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Inverse limits of simplicial complexes


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INVERSE LIMITS OF SIMPLICIAL COMPLEXES

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Summary

Eilenberg and Steenrod have shown that any compact space may be expressed as an inverse limit in $\mathcal{F}op$, the category of topological spaces, of a diagram of simplicial complexes. [1]

We show that any paracompact space may be expressed as a limit of a diagram of nerves; further that $\lbrack \mathcal{H}, K \rbrack : \mathcal{F}op \to \mathcal{E}ns$ preserves such limits when $K$ is a complex, where $\lbrack \mathcal{H}, K \rbrack$ denotes the contravariant hom-functor from $\mathcal{H}tp$ (the category of spaces and homotopy classes of maps) to $\mathcal{E}ns$ the category of sets, and $\mathcal{H} : \mathcal{F}op \to \mathcal{H}tp$ is the canonical quotient functor.

1. Introduction

Let $\mathcal{A}$ denote a small category having the property that for $A, B$ objects of $\mathcal{A}$, there exists a $C$ in $\mathcal{A}$ and maps $f : C \to A, g : C \to B$ respectively.

1.1 Definition: A diagram in a category $\mathcal{M}$ with scheme $\mathcal{A}$ will be said to be a semi inverse diagram.

1.2 Definition: $\mathcal{H} : \mathcal{M} \to \mathcal{B}$ is a semi-directly continuous contravariant functor if it takes the limit of every semi inverse diagram $(\mathcal{A}, \Phi : \mathcal{A} \to \mathcal{M})$ in $\mathcal{M}$ to the colimit of $(\mathcal{A}, \mathcal{H} \circ \Phi : \mathcal{A} \to \mathcal{B})$ in $\mathcal{B}$.

1.3 Remark: We assume familiarity with the diagram of nerves of a space arising from locally finite partitions of unity, and with the natural map which we shall denote by $\pi_j : X \to N_j$ from $X$ a space admitting a locally finite partition of unity having nerve $N_j$. We recall that if one locally finite partition of unity with nerve $N_j$ refines another with nerve $N_k$, then there is a map $\sigma_{jk} : N_j \to N_k$ which homotopy commutes with the canonical maps from $X$.

1.4 Remark: Our first proposition shows that when $X$ is paracompact we may choose maps between the nerves such that $X$ is the limit of the resulting semi-inverse diagram. This is the content of § 2. In § 3 we show that if $K$ is a complex, $\lbrack \mathcal{H}, K \rbrack : \mathcal{F}op \to \mathcal{E}ns$ preserves limits of this type. Finally in § 4 we conclude with remarks on some applications of these results.
2.

2.1 DEFINITION: Let \{f_u\}_U and \{g_v\}_V denote locally finite partitions of unity on a space \(X\), with corresponding nerves \(N\) and \(M\). The product partition has maps the non-zero functions \(\{f_u \cdot g_v\}_{U \times V}\), multiplying values in \([0, 1]\) pointwise, and the product nerve \(N \times M\) is the nerve of this cover. It is trivial to verify that the product partition is a locally finite partition of unity.

2.2 PROPOSITION: There are projections \(p_n : N \times M \to N\), \(p_m : N \times M \to M\) which commute in \(\mathcal{F}_{\text{op}}\) with the canonical maps from \(X\), \(\pi_{N \times M}\), \(\pi_N\) and \(\pi_M\).

PROOF: Let \(\{f_u\}_U\) and \(\{g_v\}_V\) be the partitions of unity on \(X\) giving rise to \(N\), \(M\) respectively. Let \(\{i_u\}_U\), \(\{j_v\}_V\) be the sets of vertices of \(N\), \(M\) respectively, corresponding to \(f_u\), \(g_v\), and let \((i_u, j_v)\) be the vertex of \(N \times M\) corresponding to the map \(f_u \cdot g_v : X \to [0, 1]\), the vertex existing only when the map is not the zero map. Define \(p_N : N \times M \to N\) on vertices by \((i_u, j_v) \mapsto i_u\). Then it is clear that when any finite collection

\[(i_1, j_1), (i_2, j_2) \cdots (i_r, j_r)\]
determines a simplex of \(N \times M\), then the set \(i_1, i_2, \cdots, i_r\) determines a simplex of \(N\), and we may extend the vertex map linearly. Hence \(p_N\) is well defined. We show it commutes with \(\pi_{N \times M}\) and \(\pi_N\). Take \(x\) in \(X\). If \(\pi_N(x)\) has non-zero co-ordinates the set \(f_1(x), f_2(x), \cdots, f_t(x)\) in \(N\), with respect to vertices \(i_1\) to \(i_t\) and if \(\pi_M(x)\) has non-zero co-ordinates the set of maps \(g_1(x), g_2(x), \cdots, g_s(x)\) in \(M\), with respect to vertices \(j_1\) to \(j_s\), then \(\pi_{N \times M}(x)\) in \(N \times M\) is in the simplex with vertices the \((ia, jb)\) for \(1 \leq a \leq t, 1 \leq b \leq s\), some of which may be missing, and co-ordinates the \(f_a \cdot g_b\) respectively. Now we have by linearity:

\[
p_N(\pi_{N \times M}(x)) = \sum_{1 \leq a \leq t} \sum_{1 \leq b \leq s} f_a(x) \cdot g_b(x) \cdot (i_a, j_b)
\]

\[
= \sum_{1 \leq a \leq t} \sum_{1 \leq b \leq s} f_a(x) \cdot g_b(x) \cdot (i_a, j_b)
\]

\[
= \sum_{1 \leq a \leq t} f_a(x) \cdot \sum_{1 \leq b \leq s} g_b(x) \cdot (i_a, j_b)
\]

\[
= \sum_{1 \leq a \leq t} f_a(x) \cdot \sum_{1 \leq b \leq s} g_b(x) \cdot i_a
\]

\[
= \sum_{1 \leq a \leq t} f_a(x) \cdot i_a \quad \text{Since } (\sum_{1 \leq b \leq s} g_b(x) = 1)
\]

\[
= \pi_N(x)
\]

2.3 DEFINITION: Let \(N\) be a nerve arising from a partition of unity on \(X\), and \(M\) a subdivision of \(N\). Then \(\pi_N\) regarded as a map to \(M\) defines a
partition of unity on $X$ which has as its nerve the subset of $M$ containing simplices intersecting $\pi_N(X)$, and $\pi_M$ composed with the inclusion is $\pi_N$. We shall call such a refinement a proper refinement, and refer to the nerve $M$ as a proper refinement of $N$.

2.4 Definition: For any space $X$, the proper diagram of nerves is the diagram containing all nerves of locally finite partitions of unity on $X$, and having maps either the inclusions of 2.3 or else the projections of 2.2.

2.5 Remark: It is clear that the diagram is a semi-inverse diagram, but in general for $N$ a proper refinement of $M$, and $N \times M$ the product, $i: N \to M$ the inclusion, it is not the case that $i \cdot p_N$ is equal to $p_M$. Hence the diagram is not an inverse system. However, we do have $i \cdot p_N$ homotopic to $p_M$, as is well known.

2.6 Proposition: A paracompact space $X$ is the limit in $\mathcal{Top}$ of its proper diagram of nerves.

Proof: The family of natural maps $\pi_N: X \to N$ is a compatible family of maps into the diagram, and hence determines a unique map $\pi: X \to L$, where $L$ is the limit of the diagram. We show that $\pi$ is a homeomorphism.

It is immediate that $\pi$ is continuous, and evident that it is injective, since any two points of $X$ may be sent to 0 and 1 in a nerve which is a copy of the unit interval. This only requires $X$ to be completely regular, which of course it is.

We show that $\pi$ is surjective: suppose there is a $y$ in $L - \pi X$. Since $L$ is a completely regular space (all the nerves are, and $L$ is a subset of their product) we take all the closed neighbourhoods of $y$ in $L$ and observe that their intersection is $y$.

The family of complements is an open cover of $\pi X$, and hence determines an open cover of $X$. Since $X$ is paracompact, there is a subordinates partition of unity, and a nerve rising from it which we shall call $N$. We consider the diagram:

$$
\begin{array}{c}
X \\
\pi_N \downarrow \pi \\
N \\
\end{array}
\xrightarrow{m} \begin{array}{c}
\downarrow m \\
L \\
\end{array}
$$

where $m$ is the canonical map from the limit into the element of the diagram. By construction the diagram commutes.

Now if $y \in L$, $my$ is in some simplex of $N$, say $(v_1, v_2, \cdots, v_k)$, and by the continuity of $m$, there is some neighbourhood $U$ of $y$ in $L$ such that $mU \subset \text{st} v_1$, where $\text{st}$ denotes the open star on the vertex. If $V_1$ is the open
set on $X$ corresponding to the vertex $v_1$, then it follows that $\pi^{-1}(mU) \subseteq V_1$, and by commutativity that $(m \cdot \pi)^{-1}(mU) \subseteq V_1$, that is to say we have that

$$\pi^{-1}(m^{-1}(mU)) \subseteq V_1 \subseteq X$$

whence

$$\pi^{-1}U \subseteq V_1 \subseteq X,$$

and

$$\pi \cdot \pi^{-1}U \subseteq \pi V_1 \subseteq L.$$  

By hypothesis, $\pi V_1$ is disjoint from $y$; moreover, by our construction of the cover of $X$, we can find a $W$ open in $L$ and containing $y$ such that $W \subseteq \pi$, $x^{-1}U = \emptyset$. Without loss of generality, we may take $W \subseteq U$. Now it follows that $W$ is disjoint from $\pi X$ in $L$, for if $w \in W \cap \pi X$, $\exists w' \in X : \pi w' = w$, and $w \in W \cap \pi \cdot \pi^{-1}U$, contra. We have shown therefore that if $y \in L - \pi X$, $y \in L - \pi \overline{X}$ also, hence that $\pi X$ is closed in $L$. Now an open set $W$ on $y$ in $L$ contains on $y$ the intersection of some open sets with $L$

$$L \cap m^{-1}_1(W_1) \cap m^{-1}_2(W_2) \cap \cdots \cap m^{-1}_t(W_t)$$

for some open sets $W_1, W_2, \cdots, W_t$ in nerves $N_1, N_2, \cdots, N_t$, where the maps $m_1, m_2, \cdots, m_t$ are the canonical maps from the limit.

Form $M = N_1 \times N_2 \times \cdots \times N_t$, the nerve product. Let $p_1, p_2, \cdots, p_t$ denote the projections from $M$ to $N_1, N_2, \cdots, N_t$ respectively. Then $p_1^{-1}(W_1) \cap p_2^{-1}(W_2) \cap \cdots \cap p_t^{-1}(W_t)$ is an open set in $M$, containing $m_M(y)$, and is clearly disjoint from $\pi_M X$. But in this event, we can take a subdivision of $M$ excluding $m_M(y)$, contradicting $y$ being an element of $L$. Hence there are no elements in $L - \pi X$, i.e. $\pi$ is surjective.

Finally to prove that $\pi$ is open, take $A$ open in $X$ and $a \in A$. If $A = X$ there is nothing to prove, so take $A \neq X$. Then there is a map $\alpha : X \to [0, 1]$ such that $\alpha(a) = 1$ and $\alpha^{-1}(0, 1] \subseteq A$. This yields a nerve $I$ of a covering with $\alpha$ as canonical map, and if $m_I$ is the map from $L$ to this nerve, then $B = m_I^{-1}(0, 1]$ is open in $L$. Clearly $\pi x \in B$, and since $\pi$ is onto, $B = \pi \cdot \pi^{-1}B$, but $\pi^{-1}B = (\pi \cdot m_I)^{-1}(0, 1] \subseteq A$, hence $B \subseteq \pi A$ and the result follows.

3.

3.1 Proposition: With the notation of 2.6, if $K$ is a simplicial complex, and a paracompact space $X$ is a limit of its proper diagram of nerves, $\Delta$, then

$$[-, K] \cdot \mathcal{H}(X) = \lim [-, K] \cdot \mathcal{H} \Delta.$$
**Proof:** For all $N_\delta, N_\epsilon$ objects of $\Delta$, and all $h_{be}: N_\delta \to N_\epsilon$ the corresponding maps of $\Delta$, we have the set

$$\{[-, K] \circ \mathcal{H}(N_\delta) : N_\delta \in \Delta\}$$

i.e. the set of homotopy classes of maps, $\{[N_\delta, K] : N_\delta \in \Delta\}$. On this set we have the relation $\sim$ given by

$$[g_\delta] \sim [g_\epsilon] \text{ iff } [g_\delta] = [g_\epsilon \circ h_{be}]$$

i.e. iff

$$([-, K] \circ \mathcal{H}(h_{be}))[g_\epsilon] = [g_\delta]$$

Then $t < e$ set $\lim_{\rightarrow} [-, K] \circ \mathcal{H} \Delta$ is the set $\{[-, K] \circ \mathcal{H}(N_\delta) : N_\delta \in \Delta\}$ factored by the smallest equivalence containing the relation $\sim$.

We write $\{g_\epsilon\}$ for the class containing $[g_\epsilon] \in [N_\epsilon, K]$. Since the canonical maps $\pi_\delta: X \to N_\delta$ commute with the maps of the diagram $\Delta$ we may define

$$\ell: \lim_{\rightarrow} [-, K] \cdot \mathcal{H} \Delta \to [-, K] \cdot \mathcal{H}X \quad \text{by} \quad \{g_\epsilon\} \mapsto [g_\epsilon \cdot \pi_\epsilon]$$

and this is well defined, i.e. it is independent of the choice of representative in $\{g_\epsilon\}$.

$\ell$ is surjective. For take $[g] \in [X, K]$. Choosing a representative $g \in [g]$, $g: X \to K$ in $\mathcal{F}op$, we note that $g$ factors through the canonical map to a nerve $N$ (pull back the star open cover of $K$; we get not only an open cover of $X$ but also a natural partition of unity. Then $N$ is a subcomplex of $K$, viz. that complex having simplices of $K$ intersecting $gX$) and we write $g = \pi_N \cdot j$ where $j: N \to K$ is the inclusion. Then $j$ certainly defines a class $\{j\}$ in the limit, and $\ell(\{j\}) = [g]$. Choosing a different representative of $[g]$ might conceivably give us a different $\{j\}$, but any one suffices. In fact this does not happen, because: $\ell$ is injective. Suppose $\ell(\{g_\epsilon\}) = \ell(\{f_\delta\})$, that is to say, we have $[g_\epsilon]$ and $[f_\delta]$ such that $[g_\epsilon \cdot \pi_\epsilon] = [f_\delta \cdot \pi_\delta]$. This assures us of a homotopy $H: X \times I \to K$ between $g_\epsilon \circ \pi_\epsilon$ and $f_\delta \circ \pi_\delta$, which determines a partition of unity on $X \times I$, arising from the star open cover of $K$. We consider the cover defined by the portion of unity. For each $x$ in $X$ and each $t$ in $I$, there are some finite number of sets of the cover, say $V_1^t, V_2^t, \cdots, V_n^t$ on $x, t$. Let $V_t$ be their intersection. Cover $x \times I$ by such sets. Since it is compact, there is a finite subcover, say $V_1, V_2, \cdots, V_s$. Project each set down to $X$ and take $V_s$ to be the intersection.

Doing this for each $x$ in $X$ we obtain a cover of $X, \chi$ such that $\forall t \in [0, 1]$, $H_t^{-1}$ st.o.c. $K$ is refined by $\chi$, where st.o.c. $K$ is of course the star open cover of $K$. Let $N$ be a nerve arising from a partition of unity subordinate to $\chi$; then $H_t: X \to K$ factors through $N$ for all $t$, i.e. $H$ factors through
Since we have refined the partition of unity corresponding to \( N_\delta \) (and \( N_\varepsilon \)) in the process, there is a map from \( N \) to each of them, say \( \alpha, \beta \) respectively, with \( g_\varepsilon \circ \beta \) homotopic to \( f_\varepsilon \circ \alpha \) by the map from \( N \times I \) to \( K \).

It is to be remembered that \( \alpha, \beta \) will not in general be in \( A \), but we may take then a nerve product \( N \times N_\varepsilon \times N_\delta \) in place of \( N \), and the result follows.

3.2 Remark: The two preceding propositions admit the following partial converse: if \( P \) is paracompact and \([-, P] \cdot \mathcal{H}\) preserves limits of proper diagrams of nerves then \( P \) has the homotopy type of a complex.

This follows immediately from Milnor [2] where it is shown that a space dominated by a CW complex has the homotopy type of one, and the observation that the identity map on \( P \) must factor through a nerve.

4.

4.1 Definition: Let \( \mathcal{C} \) denote the full subcategory of simplicial complexes; \( \mathcal{H} \mathcal{C} \) we take to be either the image of \( \mathcal{C} \) under \( \mathcal{H} \) in \( \mathcal{H} \mathcal{P} \) or the full subcategories of \( \mathcal{H} \mathcal{P} \) having as objects those spaces having the homotopy type of a simplicial complex (this includes the CW complexes, by [2]).

4.2 Proposition: Let \( \Gamma : \mathcal{F} \mathcal{O}p \to \mathcal{E}ns \) be a contravariant functor preserving limits of proper diagrams of simplicial complexes, and factoring through \( \mathcal{H} \mathcal{P} \) to give \([\Gamma]\), with \([\Gamma]\)\(\mathcal{H} \mathcal{C}\) representable by a complex \( K \). Then the representation extends to spaces having the homotopy type of a paracompact space.

Proof: We have \( \sigma : [-, K] \to [\Gamma] \) a natural transformation arising from some \( s \in \Gamma K \) by the Yoneda lemma, and \( \sigma|\mathcal{H} \mathcal{C} \) is an equivalence.

If \( P \) is a paracompact space we express it as the limit of the appropriate diagram of nerves in \( \mathcal{F} \mathcal{O}p \). Now

\[
[P, K] \approx [\lim A, K] \approx \operatorname{colim} [A, K]
\]

by continuity of \([-, K] \cdot \mathcal{H}\)

and

\[
\operatorname{colim} [A, K] \approx \operatorname{colim} [\Gamma](A) \approx \operatorname{colim} \Gamma A
\]

\[
\operatorname{colim} \Gamma A \approx \Gamma \lim A \quad \text{by continuity of } \Gamma
\]

\[
\approx \Gamma P
\]

where the \( \approx \) signs denote various natural equivalences, and some minor abuses of language have taken place.

4.3 Remark: It follows by the same kind of argument that the Čech cohomology theory is, as is well known, representable on paracompacta.
More generally, if a suitable functor is given on $\mathcal{HC}$ and extended by the generalized Čech process, then we can again expect to obtain an extension of representability.

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