O}_{y} \) are U.F.D. for every \( y \in U \), then \( H(K) \) is an U.F.D.

By a different way, in [2] a criterion is given for the factoriality of \( H(K) \) in the case that \( K \) is a compact Stein set and that every element of \( H(K) \) is expressible as product of irreducible elements.

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