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HECKE THEORY FOR $GL(3)$

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In view of the recent results of Gelfand, Kajdan, and one of the authors ([1], [6]), it appears likely that the results of [2] – the Hecke theory – will extend to all groups $GL(p)$. In this note, we present nearly complete results for the case $p = 3$. To avoid technical difficulties we restrict ourselves to the case of a function field.

1. Global computations

Let F be a commutative field, G (resp. G') the group $GL(p)$ (resp. $GL(p-1)$) regarded as an algebraic group defined over F. We regard G' as imbedded into G by the map

$$g \to \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}.$$

We denote by Z (resp. Z') the center of G (resp. G') and by ${}^tg$ the transpose of a matrix g. The entries of a matrix g in G are written as g_{ij}, the first index being the column index.

In this section we take F to be a function field whose field of constants has cardinality Q. We let $\mathbb{A}$ be the ring of adeles of F and $\mathbb{I}$ the group of ideles. Then $G_{\mathbb{A}} = GL(p, \mathbb{A})$ is a locally compact group of which $G_F = GL(p, F)$ is a discrete subgroup. A cusp form on $G_{\mathbb{A}}$ is a complex-valued function ϕ, which satisfies the following three conditions:

(1.1) for all $g \in G_{\mathbb{A}}$, $a \in Z_{\mathbb{A}} = \mathbb{I}$, and $\gamma \in G_F$, $\phi(a\gamma g) = \omega(a)\phi(g)$,

where ω is a quasi-character of $\mathbb{I}/F^{\times}$,

(1.2) the function ϕ is invariant on the right by a compact open subgroup of $G_{\mathbb{A}}$,

(1.3) if P is a proper F-parabolic subgroup of G and U its unipotent radical, then

$$\int_{U_F \backslash U_{\mathbb{A}}} \phi(ug)\mathrm{d}u = 0, \qquad \text{for all } g \in G_{\mathbb{A}}.$$

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Let N be the group of p by p upper triangular matrices whose diagonal entries are one. Let $N' = N \cap G'$ be the corresponding subgroup of G'. Choose a non-trivial character ψ of $\mathbb{A}/F$ and define a character θ of $N_{\mathbb{A}}$ by the formula

$$\theta(n) = \prod_{1 \leqq i \leqq p-1} \psi(n_{i+1,i}). \tag{1.4}$$

Let ϕ be a cusp form on $G_{\mathbb{A}}$. The function

$$W(g) = \int_{N_F \backslash N_{\mathbb{A}}} \phi(ng)\bar{\theta}(n)\,\mathrm{d}n \tag{1.5}$$

is called the 'Whittaker function' attached to ϕ. It transforms on the left according to the formula

$$W(ng) = \theta(n)W(g), \qquad \text{for } n \in N_{\mathbb{A}},$$

and the form ϕ has the following Fourier expansion:

$$\phi(g) = \sum_{\gamma \in N'_F \backslash G'_F} W(\gamma g). \tag{1.6}$$

Now let ϕ' be a cusp form on $G'_{\mathbb{A}}$ and let us compute the integral

$$\int_{G'_F \backslash G'_{\mathbb{A}}} \phi(g)\phi'(g)|\det g|^s\,\mathrm{d}g, \tag{1.7}$$

where $s \in \mathbb{C}$ and $\mathrm{d}g$ is an invariant measure on $G'_F \backslash G'_{\mathbb{A}}$. Replacing ϕ by (1.6), we obtain

$$\begin{aligned}
\int_{G'_F \backslash G'_{\mathbb{A}}} \sum_{N'_F \backslash G'_F} W(\gamma g)\phi'(g)|\det g|^s\,\mathrm{d}g &= \int_{G'_F \backslash G'_{\mathbb{A}}} \sum_{N'_F \backslash G'_F} W(\gamma g)\phi'(\gamma g)|\det (\gamma g)|^s\,\mathrm{d}g \\
&= \int_{N'_F \backslash G'_{\mathbb{A}}} W(g)\phi'(g)|\det g|^s\,\mathrm{d}g \\
&= \int_{N'_{\mathbb{A}} \backslash G'_{\mathbb{A}}} |\det g|^s\,\mathrm{d}g \int_{N'_F \backslash N'_{\mathbb{A}}} W(ng)\phi'(ng)\,\mathrm{d}n \\
&\int_{N'_{\mathbb{A}} \backslash G'_{\mathbb{A}}} |\det g|^s W(g)\,\mathrm{d}g \int_{N'_F \backslash N'_{\mathbb{A}}} \theta(n)\phi'(ng)\,\mathrm{d}n.
\end{aligned}$$

Let W' be the Whittaker function attached to ϕ' and ε the $p-1$ by $p-1$ matrix defined by

$$\varepsilon_{ij} = \delta_{i,j}\,(-1)^i.$$

Since $\theta(\varepsilon n \varepsilon^{-1}) = \theta(n^{-1})$, in the last line the inner integral is actually $W'(\varepsilon g)$. Thus (1.7) is equal to

$$\int_{N'_{\mathbb{A}}\backslash G'_{\mathbb{A}}} W(g)W'(\varepsilon g)|\det g|^s \mathrm{d}g. \tag{1.8}$$

More precisely, for Res sufficiently large, all the above integrals converge and are equal.

Since ϕ and ϕ' are compactly supported modulo $G_F Z_{\mathbb{A}}$ and $G'_F Z'_{\mathbb{A}}$ respectively, the integral (1.7) is always convergent. Hence, the integral (1.8), which converges only in a half-space, is, as a function of s, a polynomial in Q^{-s}, Q^s.

Now, let w be the element of G_F defined by

$$w_{ij} = 0 \qquad \text{if } i+j \neq p+1,\ w_{p+1-i,i} = (-1)^{i-1} \quad (1 \leqq i, j \leqq p)$$

and let w' be the corresponding element of G'_F. Clearly

$$wNw^{-1} = {}^tN, \qquad \theta(w^tn^{-1}w^{-1}) = \theta(n).$$

In particular, the functions $\tilde{\phi}$ and $\tilde{\phi}'$ defined by

$$\tilde{\phi}(g) = \phi(w^tg^{-1}) = \phi({}^tg^{-1}), \qquad \tilde{\phi}(g) = \phi'(w'^tg^{-1}) = \phi'({}^tg^{-1})$$

are automorphic forms on $G_{\mathbb{A}}$ and $G'_{\mathbb{A}}$ respectively, whose Whittaker functions are the functions $\tilde{W}$ and $\tilde{W}'$ given by

$$\tilde{W}(g) = W(w^tg^{-1}), \qquad \tilde{W}'(g) = W'(w'^tg^{-1}).$$

Changing g into ${}^tg^{-1}$ in the integral (1.7), we easily obtain that (1.7) is also equal to the following integral:

$$\int_{G'_F\backslash G'_{\mathbb{A}}} \tilde{\phi}(g)\tilde{\phi}'(g)|\det g|^{-s} \mathrm{d}g, \tag{1.9}$$

and conclude that (1.8) and the integral

$$\int_{N'_{\mathbb{A}}\backslash G'_{\mathbb{A}}} \tilde{W}(g)\tilde{W}'(g)|\det g|^{-s} \mathrm{d}g \tag{1.10}$$

are equal in the sense of analytic continuation – as polynomials in Q^{-s}, Q^s.

2. Local conjectures

From now on and until Section 5, the field F will be a non-archimedean local field whose residual field has q elements. We denote by $|x|$ or $\alpha_F(x)$ or simply $\alpha(x)$ the module of an x in F. We denote by $M(p \times r, F)$ the space of matrices with p columns and r rows whose entries belong to F, and by $\mathcal{S}(p \times r, F)$ the space of locally constant compactly supported complex-valued functions on $M(p \times r, F)$.

Fix a non-trivial character ψ of F and let θ be the character of N_F defined by (1.4). Let π be an irreducible admissible representation of G_F. We say that π is non-degenerate if it can be realized by right translations in a space $\mathscr{W}$ of functions W on G_F satisfying

$$W(ng) = \theta(n)W(g), \qquad \text{for all } n \text{ in } N_F,\ g \text{ in } G_F.$$

If π is non-degenerate the space $\mathscr{W}$ is unique and denoted by $\mathscr{W}(\pi, \psi)$. It is called the Whittaker model of π ([1], [6]).

Since ${}^tg$ and g are always conjugate in G_F, the representation $\tilde{\pi}$ contragredient to π is equivalent to the representation $g \to \pi({}^tg^{-1})$ ([1]). In particular, if π is non-degenerate so is $\tilde{\pi}$. More precisely, if W belongs to $\mathscr{W}(\pi, \psi)$ then the function $\tilde{W}$ defined by

$$\tilde{W}(g) = W(w\,{}^tg^{-1})$$

belongs to $\mathscr{W}(\tilde{\pi}, \psi)$.

Let π and π' be non-degenerate representations of G_F and G'_F respectively. For $W \in \mathscr{W}(\pi, \psi)$ and $W' \in \mathscr{W}(\pi', \psi)$ and $s \in \mathbb{C}$, we set

$$\Psi(s, W, W') = \int_{N'_F \backslash G'_F} W(g)W'(\varepsilon g)|\det g|^{s-\frac{1}{2}}\,dg.$$

The global computations of Section 1 lead to the formulation of the following conjectures.

(2.1) For Res sufficiently large the integrals $\Psi(s, W, W')$ and $\Psi(s, \tilde{W}, \tilde{W}')$ are absolutely convergent.

(2.2) They are rational functions of q^{-s}. More precisely, for $W \in \mathscr{W}(\pi, \psi)$ and $W' \in \mathscr{W}(\pi', \psi)$ the integrals $\Psi(s, W, W')$ span a fractional ideal $\mathbb{C}[q^{-s}, q^s]L(s, \pi \times \pi')$ of the ring $\mathbb{C}[q^{-s}, q^s]$; the factor $L(s, \pi \times \pi')$ has the form $1/P(q^{-s})$ where $P \in \mathbb{C}[X]$ and $P(0) = 1$.

There is a similar factor $L(s, \tilde{\pi} \times \tilde{\pi}')$.

(2.3) There is a factor $\varepsilon(s, \pi \times \pi', \psi)$ of the form cq^{-ns} such that

$$\Psi(1-s, \tilde{W}, \tilde{W}')/L(1-s, \tilde{\pi} \times \tilde{\pi}') = \varepsilon(s, \pi \times \pi', \psi)\omega'(-1)^p\Psi(s, W, W')/L(s, \pi \times \pi')$$

for $W \in \mathscr{W}(\pi, \psi)$ and $W' \in \mathscr{W}(\pi', \psi)$, where ω' is the quasi-character of $F^\times$ such that $\pi'(a1_{p-1}) = \omega'(a) \cdot 1$.

Let us emphasize that in the above statements the pairs (π, π') and $(\tilde{\pi}, \tilde{\pi}')$ play symmetrical roles, since, in fact, $\tilde{W}$ (resp. $\tilde{W}'$) belongs to $\mathscr{W}(\tilde{\pi}, \psi)$(resp. $\mathscr{W}(\tilde{\pi}', \psi)$) if W (resp. W') belongs to $\mathscr{W}(\pi, \psi)$(resp. $\mathscr{W}(\tilde{\pi}', \psi)$).

These conjectures have been proved for $p = 2$ (see below), and, in [1], for all p under the additional assumption that π is supercuspidal. Indeed, under this assumption, the restriction of a W in $\mathscr{W}(\pi, \psi)$ to G'_F belongs to the space $C_c^\infty(G', \theta)$ of all maps f from G'_F to $\mathbb{C}$ which, on the left, transform according to

$$f(ng) = \theta(n)f(g), \qquad n \in N'_F,$$

and are locally constant, and compactly supported modulo N'_F. In fact, the map $W \to W|G'_F$ is a bijection of $\mathscr{W}(\pi, \psi)$ onto $C_c^\infty(G', \theta)$. It follows that the integrals are convergent for all s and satisfy (2.2) with $L(s, \pi \times \pi') = L(s, \tilde{\pi} \times \tilde{\pi}') = 1$. In order to prove (2.3) one need only show that there is a constant c such that, for all W in $\mathscr{W}(\pi, \psi)$ and W' in $\mathscr{W}(\pi', \psi)$,

$$\int_{N'_F \backslash G'_F} \tilde{W}(g)\tilde{W}'(\varepsilon g)\,\mathrm{d}g = c \int_{N'_F \backslash G'_F} W(g)W'(\varepsilon g)\,\mathrm{d}g.$$

Indeed, the left hand side defines a bilinear form B on the product $C_c^\infty(G', \theta) \times W(\pi', \psi)$. It satisfies the following invariance condition:

$$B(\rho(g)f, \pi'(g)W') = B(f, W'), \qquad \text{where } (\rho(g)f)(h) = f(hg).$$

Therefore, it must of the form

$$B(f, W') = \int_{N'_F \backslash G'_F} \lambda(\pi'(g)W')f(g)\,\mathrm{d}g,$$

where λ is a linear form on $\mathscr{W}(\pi', \psi)$ such that

$$\lambda(\pi'(n)W') = \bar{\theta}(n)\lambda(W'), \qquad \text{for } n \in N'_F.$$

The uniqueness of the Whittaker model shows that λ must be proportional to the linear form $W' \to W'(\varepsilon)$ and our assertion follows.

In the next sections we call the factor $L(s, \pi \times \pi')$ the 'g.c.d.' of the integrals $\Psi(s, W, W')$ and set

$$\varepsilon'(s, \pi \times \pi', \psi) = \varepsilon(s, \pi \times \pi', \psi)L(1-s, \tilde{\pi} \times \tilde{\pi}')/L(s, \pi \times \pi').$$

We shall use also the same notation and terminology for other integrals which have similar properties.

3. The case $p = 2$

In this section we review the case $p = 2$. Accordingly, $G = GL(2)$, $G' = GL(1)$, $G'_F = F^\times$,

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$w' = 1$. Moreover, $w\,{}^tg^{-1}w^{-1} = \det g^{-1} \cdot g$ for g in G_F.

Let π and π' be as above. Let ω be the quasi-character of $F^\times$ such that

$$\pi\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \omega(a)\cdot 1.$$

Then $\tilde{\pi}$ is equivalent to the representation $\pi\otimes\omega^{-1}$ (i.e. the representation $g\to\pi(g)\omega^{-1}(\det g)$). Moreover, for W in $\mathscr{W}(\pi,\psi)$ the function $\tilde{W}$ is given by

$$\tilde{W}(g) = W(gw)\omega^{-1}(\det g).$$

On the other hand, the representation π' is just a quasi-character of $F^\times$ and the function W' coincides with the function π'. Therefore the integral $\Psi(s, W, W')$ reduces to the integral

$$\int_{F^\times} W\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-\frac{1}{2}}\pi'(a)\mathrm{d}^\times a$$

and the integral $\Psi(s, \tilde{W}, \tilde{W}')$ to

$$\int_{F^\times} W\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} w\right] |a|^{s-\frac{1}{2}}\omega^{-1}\pi'^{-1}(a)\mathrm{d}^\times a.$$

Hence, the results of [2] show that (2.1) to (2.3) are true with

$$L(s,\pi\times\pi') = L(s,\pi\otimes\pi'),\qquad \varepsilon(s,\pi\times\pi',\psi) = \varepsilon(s,\pi\otimes\pi',\psi).$$

We shall need the fact that these factors are related to other integrals as well. Indeed, let V be the space of π, $\tilde{V}$ the space of $\tilde{\pi}$, and $\langle\cdot,\cdot\rangle$ the invariant bilinear form on $V\times\tilde{V}$. A coefficient of π is a function f of the form

$$f(g) = \langle \pi(g)v, \tilde{v}\rangle$$

where v is in V and $\tilde{v}$ in $\tilde{V}$. The function $\check{f}$ defined by $\check{f}(g) = f(g^{-1})$ is a coefficient of $\tilde{\pi}$.

Let f be a coefficient of π, Φ a function in $\mathscr{S}(2\times 2, F)$ and s a complex number; we set

$$Z(\Phi, s, f) = \int_{G_F} f(x)|\det x|^s\Phi(x)\mathrm{d}^\times x,$$

where $\mathrm{d}^\times x$ is a multiplicative Haar measure on G_F. These integrals converge for Re s large, the 'g.c.d.' of the integrals $Z(\Phi, s+\frac{1}{2}, f)$ is the factor $L(s,\pi) = L(s,\pi\times 1)$, and the integrals satisfy the functional equation

$$Z(\hat{\Phi}, \tfrac{3}{2}-s, \check{f}) = \varepsilon'(s,\pi,\psi)Z(\Phi, s+\tfrac{1}{2}, f) \tag{3.1}$$

where $\hat{\Phi}$ denotes the Fourier transform of Φ:

$$\hat{\Phi}(x) = \int_{M(2,F)} \Phi(y)\psi(Tr(yx))\mathrm{d}y,$$

Tr denoting the trace of an element of $M(2\times 2, F)$ and dy being the self-dual Haar measure on $M(2\times 2, F)$.

At least when π is supercuspidal, this can be derived from the following lemma combined with the fact that on $V\times\tilde{V} = \mathscr{W}(\pi, \psi)\times\mathscr{W}(\tilde{\pi}, \psi)$ the invariant bilinear form is given by

$$\langle W_1, W_2\rangle = \int_{F^\times} W_1\begin{pmatrix} a & 0\\ 0 & 1\end{pmatrix} W_2\begin{pmatrix} -a & 0\\ 0 & 1\end{pmatrix} d^\times a.$$

LEMMA (3.2): *Suppose Φ is in $\mathscr{S}(2\times 2, F)$. Then*

$$\int_{F^3} \Phi\left[\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0\\ b & -1\end{pmatrix}\right]\psi(-b)\psi(-x)\,da\,db\,dx$$

$$= \int_F \hat{\Phi}\left[w^{-1}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right]\psi(x)\,dx.$$

The lemma itself is a simple consequence of the Fourier theorem in one variable.

Finally, we shall need the results of [3] that we now state in a form appropriate to our purposes. Let π_1 and π_2 be non-degenerate representations of $G_F = GL(2, F)$. For $\Phi\in\mathscr{S}(1\times 2, F)$, $W_1\in\mathscr{W}(\pi_1, \psi)$, and $W_2\in\mathscr{W}(\pi_2, \psi)$, we set

$$\Psi(s, W_1, W_2, \phi) = \int_{N_F\backslash G_F} W_1(g)W_2(\varepsilon g)\phi[(0, 1)g]|\det g|^s dg$$

where

$$\varepsilon = \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}.$$

Then again, the integrals are convergent for Res large, have a g.c.d. $L(s, \pi\times\pi')$, and satisfy a functional equation

(3.3) $\Psi(1-s, \tilde{W}_1, \tilde{W}_2, \hat{\phi}) = \omega_2(-1)\varepsilon'(s, \pi_1\times\pi_2, \psi)\Psi(s, W_1, W_2, \phi)$

where $\hat{\phi}$ is the Fourier transform of ϕ:

$$\hat{\phi}(x, y) = \int_{F^2}\phi(u, v)\psi(ux+vy)\,du\,dv,$$

and ω_i, for $i = 1, 2$, is the quasi-character of $F^\times$ defined by

$$\pi_i\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} = \omega_i(a)\cdot 1.$$

4. The case $p = 3$

In this section $p = 3$; accordingly, $G = GL(3)$, $G' = GL(2)$,

$$w = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad w' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In addition, we let U be the subgroup of matrices of the form

$$\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}.$$

Fix an admissible irreducible representation σ of G'_F and a quasi-character μ of $F^\times$. Let $\mathscr{H}$ be the space of σ and denote by $\mathscr{V}$ the space of all locally constant mappings f from G_F to $\mathscr{H}$, such that

$$f\left[\begin{pmatrix} m & 0 \\ u & a \end{pmatrix} g\right] = \sigma(m)|\det m|^{-\frac{1}{2}}\mu\alpha(a) f(g)$$

for all m in $G'_F = GL(2, F)$, a in $F^\times$, and g in G_F. We call π_0 the representation of G_F in $\mathscr{V}$ defined by right translations. From the results of [4], one can prove that any irreducible admissible representation of G_F which is not supercuspidal is a component of π_0, for a suitable choice of σ and μ. Furthermore, according to a result of Rodier, if π_0 has a non-degenerate component, then σ itself is non-degenerate. So, from now on, we assume σ to be so. We also let $\mathscr{H}$ be the space $\mathscr{W}(\sigma, \psi)$. Then, one can show that π_0 has exactly one non-degenerate component, π say.

In order to show that (2.1) to (2.3) are true for any non-degenerate representation of G_F, it suffices to show that they are true for such a π. In these notes, we assume in addition that π_0 is irreducible and sketch a proof of the fact that $\pi = \pi_0$ is then non-degenerate and that the conjectures are true for π and $\tilde{\pi}$.

Let $\mathscr{S} = \mathscr{S}(3 \times 2, F) \otimes \mathscr{H}$ be the space of all locally constant compactly supported maps from $M(3 \times 2, F)$ *to* $\mathscr{H}$. If Φ is in $\mathscr{S}$, then the function f defined by

$$(4.1) \qquad f(g) = \mu\alpha(\det g) \int_{G'_F} \mu\alpha^{\frac{3}{2}}(\det m)\sigma(m^{-1})\Phi(vmg)\,d^\times m,$$

where

$$v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

is an element of $\mathscr{V}$ – provided the integral converges. For $g \in G_F$, $f(g)$ is a function on G'_F whose value at $h \in G'_F$ is denoted by $f(g; h)$. A similar notation is used for Φ. Accordingly, $f(g; e)$ is simply

$$f(g; e) = \mu\alpha(\det g) \int_{G'_F} \mu\alpha^{\frac{3}{2}}(\det m)\Phi(vmg; m^{-1})\mathrm{d}^\times m.$$

Set now

$$W(g) = \int_{U_F} f(ug; e)\bar{\theta}(u)\mathrm{d}u. \tag{4.2}$$

Then, when W is defined, it satisfies

$$W(ng) = \theta(n)W(g), \qquad \text{for } n \in N_F.$$

Suppose now that Φ has the form

$$\Phi\left[\begin{pmatrix} a & b & x \\ c & d & y \end{pmatrix}; g\right] = \phi_1\begin{pmatrix} a & b \\ c & d \end{pmatrix}\phi_2(x, y)W_1(g), \tag{4.3}$$

where ϕ_1 is in $\mathscr{S}(2\times 2, F)$, ϕ_2 in $\mathscr{S}(2\times 1, F)$, and W_1 in $\mathscr{W}(\sigma, \psi)$. Then, for $g \in G'_F$,

$$\begin{aligned} W(g) &= \int_{U_F} f(gg^{-1}ug; e)\bar{\theta}(u)\mathrm{d}u \\ &= \alpha^{\frac{1}{2}}(\det g) \int_{U_F} f(u; g)\bar{\theta}(gug^{-1})\mathrm{d}u \\ &= \alpha^{\frac{1}{2}}(\det g) \int_{G'_F \times F^2} \phi_1(m)\phi_2[(x, y)^t m]W_1(gm^{-1})\psi[(0, -1)g^t(x, y)] \\ &\qquad\qquad \times \mu\alpha^{\frac{3}{2}}(\det m)d^\times m\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

Let ϕ be the co-Fourier-transform of ϕ_2:

$$\phi(u, v) = \int_{F^2} \phi_2(x, y)\psi[-(u, v)^t(x, y)]\,\mathrm{d}x\,\mathrm{d}y.$$

Then, we may write the above integral as

$$W(g) = \alpha^{\frac{1}{2}}(\det g) \int_{G'_F} \phi_1(m)\phi[(0, 1)gm^{-1}]W_1(gm^{-1})\mu\alpha^{\frac{1}{2}}(\det m)\mathrm{d}^\times m, \tag{4.4}$$

an integral which converges 'better' than (4.2). It can be used to define W even though (4.2) might not converge. Then, it can be shown that π is non-degenerate and W belongs to $\mathscr{W}(\pi, \psi)$. Actually, since the space $\mathscr{S}$ is spanned by functions of the form (4.3), the space $\mathscr{W}(\pi, \psi)$ is spanned by functions W for which $W(g)$, $g \in G'_F$, is given by (4.4).

We also need an integral representation for $\tilde{W}(g)$, $g \in G'_F$. Now, if g is in G'_F,

$$\tilde{W}(g) = W(w\,{}^t g^{-1}) = \int_{U_F} f(uw\,{}^t g^{-1}; e)\bar{\theta}(u)\,\mathrm{d}u$$

$$= \mu\alpha(\det g^{-1}) \int_{G'_F \times U_F} \mu\alpha^{\frac{3}{2}}(\det m)\Phi[vmuw\,{}^t g^{-1}; m^{-1}]\bar{\theta}(u)\,\mathrm{d}^\times m\,\mathrm{d}u.$$

More explicitly,

$$u = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad \theta(u) = \psi(b), \qquad \mathrm{d}u = \mathrm{d}a\,\mathrm{d}b,$$

and $vmuw^t g^{-1}$ is the 3 by 2 matrix obtained by the juxtaposition of the 2 by 2 matrix

$$m \begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix} {}^t g^{-1}$$

and the 1 by 2 matrix $m\binom{1}{0}$. Hence, the above integral may be written in the form

$$\mu\alpha(\det g^{-1}) \int_{G'_F \times F^2} \mu\alpha^{\frac{3}{2}}(\det m)\phi_1\left[m\begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix}{}^t g^{-1}\right]\phi_2[(1, 0)\,{}^t m]$$
$$\times W_1(m^{-1})\psi(-b)\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}^\times m.$$

The integral on G'_F can be realized as an integral on N'_F and then on G'_F/N'_F. In this way, one obtains

$$\mu\alpha(\det g^{-1}) \int_{G'_F/N'_F} \mu\alpha^{\frac{3}{2}}(\det m)\phi_2[(1, 0)\,{}^t m]W_1(m^{-1})\,\mathrm{d}^\times m$$
$$\times \int \phi_1\left[mn\begin{pmatrix} a & 0 \\ b & -1 \end{pmatrix}{}^t g^{-1}\right]\psi(-b)\,\mathrm{d}a\,\mathrm{d}b\,\bar{\theta}(n)\,\mathrm{d}n.$$

Using Lemma (3.2) to transform the inner integral, one arrives at

$$\alpha\mu^{-1}(\det g) \int_{G'_F/N'_F} \mu\alpha^{-\frac{1}{2}}(\det m)\phi_2[(1, 0)\,{}^t m]W_1(m^{-1})\,\mathrm{d}^\times m$$
$$\times \int_{N'_F} \hat{\phi}_1[{}^t g w'^{-1} n m^{-1}]\theta(n)\,\mathrm{d}n.$$

At this point, one puts back together the integrations on G'_F/N'_F and N'_F and changes m into ${}^t m^{-1}\,{}^t g w'^{-1}$ to arrive at the following formula:

$$(4.5) \quad \tilde{W}(g) = \alpha^{\frac{1}{2}}(\det g) \int_{G'_F} \mu^{-1}\alpha^{\frac{1}{2}}(\det m)\hat{\phi}_1{}^t(m)\hat{\phi}[(0, 1)gm^{-1}]$$
$$\times \tilde{W}_1(gm^{-1})\,\mathrm{d}^\times m.$$

Here $\hat{\phi}_1{}^t(m) = \hat{\phi}_1({}^tm)$ and $\hat{\phi} = \phi_2$ is the Fourier transform of ϕ.

Now let π' be a non-degenerate representation of G'_F. For W' in $\mathscr{W}(\pi', \psi)$, the integral $\Psi(s, W, W')$ may be written as a double integral with respect to $g \in N'_F \backslash G'_F$ and $m \in G'_F$. After changing g into gm we find that

$$(4.6) \qquad \Psi(s, W, W') = \int_{G'_F} \phi_1(m)\mu\alpha^{s+\frac{1}{2}}(\det m) f(s, m) \mathrm{d}^\times m$$

where

$$f(s, m) = \Psi(s, W_1, \pi'(m)W', \phi).$$

Since, for a fixed s, the function $m \to f(s, m)$ is a coefficient of π', it is not too hard to see that (2.1) and (2.2) are true with

$$L(s, \pi \times \pi') = L(s, \pi' \times \sigma) L(s, \pi' \otimes \mu).$$

Similarly, one finds that

$$(4.7) \qquad \Psi(s, \tilde{W}, \tilde{W}') = \int_{G'_F} \hat{\phi}_1(m)\mu^{-1}\alpha^{s+\frac{1}{2}}(m) f'(s, {}^tm) \mathrm{d}^\times m$$

where

$$f'(s, m) = \Psi(s, \tilde{W}_1, \tilde{\pi}'(m)\tilde{W}', \hat{\phi}).$$

Now one observes that

$$\tilde{\pi}'(m)\tilde{W}' = (\pi'({}^tm^{-1})W')^{\sim}$$

and therefore that, by (3.3),

$$f'(1-s, {}^tm) = \omega'(-1)\varepsilon'(s, \pi \times \pi', \psi) f(s, m^{-1}).$$

Combining this with (3.1), one arrives at the functional equation (2.3) with

$$\varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \pi' \times \sigma, \psi)\varepsilon(s, \pi' \otimes \mu, \psi).$$

Hence (2.1) to (2.3) are now completely proved for $p = 3$. For instance, if μ, π, and π' are 'unramified' – contain the trivial representation of a maximal compact subgroup, the various factors have the following form:

$$L(s, \mu) = (1 - cq^{-s})^{-1}, \qquad L(s, \sigma) = (1 - aq^{-s})^{-1}(1 - bq^{-s})^{-1},$$

$$(4.5) \qquad L(s, \pi') = (1 - a'q^{-s})^{-1}(1 - b'q^{-s})^{-1},$$

$$L(s, \pi \times \pi')^{-1} = (1 - a'cq^{-s})(1 - b'cq^{-s})(1 - aa'q^{-s})(1 - ba'q^{-s})(1 - ab'q^{-s}) \times (1 - bb'q^{-s}).$$

If, in addition, the character ψ is of order zero, then the ε factor is one.

5. Conclusion

Let us go back to the situation and the notations of Section 1. Let π be an admissible irreducible representation of $G_{\mathbb{A}}$. Then, π is, in a certain sense, an infinite tensor product $\otimes \pi_v$, where, for each place v, π_v is an admissible irreducible representation of $G_v = GL(3, F_v)$; almost all the representations π_v are unramified. We shall assume that π is non-degenerate (i.e. that each π_v is non-degenerate) and also that there is a character ω of $\mathbb{I}/F^{\times}$ such that

$$\pi\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \omega(a) \cdot 1 \qquad \text{for } a \text{ in } \mathbb{I}.$$

Let also π' be a representation of $G'_{\mathbb{A}}$ satisfying the same assumptions. Then, we define

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v), \qquad \varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

The first product is an infinite Euler product whose factors are almost all of the form (4.5); if, for instance, the representations are pre-unitary, then it converges for Res sufficiently large. In the second factor, we have set $\psi(x) = \prod \psi_v(x_v)$; the product has only a finite number of terms $\neq 1$ and is actually independent of the choice of ψ.

Suppose now that π and π' are contained in the corresponding space of cusp-forms. They are then both pre-unitary and non-degenerate. The factor $L(s, \pi \times \pi')$ is a polynomial in Q^{-s}, Q^{s}, and satisfies the functional equation

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi')L(1-s, \tilde{\pi} \times \tilde{\pi}').$$

The proof follows step by step the proof of (11.1) in [3].

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