Composition Mathematica

M. D. Alder

Inverse limits of simplicial complexes

Compositio Mathematica, tome 29, nº 1 (1974), p. 1-7
<http://www.numdam.org/item?id=CM_1974__29_1_1_0>

COMPOSITIO MATHEMATICA, Vol. 29, Fasc. 1, 1974, pag. 1–7
Noordhoff International Publishing
Printed in the Netherlands

INVERSE LIMITS OF SIMPLICIAL COMPLEXES

M. D. Alder

Summary

Eilenberg and Steenrod have shown that any compact space may be expressed as an inverse limit in $\mathscr{T}op$, the category of topological spaces, of a diagram of simplicial complexes. [1]

We show that any paracompact space may be expressed as a limit of a diagram of nerves; further that $[-, K] \circ \mathscr{H} : \mathscr{T}op \to \mathscr{E}ns$ preserves such limits when K is a complex, where $[-, K]$ denotes the contravariant hom-functor from $\mathscr{H}tp$ (the category of spaces and homotopy classes of maps) to $\mathscr{E}ns$ the category of sets, and $\mathscr{H} : \mathscr{T}op \to \mathscr{H}tp$ is the canonical quotient functor.

1. Introduction

Let Δ denote a small category having the property that for A, B objects of Δ, there exists a C in Δ and maps $f: C \to A$, $g: C \to B$ respectively.

1.1 DEFINITION: A diagram in a category $\mathscr{M}$ with scheme Δ will be said to be a semi inverse diagram.

1.2. DEFINITION: $\Gamma: \mathscr{M} \to \mathscr{B}$ is a *semi-directly continuous* contravariant functor if it takes the limit of every semi inverse diagram $(\Delta, \Phi: \Delta \to \mathscr{M})$ in $\mathscr{M}$ to the colimit of $(\Delta, \Gamma \circ \Phi: \Delta \to \mathscr{B})$ in $\mathscr{B}$.

1.3 REMARK: We assume familiarity with the diagram of nerves of a space arising from locally finite partitions of unity, and with the natural map which we shall denote by $\pi_j: X \to N_j$ from X a space admitting a locally finite partition of unity having nerve N_j. We recall that if one locally finite partition of unity with nerve N_j refines another with nerve N_k, then there is a map $\alpha_{jk}: N_j \to N_k$ which homotopy commutes with the canonical maps from X.

1.4 REMARK: Our first proposition shows that when X is paracompact we may choose maps between the nerves such that X is the limit of the resulting semi-inverse diagram. This is the content of §2. In §3 we show that if K is a complex, $[-, K] \circ \mathscr{H} : \mathscr{T}op \to \mathscr{E}ns$ preserves limits of this type. Finally in §4 we conclude with remarks on some applications of these results.

2.

2.1 DEFINITION: Let $\{f_u\}_U$ and $\{g_v\}_V$ denote locally finite partitions of unity on a space X, with corresponding nerves N and M. The *product partition* has maps the non-zero functions $\{f_u \cdot g_v\}_{U \times V}$, multiplying values in $[0, 1]$ pointwise, and the *product nerve* $N \times M$ is the nerve of this cover. It is trivial to verify that the product partition is a locally finite partition of unity.

2.2 PROPOSITION: *There are projections* $p_n : N \times M \to N$, $p_m : N \times M \to M$ *which commute in* $\mathscr{T}op$ *with the canonical maps from* X, $\pi_{N \times M}$, π_N *and* π_M.

PROOF: Let $\{f_u\}_U$ and $\{g_v\}_V$ be the partitions of unity on X giving rise to N, M respectively. Let $\{i_u\}_U$, $\{j_v\}_V$ be the sets of vertices of N, M respectively, corresponding to f_u, g_v, and let (i_u, j_v) be the vertex of $N \times M$ corresponding to the map $f_u \cdot g_v : X \to [0, 1]$, the vertex existing only when the map is not the zero map. Define $p_N : N \times M \to N$ on vertices by $(i_u, j_v) \mapsto i_u$. Then it is clear that when any finite collection

$$(i_1, j_1), (i_2, j_2) \cdots (i_r, j_r)$$

determines a simplex of $N \times M$, then the set $i_1, i_2, \cdots, i_r$ determines a simplex of N, and we may extend the vertex map linearly. Hence p_N is well defined. We show it commutes with $\pi_{N \times M}$ and π_N. Take x in X. If $\pi_N(x)$ has non-zero co-ordinates the set $f_1(x), f_2(x), \cdots, f_t(x)$ in N, with respect to vertices i_1 to i_t and if $\pi_M(x)$ has non-zero co-ordinates the set of maps $g_1(x), g_2(x), \cdots, g_s(x)$ in M, with respect to vertices j_1 to j_s, then $\pi_{N \times M}(x)$ in $N \times M$ is in the simplex with vertices the (i_a, j_b) for $1 \leqq a \leqq t$, $1 \leqq b \leqq s$, some of which may be missing, and co-ordinates the $f_a \cdot g_b$ respectively. Now we have by linearity:

$$\begin{aligned}
p_N(\pi_{N \times M}(x)) &= p_N \sum_{\substack{1 \leqq a \leqq t \\ 1 \leqq b \leqq s}} (f_a \cdot g_b)(x) \cdot (i_a, j_b) \\
&= \sum_{1 \leqq a \leqq t} p_N \sum_{1 \leqq b \leqq s} f_a(x) \cdot g_b(x) \cdot (i_a, j_b) \\
&= \sum_{1 \leqq a \leqq t} f_a(x) \cdot p_N \sum_{1 \leqq b \leqq s} g_b(x) \cdot (i_a, j_b) \\
&= \sum_{1 \leqq a \leqq t} f_a(x) \cdot \sum_{1 \leqq b \leqq s} g_b(x) \cdot i_a \\
&= \sum_{1 \leqq a \leqq t} f_a(x) \cdot i_a \quad \text{Since } \Big(\sum_{1 \leqq b \leqq s} g_b(x) = 1 \Big) \\
&= \pi_N(x).
\end{aligned}$$

2.3 DEFINITION: Let N be a nerve arising from a partition of unity on X, and M a subdivision of N. Then π_N regarded as a map to M defines a partition of unity on X which has as its nerve the subset of M containing simplices intersecting $\pi_N(X)$, and π_M composed with the inclusion is π_N. We shall call such a refinement a *proper refinement*, and refer to the nerve M as a *proper refinement* of N.

2.4 DEFINITION: For any space X, the *proper diagram of nerves* is the diagram containing all nerves of locally finite partitions of unity on X, and having maps either the inclusions of 2.3 or else the projections of 2.2.

2.5 REMARK: It is clear that the diagram is a semi-inverse diagram, but in general for N a proper refinement of M, and $N \times M$ the product, $i: N \to M$ the inclusion, it is not the case that $i \cdot p_N$ is equal to p_M. Hence the diagram is not an inverse system. However, we do have $i \cdot p_N$ homotopic to p_M, as is well known.

2.6 PROPOSITION: *A paracompact space X is the limit in $\mathscr{T}op$ of its proper diagram of nerves.*

PROOF: The family of natural maps $\pi_N: X \to N$ is a compatible family of maps into the diagram, and hence determines a unique map $\pi: X \to L$, where L is the limit of the diagram.We show that π is a homeomorphism.

It is immediate that π is continuous, and evident that it is injective, since any two points of X may be sent to 0 and 1 in a nerve which is a copy of the unit interval. This only requires X to be completely regular, which of course it is.

We show that π is surjective: suppose there is a y in $L - \pi X$. Since L is a completely regular space (all the nerves are, and L is a subset of their product) we take all the closed neighbourhoods of y in L and observe that their intersection is y.

The family of complements is an open cover of πX, and hence determines an open cover of X. Since X is paracompact, there is a subordinates partition of unity, and a nerve rising from it which we shall call N. We consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi} & L \\ & \underset{\pi_N}{\searrow} & \downarrow m \\ & & N \end{array}$$

where m is the canonical map from the limit into the element of the diagram. By construction the diagram commutes.

Now if $y \in L$, my is in some simplex of N, say $(v_1, v_2, \cdots, v_s)$, and by the continuity of m, there is some neighbourhood U of y in L such that $mU \subset \operatorname{st} v_1$, where st denotes the open star on the vertex. If V_1 is the open set on X corresponding to the vertex v_1, then it follows that $\pi_N^{-1}(mU) \subset V_1$, and by commutativity that $(m \cdot \pi)^{-1}(mU) \subset V_1$, that is to say we have that

$$\pi^{-1}(m^{-1}(mU)) \subset V_1 \subset X$$

whence

$$\pi^{-1}U \subset V_1 \subset X,$$

and

$$\pi \cdot \pi^{-1}U \subset \pi V_1 \subset L.$$

By hypothesis, πV_1 is disjoint from y; moreover, by our construction of the cover of X, we can find a W open in L and containing y such that $W \subset \pi$, $x^{-1}U = \emptyset$. Without loss of generality, we may take $W \subset U$. Now it follows that W is disjoint from πX in L, for if $w \in W \cap \pi X$, $\exists w' \in X : \pi w' = w$, and $w \in W \cap \pi \cdot \pi^{-1}U$, contra. We have shown therefore that if $y \in L - \pi X$, $y \in L - \overline{\pi X}$ also, hence that πX is closed in L. Now an open set W on y in L contains on y the intersection of some open sets with L

$$L \cap m_1^{-1}(W_1) \cap m_2^{-1}(W_2) \cap \cdots, m_t^{-1}(W_t)$$

for some open sets $W_1, W_2, \cdots, W_t$ in nerves $N_1, N_2, \cdots, N_t$, where the maps $m_1, m_2, \cdots, m_t$ are the canonical maps from the limit.

Form $M = N_1 \times N_2 \times \cdots \times N_t$, the nerve product. Let $p_1, p_2, \cdots, p_t$ denote the projections from M to $N_1, N_2, \cdots, N_t$ respectively. Then $p_1^{-1}(W_1) \cap p_2^{-1}(W_2) \cap \cdots \cap p_t^{-1}(W_t)$ is an open set in M, containing $m_M(y)$, and is clearly disjoint from $\pi_M X$. But in this event, we can take a subdivision of M excluding $m_M(y)$, contradicting y being an element of L. Hence there are no elements in $L - \pi X$, i.e. π is surjective.

Finally to prove that π is open, take A open in X and $a \in A$. If $A = X$ there is nothing to prove, so take $A \neq X$. Then there is a map $\alpha : X \to [0, 1]$ such that $\alpha(a) = 1$ and $\alpha^{-1}(0, 1] \subset A$. This yields a nerve I of a covering with α as canonical map, and if m_I is the map from L to this nerve, then $B = m_I^{-1}(0, 1]$ is open in L. Clearly $\pi x \in B$, and since π is onto, $B = \pi \cdot \pi^{-1}B$, but $\pi^{-1}B = (\pi \cdot m_I)^{-1}(0, 1] \subset A$, hence $B \subset \pi A$ and the result follows.

3.

3.1 PROPOSITION: *With the notation of* 2.6, *if K is a simplicial complex, and a paracompact space X is a limit of its proper diagram of nerves, Δ, then*

$$[-, K] \cdot \mathscr{H}(X) = \varinjlim [-, K] \cdot \mathscr{H}\Delta.$$

PROOF: For all N_δ, N_ε objects of Δ, and all $h_{\delta\varepsilon}: N_\delta \to N_\varepsilon$ the corresponding maps of Δ, we have the set

$$\{[-, K] \circ \mathscr{H}(N_\delta): N_\delta \in \Delta\}$$

i.e. the set of homotopy classes of maps, $\{[N_\delta, K]: N_\delta \in \Delta\}$. On this set we have the relation $\sim$ given by

$$[g_\delta] \sim [g_\varepsilon] \text{ iff } [g_\delta] = [g_\varepsilon \circ h_{\delta\varepsilon}]$$

i.e. iff

$$([-, K] \circ \mathscr{H}(h_{\delta\varepsilon}))[g_\varepsilon] = [g_\delta]$$

Then $t <$ e set $\lim_{\to}[-, K] \circ \mathscr{H}\Delta$ is the set $\{[-, K] \circ \mathscr{H}(N_\delta): N_\delta \in \Delta\}$ factored by the smallest equivalence containing the relation $\sim$.

We write $\{g_\varepsilon\}$ for the class containing $[g_\varepsilon] \in [N_\varepsilon, K]$. Since the canonical maps $\pi_\delta: X \to N_\delta$ commute with the maps of the diagram Δ we may define

$$\ell: \underrightarrow{\lim}\,[-, K] \cdot \mathscr{H}\Delta \to [-, K] \cdot \mathscr{H}X \quad \text{by} \quad \{g_\varepsilon\} \mapsto [g_\varepsilon \cdot \pi_\varepsilon]$$

and this is well defined, i.e. it is independent of the choice of representative in $\{g_\varepsilon\}$.

ℓ is surjective. For take $[g] \in [X, K]$. Choosing a representative $g \in [g]$, $g: X \to K$ in $\mathscr{T}op$, we note that g factors through the canonical map to a nerve N (pull back the star open cover of K; we get not only an open cover of X but also a natural partition of unity. Then N is a subcomplex of K, viz. that complex having simplices of K intersecting gX) and we write $g = \pi_N \cdot j$ where $j: N \to K$ is the inclusion. Then j certainly defines a class $\{j\}$ in the limit, and $\ell(\{j\}) = [g]$. Choosing a different representative of $[g]$ might conceivably give us a different $\{j\}$, but any one suffices. In fact this does not happen, because: ℓ is injective. Suppose $\ell(\{g_\varepsilon\}) = \ell(\{f_\delta\})$, that is to say, we have $[g_\varepsilon]$ and $[f_\delta]$ such that $[g_\varepsilon \cdot \pi_\varepsilon] = [f_\delta \cdot \pi_\delta]$. This assures us of a homotopy $H: X \times I \to K$ between $g_\varepsilon \circ \pi_\varepsilon$ and $f_\delta \circ \pi_\delta$, which determines a partition of unity on $X \times I$, arising from the star open cover of K. We consider the cover defined by the portion of unity. For each x in X and each t in I, there are some finite number of sets of the cover, say $V_t^1, V_t^2, \cdots, V_t^n$ on x, t. Let V_t be their intersection. Cover $x \times I$ by such sets. Since it is compact, there is a finite subcover, say $V_1, V_2, \cdots, V_s$. Project each set down to X and take V_x to be the intersection.

Doing this for each x in X we obtain a cover of X, χ such that $\forall t \in [0, 1]$, H_t^{-1} st.o.c. K is refined by χ, where st.o.c. K is of course the star open cover of K. Let N be a nerve arising from a partition of unity subordinate to χ; then $H_t: X \to K$ factors through N for all t, i.e. H factors through

$N \times I$. Since we have refined the partition of unity corresponding to N_δ (and N_ε) in the process, there is a map from N to each of them, say α, β respectively, with $g_\varepsilon \circ \beta$ homotopic to $f_\delta \circ \alpha$ by the map from $N \times I$ to K. It is to be remembered that α, β will not in general be in Δ, but we may take then a nerve product $N \times N_\varepsilon \times N_\delta$ in place of N, and the result follows.

3.2 REMARK: The two preceding propositions admit the following partial converse: if P is paracompact and $[-, P] \cdot \mathscr{H}$ preserves limits of proper diagrams of nerves then P has the homotopy type of a complex. This follows immediately from Milnor [2] where it is shown that a space dominated by a CW complex has the homotopy type of one, and the observation that the identity map on P must factor through a nerve.

4.

4.1 DEFINITION: Let $\mathscr{C}$ denote the full subcategory of simplicial complexes; $\mathscr{H}\mathscr{C}$ we take to be either the image of $\mathscr{C}$ under $\mathscr{H}$ in $\mathscr{H}tp$ or the full subcategories of $\mathscr{H}tp$ having as objects those spaces having the homotopy type of a simplicial complex (this includes the CW complexes, by [2]).

4.2 PROPOSITION: *Let $\Gamma : \mathscr{T}op \to \mathscr{E}ns$ be a contravariant functor preserving limits of proper diagrams of simplicial complexes, and factoring through $\mathscr{H}tp$ to give $[\Gamma]$, with $[\Gamma]|\mathscr{H}\mathscr{C}$ representable by a complex K. Then the representation extends to spaces having the homotopy type of a paracompact space.*

PROOF: We have $\sigma : [-, K] \to [\Gamma]$ a natural transformation arising from some $s \in \Gamma K$ by the Yoneda lemma, and $\sigma|\mathscr{H}\mathscr{C}$ is an equivalence.

If P is a paracompact space we express it as the limit of the appropriate diagram of nerves in $\mathscr{T}op$. Now

$$[P, K] \approx [\lim \Delta, K] \approx \operatorname{colim} [\Delta, K] \quad \text{by continuity of } [-, K] \cdot \mathscr{H}$$

and

$$\operatorname{colim} [\Delta, K] \approx \operatorname{colim} [\Gamma](\Delta) \approx \operatorname{colim} \Gamma\Delta$$

$$\operatorname{colim} \Gamma\Delta \approx \Gamma \lim \Delta \quad \text{by continuity of } \Gamma$$

$$\approx \Gamma P$$

where the $\approx$ signs denote various natural equivalences, and some minor abuses of language have taken place.

4.3 REMARK: It follows by the same kind of argument that the Čech cohomology theory is, as is well known, representable on paracompacta.

More generally, if a suitable functor is given on $\mathscr{H}\mathscr{C}$ and extended by the generalized Čech process, then we can again expect to obtain an extension of representability.

I wish to express my gratitude to Professor C. T. C. Wall for his help and supervision of the doctoral thesis of which this is a part.

REFERENCES

[1] EILENBERG and STEENROD: *Foundation of Algebraic Topology*. Princeton 1952.

[2] J. MILNOR: On Spaces having the Homotopy Type of a CW Complex. *Trans. Amer. Math. Soc. 90* (1959) 272–280.

(Oblatum 10–II–1972 & 23–IV–1974)

University of Western Australia
Dept. of Mathematics
Australia WA 6009