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## Classification of algebraic varieties, I

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## CLASSIFICATION OF ALGEBRAIC VARIETIES, I<sup>1</sup>

Kenji Ueno<sup>2</sup>

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### Introduction

By an algebraic variety we shall mean an irreducible reduced *proper* algebraic  $\mathbb{C}$  scheme. By a fibre space  $f: V \rightarrow W$  of an algebraic variety  $V$  over an algebraic variety  $W$ , we mean that the morphism  $f$  is surjective and every fibre of the morphism  $f$  is connected. A fibre  $V_w = f^{-1}(w)$ ,  $w \in W$  is called a general fibre if there exists a union  $S$  of at most countably many nowhere dense algebraic subsets of  $W$  such that  $w \in W - S$ . The nowhere dense subsets of  $W$  depend on a situation that we consider. Note that if  $V$  and  $W$  are both smooth, there exists an algebraic subset  $T$  of  $W$  such that a fibre  $V_w = f^{-1}(w)$  is smooth for any  $w \in W - T$ .

In his paper [13], Iitaka introduced the notion of Kodaira dimension  $\kappa(V)$  of an algebraic variety  $V$ . (See also Definition 1.4 below.) He studied the pluri-canonical maps of a smooth algebraic variety  $V$  and showed that if the Kodaira dimension  $\kappa(V)$  of  $V$  is *positive*, then a certain birationally equivalent model  $V^*$  of  $V$  has a structure of a fibre space (unique up to birational equivalence) over an algebraic variety  $W$  such that  $\dim W = \kappa(W)$  and general fibres of the fibre space have the *Kodaira di-*

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*mensions zero.* (See also Theorem 1.14 below.) Hence the study of the birational classification of algebraic varieties is reduced to

- (1) studies of algebraic varieties  $V$  such that  $\kappa(V) = \dim V$ ,  $\kappa(V) = 0$  or  $\kappa(V) = -\infty$ ;
- (2) studies of fibre spaces whose general fibres are of Kodaira dimension zero.

In this paper we are mainly interested in algebraic varieties of Kodaira dimension zero. For such a variety  $V$  we cannot introduce a structure of a fibre space by the pluri-canonical maps. But if the irregularity  $q(V) = \dim_{\mathbb{C}} H^0(V, \Omega_V^1)$  is positive, we can introduce a structure of a fibre space associated to the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$ . (See Remark 2.12.) One of the main purposes of the present paper is to study this fibre space.

Let us recall a conjecture and problems raised by Iitaka in [14].

*Conjecture  $I_n$ .* Let  $V$  be an  $n$ -dimensional algebraic variety. If  $\kappa(V) = 0$ , then we have  $q(V) \leq n$ .

*Problem.* If  $\kappa(V) = 0$  and  $q(V) = \dim V$ , is the Albanese map of  $V$  birational?

*Problem.* If  $\kappa(V) = 0$  and  $q(V) = \dim V - 1$ , does there exist a finite unramified covering  $\tilde{V}$  of  $V$  such that  $q(\tilde{V}) = \dim V$ ?

These conjecture and problems are true for algebraic surfaces and for all known examples.

Here we will propose a much stronger but more geometrical conjecture.

*Conjecture  $K_n$ .* Let  $V$  be an  $n$ -dimensional smooth algebraic variety. If  $\kappa(V) = 0$ , then the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  is surjective and all fibres of  $\alpha$  are connected. Moreover the fibre space  $\alpha : V \rightarrow \text{Alb}(V)$  is birationally equivalent to an analytic fibre bundle over  $\text{Alb}(V)$  whose fibre is an algebraic variety of Kodaira dimension zero.

If  $K_n$  is true, the above Conjecture  $I_n$  and problems are true. By the classification of algebraic surfaces,  $K_2$  is true. (See Example 1.12 (4) below. For the structure of hyperelliptic surfaces see Suwa [33].) Moreover Matsushima [24] showed that if  $V$  is a smooth projective variety and there exists an integer  $m$  such that the  $m$  times tensor product of the canonical bundle  $K_V^{\otimes m}$  is analytically trivial (more generally the first Chern class  $c_1(V)$  of  $V$  vanishes), the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  has the above mentioned structure of a fibre bundle. (See also Calabi [2].)

Hence there arises the problem that whether an algebraic variety  $V$  of Kodaira dimension zero has a smooth projective model  $V^*$  such that  $K_{V^*}^{\otimes m}$  is trivial for an integer  $m$ . When  $V$  is a surface this is true as a corollary of the classification theory. But in higher dimensional case this is not true in general. (See Corollary 8.4 and Corollary 8.6 below. In both cases varieties are simply connected. Products of these varieties and abelian varieties give examples of positive irregularities.) On the other hand

in Section 7 we shall show that even if the canonical bundle and any multiple of the canonical bundle are not trivial, for Kummer manifolds (see Definition 7.1)  $K_n$  is true.

Because of this new phenomenon about the canonical bundles of higher dimensional varieties, it is important to analyze the Albanese maps. In Section 3 we shall show the following.

*'For any smooth algebraic variety  $V$ , the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  is surjective if and only if*

$$\kappa(\alpha(V)) = 0.$$

*Otherwise we have  $\kappa(\alpha(V)) > 0$ . Moreover if  $\kappa(V) \geq 0$ , the Kodaira dimension of any irreducible component of general fibres of the morphism  $\alpha$  is non-negative.'*

Thus we arrive at the new conjecture, which is also due to Iitaka [14]. (He raised the conjecture in the quite different context.)

*Conjecture  $C_n$ . Let  $f : V \rightarrow W$  be a fibre space of an  $n$ -dimensional algebraic variety  $V$ . Then we have*

$$\kappa(V) \geq \kappa(V_w) + \kappa(W),$$

*where  $V_w = f^{-1}(w)$ ,  $w \in W$  is a general fibre of  $f$ .*

It is easily shown that Conjecture  $C_n$  implies Conjecture  $I_n$ . Moreover if  $\kappa(V) = 0$ ,  $C_n$  implies that the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  of  $V$  is surjective.  $C_2$  is true in view of the classification theory of algebraic surfaces. (See Appendix to Section 1.)

It is one of the most important problems of the classification of algebraic varieties to prove  $C_n$  or to give good sufficient conditions that  $C_n$  holds. We already have the affirmative answer to  $C_n$  in the case that a fibre space  $f : V \rightarrow W$  has a structure of an analytic fibre bundle. (Nakamura-Ueno [26].) In the present paper we shall show that  $C_3$  is true for a certain kind of elliptic threefolds. (See Corollary 6.3.) More general elliptic threefolds will be treated in the forthcoming paper [36].

To solve Conjecture  $C_n$  more detailed studies of fibre spaces whose general fibres have non-negative Kodaira dimensions are needed. We are studying such fibre spaces in the case that general fibres are principally polarized abelian surfaces and show that  $C_3$  is true in this case. [34], [36].

On the other hand when general fibres have positive Kodaira dimensions, even if  $V$  is a surface, such a fibre space was not studied for a long time. Recently Namikawa and Ueno began the studies of fibre spaces of curves of genus two. [28], [29].

We will give one more remark about Conjecture  $C_n$ . Using again the Albanese maps, it is easily shown that if  $\kappa(V) = -\infty$ ,  $C_n$  implies that any irreducible component of general fibres of the Albanese map  $\alpha : V$

→  $\text{Alb}(V)$  has the Kodaira dimension  $-\infty$ . In Section 7 we shall show that if  $V$  is a generalized Kummer manifold,  $\kappa(V) = -\infty$  and  $q(V) = \dim V - 1$ , then any general fibre of the Albanese map of  $V$  is  $\mathbf{P}^1$ . (See Theorem 7.17.) Hence this supports Conjecture  $C_n$ .

Now we shall give the outline of the present paper.

In Section 1 we shall give the definition of Kodaira dimensions and certain birational invariants of algebraic varieties. Then we shall give some basic properties of these birational invariants.

In Section 2 we shall collect basic facts about Albanese maps which we shall use later.

In Section 3 we shall study a subvariety of an abelian variety, using the birational invariant introduced in Section 1. The study of subvarieties gives certain informations about Albanese maps, which we already mentioned above. Moreover we shall show that under a finite unramified covering, every subvariety becomes a product of an abelian subvariety and a variety  $W$  such that  $\kappa(W) = \dim W$ . (See Theorem 3.10.)

Section 4 and Section 5 are devoted to construct certain elliptic threefolds, which play the important roles in the theory of elliptic threefolds. Using the explicit construction, in Section 6 we shall prove the canonical bundle formula for such elliptic threefolds. This formula implies that  $C_3$  is valid in this case. The formula is a generalization of the formula for elliptic surfaces due to Kodaira [19]. The present proof is quite different from Kodaira's proof. We need a lot of computations but one merit of our proof is that we can generalize the arguments not only for higher dimensional elliptic fibre spaces but also for fibre spaces of polarized abelian varieties, which will be treated in the forthcoming paper [36]. The arguments in Section 4–Section 6 are also used to prove the above mentioned Theorem 7.1. Moreover contrary to the case of elliptic surfaces, pluri-canonical systems of elliptic threefolds may have fixed components.

In Section 7 we shall study generalized Kummer manifolds. We shall show that Conjecture  $K_n$  is true for such varieties. Also we shall give a structure of a generalized Kummer manifold  $V$  such that  $\kappa(V) = -\infty$  and  $q(V) = \dim V - 1$ .

In Section 8 we shall give a few examples of Kummer manifolds of Kodaira dimension zero. We remark here that the algebraic variety  $N^{(3)}$  in Example 8.10 has the following properties.

- (a) The canonical bundle is trivial.
- (b)  $H^1(N^{(3)}, \mathcal{O}) = 0$ .
- (c)  $N^{(3)}$  is simply connected.

This shows another new phenomenon in higher dimensional case. In the case of surfaces if the canonical bundle of a surface is trivial, such a surface has a lot of deformations.

### Notations

$a(M) = \text{tr. deg}_{\mathbf{C}} C(M);$	the algebraic dimension of a compact complex manifold $M$ .
$\alpha : V \rightarrow \text{Alb}(V);$	the Albanese map of a smooth algebraic variety $V$ . (Section 2)
$\mu : B \rightarrow W;$	a basic elliptic threefold. (Section 4)
$\hat{\mu} : \hat{B} \rightarrow W;$	a non-singular model of a basic elliptic threefold $\mu : B \rightarrow W$ . (Section 5)
$\mu_{\eta} : B^{\eta} \rightarrow W;$	the elliptic threefold associated to an element $\eta \in H_{\text{adm}}^1(W_0, \mathcal{O}(B_0^{\#}))$ . (Section 5, 5.10)
$C(M);$	the field consisting of all meromorphic functions on a compact complex manifold $M$ .
$[D];$	the line bundle associated to a Cartier divisor $D$ .
$e_m = \exp(2\pi i/m)$	
$g_k(V) = \dim_{\mathbf{C}} H^0(V, \Omega_V^k)$	
$g_k^m(V) = \dim_{\mathbf{C}} H^0(V, S^m(\Omega_V^k))$	
$K_V = K(V);$	the canonical line bundle (a canonical divisor) of $V$ .
$mK_V$ means $K_V^{\otimes m}$ if $K_V$ is the canonical line bundle of $V$ .	
$P_g(V) = \dim_{\mathbf{C}} H^0(V, \mathcal{O}(K_V));$	the geometric genus.
$P_m(V) = \dim_{\mathbf{C}} H^0(V, \mathcal{O}(mK_V));$	the $m$ -genus, $m \geq 1$ .
$q(V) = \dim_{\mathbf{C}} H^0(V, \Omega_V^1);$	the irregularity.
$S^m(\mathcal{F});$	the $m$ -th symmetric tensor product of a locally free sheaf $\mathcal{F}$ .
$\kappa(V);$	the Kodaira dimension of an algebraic variety (a compact complex space) $V$ . (Definition 1.4 and Definition 1.7.)
$\Omega_V^k;$	the sheaf of germs of holomorphic $k$ forms on $V$ .
$\Theta;$	the sheaf of germs of holomorphic vector fields on a complex manifold.

## 1. Kodaira dimensions and certain birational invariants

In what follows by an algebraic variety we shall mean an irreducible reduced proper algebraic  $\mathbf{C}$  scheme. In view of GAGA we shall consider an algebraic variety to be a compact complex space. Moreover in what follows any complex space is always assumed to be reduced and irreducible.

ible. By a fibre space  $f: V \rightarrow W$ , we shall mean that  $V$  and  $W$  are algebraic varieties (or compact complex spaces) and  $f$  is surjective and has connected fibres.

In this section we shall define the *Kodaira dimension*  $\kappa(V)$ , the *irregularity*  $q(V)$  and certain birational (or bimeromorphic) invariants  $g_k^m(V)$  of an algebraic variety (or a compact complex space)  $V$ . Then we shall study fundamental properties of these birational (bimeromorphic) invariants.

Let  $V$  be a smooth algebraic variety (a compact complex manifold) of complex dimension  $n$  and let  $K_V$  and  $\Omega_V^k$  be the canonical line bundle of  $V$  and the sheaf of germs of holomorphic  $k$  forms on  $V$ , respectively. For any positive integer  $m$ , we set

$$g_k^m(V) = \dim_{\mathbb{C}} H^0(V, S^m(\Omega_V^k)), \quad k = 1, 2, \dots, n,$$

where  $S^m(\Omega_V^k)$  is the  $m$ -th symmetric product of the locally free sheaf  $\Omega_V^k$ . When  $k = \dim V = n$ , we use the notation  $P_m(V)$  instead of  $g_n^m(V)$  and call it the  $m$ -genus of  $V$ . Moreover we use the notation  $P_g(V)$  instead of  $P_1(V)$  and call it the *geometric genus* of  $V$ . Hence we write

$$\begin{aligned} P_g(V) &= \dim_{\mathbb{C}} H^0(V, \mathcal{O}(K_V)), \\ P_m(V) &= \dim_{\mathbb{C}} H^0(V, \mathcal{O}(mK_V)), \quad m = 2, 3, \dots \end{aligned}$$

We use the notation  $q(V)$  instead of  $g_1^1(V)$  and call it the *irregularity* of  $V$ . Moreover we use the notation  $g_k(V)$  instead of  $g_k^1(V)$ .

Let  $\mathfrak{B}$  be a coordinate neighborhood of a point  $x \in V$  and let  $(z_1, z_2, \dots, z_n)$  be local coordinates of  $\mathfrak{B}$  with center  $x$ . Let  $I_1, I_2, \dots, I_s$ ,  $s = \binom{n}{k}$  be all subsets of  $\{1, 2, \dots, n\}$ , consisting of  $k$  elements. We set

$$dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_k}, \quad i_1 < i_2 < \dots < i_k,$$

where  $I = \{i_1, i_2, \dots, i_k\}$ . Any element of the germ  $S^m(\Omega_V^k)_x$  at  $x$  is written as a polynomial of  $dz_{I_1}, dz_{I_2}, \dots, dz_{I_s}$  of degree  $m$  with coefficient in  $\mathcal{O}_{V,x}$ . Hence any element of  $H^0(\mathfrak{B}, S^m(\Omega_V^k))$  is written in the form

$$(1.1) \quad \sum_{\substack{m_1 + m_2 + \dots + m_s = m \\ m_i \geq 0}} a_{m_1, \dots, m_s}(z) (dz_{I_1})^{m_1} \cdot (dz_{I_2})^{m_2} \cdot \dots \cdot (dz_{I_s})^{m_s}$$

where  $a_{m_1, \dots, m_s}(z)$  is a holomorphic function on  $\mathfrak{B}$ . We call this form an  $m$ -tuple  $k$  form.

**PROPOSITION 1.2:** *Let  $f: U \rightarrow V$  be a rational (meromorphic) map between smooth varieties (compact complex manifolds)  $U$  and  $V$ . Then for any element  $\varphi \in H^0(V, S^m(\Omega_V^k))$  we can define the pull back  $f^*(\varphi)$  of  $\varphi$  such that  $f^*(\varphi) \in H^0(U, S^m(\Omega_U^k))$ .  $f^*$  is linear. Moreover if  $f$  is generically surjective,  $f^*$  is injective.*

PROOF: Let  $S$  be the maximal analytic subset of  $U$  such that  $f$  is not holomorphic at any point of  $S$ . The complex codimension of  $S$  is at least two. Since  $S^m(\Omega_U^k)$  is locally free, in view of Hartogs' theorem, any element of  $H^0(U-S, S^m(\Omega_U^k))$  can be uniquely extended to an element of  $H^0(U, S^m(\Omega_U^k))$ .

Let  $\mathfrak{U}$  and  $\mathfrak{B}$  be an open neighborhood of a point  $x \in U-S$  and an open neighborhood of a point  $f(x) \in V$  such that  $f(\mathfrak{U}) \subset \mathfrak{B}$ . Let  $(y_1, \dots, y_l)$  and  $(z_1, \dots, z_n)$  be local coordinates of  $\mathfrak{U}$  and  $\mathfrak{B}$  with respective centers  $x$  and  $f(x)$ . Then

$$(f^*(dz_{I_1}))^{n_1} \cdot (f^*(dz_{I_2}))^{n_2} \cdots (f^*(dz_{I_s}))^{n_s},$$

$$n_1 + n_2 + \cdots + n_s = m,$$

is represented by a polynomial of  $dy_{J_1}, \dots, dy_{J_t}$  of degree  $m$  with coefficient in  $H^0(\mathfrak{U}, \mathcal{O}_U)$ , where  $J_1, \dots, J_t$  are all subsets of  $\{1, 2, \dots, l\}$  consisting of  $k$  elements. For

$$\varphi = \sum_{\substack{n_1 + \cdots + n_s = m \\ n_i \geq 0}} a_{n_1, \dots, n_s}(z)(dz_{I_1})^{n_1} \cdots (dz_{I_s})^{n_s},$$

we define

$$f^*(\varphi) = \sum_{\substack{n_1 + \cdots + n_s = m \\ n_i \geq 0}} a_{n_1, \dots, n_s}(f(y))(f^*(dz_{I_1}))^{n_1} \cdots (f^*(dz_{I_s}))^{n_s}.$$

Then  $f^*(\varphi)$  is an  $m$ -tuple  $k$  form on  $U-S$ , hence an element of  $H^0(U, S^m(\Omega_U^k))$ . By the definition,  $f^*$  is linear. Moreover by the unique continuation property of holomorphic functions,  $f^*$  is injective, if  $f$  is generically surjective. q.e.d.

COROLLARY 1.3: *If two smooth algebraic varieties (compact complex manifolds)  $V_1$  and  $V_2$  are birationally (bimeromorphically) equivalent, then we have*

$$P_g(V_1) = P_g(V_2), \quad P_m(V_1) = P_m(V_2),$$

$$q(V_1) = q(V_2), \quad g_k^m(V_1) = g_k^m(V_2).$$

Now we define a subset  $N(V)$  of positive integers by

$$N(V) = \{m | P_m(V) > 0\}.$$

First we assume  $N(V) \neq \emptyset$ . Then for any positive integer  $m \in N(V)$ , we can define a rational (meromorphic) map  $\Phi_{mK} : V \rightarrow \mathbf{P}^N$  of  $V$  into the complex projective space  $\mathbf{P}^N$  by

$$\Phi_{mK} : V \longrightarrow \mathbf{P}^N$$

$$\begin{matrix} \cup & & \cup \\ z & \mapsto & (\varphi_0(z); \varphi_1(z); \cdots; \varphi_N(z)), \end{matrix}$$

where  $\{\varphi_0, \varphi_1, \dots, \varphi_N\}$  is a basis of  $H^0(V, \mathcal{O}(mK_V))$ . We call this map the *m-th canonical map*.

DEFINITION 1.4: The *Kodaira dimension*  $\kappa(V)$  of a smooth algebraic variety (a compact complex manifold)  $V$  is defined by

$$\kappa(V) = \begin{cases} \max_{m \in \mathbf{N}(V)} \dim \Phi_{mK}(V), & \text{if } N(V) \neq \emptyset, \\ -\infty, & \text{if } N(V) = \emptyset. \end{cases}$$

REMARK 1.5: This definition is different from Iitaka's original definition [13]. But both definitions are equivalent. (See Ueno [35]).

LEMMA 1.6: *If two smooth algebraic varieties (compact complex manifolds)  $V_1$  and  $V_2$  are birationally (bimeromorphically) equivalent, then we have*

$$\kappa(V_1) = \kappa(V_2).$$

PROOF: This is an easy consequence of Proposition 1.2.

DEFINITION 1.7: Let  $V$  be a singular algebraic variety (a compact complex space). We define

$$\begin{aligned} P_g(V) &= P_g(V^*), \quad P_m(V) = P_m(V^*), \quad q(V) = q(V^*), \\ g_k^m(V) &= g_k^m(V^*), \quad \kappa(V) = \kappa(V^*), \end{aligned}$$

where  $V^*$  is a non-singular model of  $V$ .

Note that these are well defined in view of (1.3), (1.6). These are *birational (bimeromorphic) invariants* of algebraic varieties (compact complex spaces).

REMARK 1.8: The *algebraic dimension*  $a(V)$  of a compact complex space  $V$  is, by definition, the transcendental degree of the field of all meromorphic functions on  $V$ . Then by the definition we have

$$\kappa(V) \leq a(V).$$

On the other hand it is known

$$a(V) \leq \dim V.$$

PROPOSITION 1.9: *Let  $f: V \rightarrow W$  be a generically surjective rational (meromorphic) map between two algebraic varieties (compact complex spaces) of same dimensions. Then we have*

$$\kappa(V) \geq \kappa(W).$$

PROOF: This is an easy consequence of Proposition 1.2.

The following fact due to Freitag will be used later.

PROPOSITION 1.10: *Let  $V$  be a compact complex manifold and let  $G$  be a finite group of analytic automorphisms of  $V$ . Then for a quotient space  $V/G$ , we have*

$$\begin{aligned} p_g(V/G) &= \dim_{\mathbb{C}} H^0(V, \mathcal{O}(K_V))^G, \\ g_k(V/G) &= \dim_{\mathbb{C}} H^0(V, \Omega_V^k)^G, \\ q(V/G) &= \dim_{\mathbb{C}} H^0(V, \Omega_V^1)^G. \end{aligned}$$

PROOF: See Freitag [6], Satz 1. p 99.

REMARK 1.11: In general we have

$$g_k^m(V/G) \leq \dim_{\mathbb{C}} K^0(V, S^m(\Omega_V^k))^G.$$

If  $m \geq 2$ , the equality does not necessary hold.

EXAMPLES 1.12:

(1) An  $n$ -dimensional algebraic variety  $V$  is called *rational (unirational)*, if  $V$  is birationally equivalent to  $\mathbb{P}^n$  (if there exists a generically surjective rational map  $g : \mathbb{P}^n \rightarrow V$  of  $\mathbb{P}^n$  onto  $V$ ). Then we have

$$\begin{aligned} g_k^m(V) &= 0, \quad m = 1, 2, 3, \dots, \\ &\quad k = 1, 2, \dots, n, \\ \kappa(V) &= -\infty. \end{aligned}$$

PROOF: By (1.2) and (1.9), it is enough to consider the case  $V = \mathbb{P}^n$ . In this case this is clear by direct computations. Or we can use the fact that the tangent bundle of  $\mathbb{P}^n$  is *ample*. See below (3).

(2) An  $n$ -dimensional algebraic variety  $V$  is called a *ruled variety*, if  $V$  is birationally equivalent to  $\mathbb{P}^1 \times W$ , where  $W$  is an  $n - 1$  dimensional algebraic variety. Then we have

$$\begin{aligned} q(V) &= q(W), \quad g_k^m(V) = g_k^m(W), \quad k = 1, 2, \dots, n - 1, \\ \kappa(V) &= -\infty. \end{aligned}$$

(3) Assume that an algebraic variety  $V$  has an *ample* tangent bundle  $T_V$ . (Hartshorne [9].) Then we have

$$\begin{aligned} g_k^m(V) &= 0, \quad k = 1, 2, \dots, n = \dim V, \\ &\quad m = 1, 2, \dots, \\ \kappa(V) &= -\infty. \end{aligned}$$

PROOF: As  $S^m(\wedge^k T_V)$  is ample for  $k = 1, 2, \dots, n, m = 1, 2, \dots$ , the dual bundle  $S^m(\wedge^k T_V)^*$  has no non-zero sections. If  $\dim V \leq 2$ , by the classification theory,  $V$  is  $\mathbb{P}^1$  or  $\mathbb{P}^2$ . Kobayashi-Ochiai [17] show that if  $\dim V = 3$ , then

$$\dim_{\mathbb{C}} H^0(V, H) \geq 6.$$

Hence by Matsumura,  $V$  is birationally equivalent to  $\mathbf{P}^1 \times W$  [23]. By (2) we have

$$P_2(W) = g_2^2(V) = 0, q(W) = q(V) = 0.$$

Hence the surface  $W$  is *rational*. For a quite different proof, see Iitaka [15]. Recently Kobayashi-Ochiai show that if  $\dim V = 3$ ,  $V$  is  $\mathbf{P}^3$  under the slightly stronger condition that  $V$  has positive holomorphic bisectonal curvatures [18].

(4) The following is the table of the classification of algebraic surfaces. (Kodaira [20], Safarebic [29a].)

$\kappa$	$q$	structure
2		algebraic surfaces of general type
1		elliptic surfaces of general type
	2	abelian surfaces
	1	hyperelliptic surfaces
0	0	$K$ 3 surfaces Enriques surfaces
	0	rational surfaces
$-\infty$	+	ruled surfaces

(5) Let  $f: V \rightarrow W$  be a fibre bundle over a complex torus  $W$  whose fibre is a complex torus  $T$ . Then we have

$$\kappa(V) \leq 0.$$

PROOF: As  $f_*(\mathcal{O}(mK_V))$  is a flat line bundle, we have

$$\begin{aligned} \dim H^0(W, f_*(\mathcal{O}(mK_V))) &\leq 1. \\ \dim H^0(V, \mathcal{O}(mK_V)) = \dim H^0(W, f_*(\mathcal{O}(mK_V))) &\leq 1. \end{aligned}$$

See also Nakamura-Ueno [26] Remark 4.

THEOREM 1.13: *Let  $f: \tilde{V} \rightarrow V$  be a finite unramified covering of an algebraic variety (a compact complex space)  $V$ . Then we have*

$$\kappa(\tilde{V}) = \kappa(V).$$

PROOF: In view of Proposition 1.9, we can assume that  $f: \tilde{V} \rightarrow V$  is a finite Galois covering with a Galois group  $G$ . When  $\tilde{V}$  is smooth, this is proved by Iitaka [13]. If  $\tilde{V}$  is singular, by Hironaka [11], there exists a non-singular model  $\tilde{V}^*$  of  $V$  such that  $G$  can be lifted to a group  $G^*$  of analytic automorphisms of  $\tilde{V}^*$ . As  $G$  acts freely on  $\tilde{V}$ ,  $G^*$  acts freely on  $\tilde{V}^*$ . Hence the quotient map  $\tilde{V}^* \rightarrow \tilde{V}^*/G^*$  is a finite unramified

covering. As  $\tilde{V}^*/G^*$  is bimeromorphically equivalent to  $V$ , we have

$$\kappa(\tilde{V}) = \kappa(\tilde{V}^*/G^*) = \kappa(V^*) = \kappa(V). \quad \text{q.e.d.}$$

The following theorem due to Iitaka is fundamental for the classification theory.

**THEOREM 1.14:** *Let  $V$  be an algebraic variety (a compact complex space) of positive Kodaira dimension. Then there exist a smooth projective variety (a compact complex manifold)  $V^*$ , a smooth projective variety  $W$  and a surjective morphism  $f: V^* \rightarrow W$ , which has the following properties.*

- (1)  $V^*$  is birationally (bimeromorphically) equivalent to  $V$ .
- (2)  $\dim W = \kappa(V)$ .
- (3) There exists an open dense subset  $U$  (in the usual complex topology of  $W$ ) such that for any fibre  $V_w = f^{-1}(w)$ ,  $w \in U$  is irreducible and smooth.
- (4)  $\kappa(V_w) = 0$ , for  $w \in U$ .
- (5) If there exists a fibre space  $f^*: V^* \rightarrow W^*$ , which satisfies the above conditions (1) ~ (4), then there exist birational (bimeromorphic) maps  $g: V^* \rightarrow V^*$ ,  $h: W \rightarrow W^*$  such that  $h \circ f = f^* \circ g$ .

That is, the fibre space  $f: V^* \rightarrow W$  is unique up to the birational (bimeromorphic) equivalence.

$$\begin{array}{ccc} V^* & \xrightarrow{g} & V^* \\ f \downarrow & & \downarrow f^* \\ W & \xrightarrow{h} & W^* \end{array}$$

Moreover if  $V$  is smooth, there exists a positive integer  $m_0$  such that for any  $m \geq m_0$ ,  $m \in N(V)$  the  $m$ -th canonical map  $\Phi_{mK_V}: V \rightarrow W_m = \Phi_{mK_V}(V)$  is birationally (bimeromorphically) equivalent to the morphism  $f: V^* \rightarrow W$ .

**PROOF:** See Iitaka [13] and Ueno [35].

**PROPOSITION 1.15:** *Let  $f: V \rightarrow W$  be a fibre space of smooth algebraic varieties (compact complex manifolds). If  $\kappa(V) \geq 0$ , then for any general fibre  $V_w = f^{-1}(w)$ , we have  $\kappa(V_w) \geq 0$ .*

**PROOF:** As  $V$  and  $W$  are smooth, there exists a nowhere dense algebraic subset  $T$  of  $W$  such that  $f|_{V'}: V' = W - f^{-1}(T) \rightarrow W' = W - T$  is of maximal rank at any point of  $V'$ . Hence for  $w \in W'$ ,  $V_w = f^{-1}(w)$  is irreducible and smooth.

In general if  $M$  is a submanifold of a complex manifold  $V$ , then we have

$$K_M = K_{V|M} \otimes (\wedge^d N_M),$$

where  $N_M$  is the normal bundle of  $M$  in  $V$  and  $d$  is the codimension of  $M$  in  $V$ . In our case, as the normal bundle of  $V_w$  in  $V$  is trivial, we have

$$K(V_w) = K_{V|V_w}.$$

Let  $\varphi \in H^0(V, \mathcal{O}(mK_V))$  be a non-zero section and let  $D = \sum_{i=1}^m n_i D_i$  be a divisor defined by  $\varphi = 0$ , where  $D_i$  is irreducible. We can assume that for  $i = 1, 2, \dots, l$ ,  $f(D_i) \not\subseteq W$  and for  $i = l+1, \dots, m$ ,  $f(D_i) = W$ . We set  $S = \bigcup_{i=1}^l f(D_i) \cap T$ . Then for any  $w \in W - S$ ,  $\varphi$  is not identically zero on  $V_w$  and induces a non-zero element of  $H^0(V_w, \mathcal{O}(mK(V_w)))$ . Hence  $\kappa(V_w) \geq 0$ .

**COROLLARY 1.16:** *Let  $f: V \rightarrow W$  be a fibre space of smooth algebraic varieties (compact complex manifolds). If  $\kappa(V_w) = -\infty$ , for any general fibre, we have  $\kappa(V) = -\infty$ .*

**THEOREM 1.17:** *Let  $f: V \rightarrow W$  be a fibre space. Then we have*

$$\kappa(V) \leq \dim W + \kappa(V_w),$$

where  $V_w = f^{-1}(w)$  is a general fibre of  $f$ .

**PROOF:** See Iitaka [13].

*Appendix. A proof of  $C_2$ .*

We can assume that both  $V$  and  $W$  are smooth. Moreover by Cor. 1.16, we can assume that the genera of any general fibre and the curve  $W$  are *positive*. Hence the surface  $V$  does not contain infinitely many rational curves. Hence  $V$  is neither rational nor ruled. That is  $\kappa(V) \geq 0$ .

*Case 1.  $\kappa(V) = 0$ .*

From Example 1.12 (4), we have

$$2 \geq q(V) \geq q(W) = 1,$$

and  $V$  is birationally equivalent to an abelian surface or a hyperelliptic surfaces and  $W$  is an elliptic curve. (Note that there is no surjective morphism of an abelian variety to a curve of genus  $g \geq 2$ . And a minimal hyperelliptic surface has a finite unramified covering surface which is a product of two elliptic curves [33].) Hence there exists a finite unramified covering surface  $\tilde{V}$  of  $V$  such that  $\tilde{V}$  is obtained by successive blowing ups of a product of two elliptic curves. Then any general fibre  $V_w$  of  $f$  over  $w \in W$  is an elliptic curve. Hence  $C_2$  holds in this case.

*Case 2.  $\kappa(V) = 1$ .*

The surface  $V$  is an elliptic surface. If general fibres of  $f$  are elliptic curves, then  $C_2$  is true by canonical bundle formulas for elliptic surfaces. (Kodaira [20] I, p 772).

Assume that a general fibre of  $f$  is a curve of genus  $g \geq 2$ . As  $V$  is an elliptic surface, there exists a fibre space  $g : V \rightarrow C$  over a curve  $C$  such that general fibres are elliptic curves. Let  $V_a = g^{-1}(a)$ ,  $a \in C$  be a general fibre of  $g$ . Then  $f(V_a) = W$ , otherwise  $V_a$  is contained in a fibre of  $f$ , but it is impossible. Hence by our assumption  $W$  is an elliptic curve. Hence  $C_2$  holds.

*Case 3.*  $\kappa(V) = 2$ .

In this case  $C_2$  is trivially true.

**REMARK:**  $C_2$  is also valid for any compact analytic surface. As there exists a surjective morphism of  $V$  onto a curve  $W$ , we have  $a(V) \geq 1$ . If  $a(V) = 2$ , then  $V$  is algebraic and we have already done. If  $a(V) = 1$ , then  $V$  is an elliptic surface. Hence there exists a surjective morphism  $g : V \rightarrow C$  of  $V$  onto a curve  $C$  such that general fibres are elliptic curves and  $C(V) = C(C)$ , where  $C(V)$ ,  $C(C)$  are the field of all meromorphic functions on  $V$  and  $C$ , respectively. Hence we have  $C(C) \supset C(W)$ . As fibres of  $f$  are connected we have  $C(C) = C(W)$ . Hence  $C = W$  and  $g = f$ . Hence  $C_2$  holds. (Kodaira [20] I, p 772).

## 2. Albanese maps

For readers' convenience, in this section we shall collect elementary facts about Albanese maps, which we shall use later.

**LEMMA 2.1:** *Let  $V$  be an  $n$ -dimensional compact complex manifold. Assume that there exists a surjective morphism  $f : U \rightarrow V$  of an  $n$ -dimensional compact Kähler manifold  $U$  onto  $V$ . Then the Hodge spectral sequence*

$$E_1^{p,q} = H^q(V, \Omega_V^p) \Rightarrow H^{p+q}(V, C),$$

*is degenerate and we have*

$$\dim H^p(V, \Omega_V^q) = \dim H^q(V, \Omega_V^p).$$

*Hence we have*

$$(2.2) \quad q(V) = \dim H^1(V, \mathcal{O}) = \frac{1}{2}b_1(V),$$

*where  $b_1(V)$  is the first Betti number of  $V$ .*

**PROOF:** If we know that by  $f$ ,  $H^p(V, \Omega_V^q)$  is mapped injectively into  $H^p(U, \Omega_U^q)$ , we can use the argument of Deligne ([4] p 122). The injectivity is proved as follows. We consider  $H^p(V, \Omega_V^q)$  as the Dolbeault cohomology group. Assume that a type  $(p, q)$  form  $\omega \in H^p(V, \Omega_V^q)$  is mapped to zero. Then for any type  $(n-p, n-q)$  form  $\omega' \in H^{n-p}(V, \Omega_V^{n-q})$ , we have

$$0 = \int_U f^*(\omega) \wedge f^*(\omega') = \deg(f) \int_V \omega \wedge \omega'.$$

Hence by the Serre duality we conclude that  $\omega = 0$ . q.e.d.

REMARK 2.3: If  $V$  is a smooth algebraic variety, we can always find a surjective morphism  $f : U \rightarrow V$  of a smooth projective variety  $U$  to  $V$ . If  $a(V) = \dim V$ , such  $U$  also exists. (Moiřezon [25] Chapter II.)

Now let us assume that a compact complex manifold  $V$  satisfies (2.2). Let  $\{\gamma_1, \gamma_2, \dots, \gamma_{2q}\}$  be a basis of the free part of  $H_1(V, \mathbf{Z})$  and let  $\{\omega_1, \omega_2, \dots, \omega_q\}$  be a basis of  $H^0(V, \Omega_V^1)$ . Let  $\Delta$  be a lattice in  $\mathbf{C}^q$  generated by  $2q$  vectors

$$\begin{pmatrix} \int_{\gamma_1} \omega_1 \\ \int_{\gamma_1} \omega_2 \\ \vdots \\ \int_{\gamma_1} \omega_q \end{pmatrix}, \begin{pmatrix} \int_{\gamma_2} \omega_1 \\ \int_{\gamma_2} \omega_2 \\ \vdots \\ \int_{\gamma_2} \omega_q \end{pmatrix}, \dots, \begin{pmatrix} \int_{\gamma_{2q}} \omega_1 \\ \int_{\gamma_{2q}} \omega_2 \\ \vdots \\ \int_{\gamma_{2q}} \omega_q \end{pmatrix}.$$

The complex torus  $\mathbf{C}^q/\Delta$  is called the *Albanese variety* of  $V$  and is written as  $\text{Alb}(V)$ . If  $V$  is algebraic (or more generally  $a(V) = \dim V$ ),  $\text{Alb}(V)$  is an abelian variety.

Now we fix a point  $x_0$  of  $V$ . Then we can define a morphism  $\alpha_{x_0} : V \rightarrow \text{Alb}(V)$  of  $V$  into the  $\text{Alb}(V)$  by

$$\alpha_{x_0} : V \rightarrow \text{Alb}(V) \\ \begin{matrix} \Psi \\ x \mapsto \end{matrix} \begin{pmatrix} \int_{x_0}^x \omega_1 \\ \int_{x_0}^x \omega_2 \\ \vdots \\ \int_{x_0}^x \omega_q \end{pmatrix}.$$

We call this morphism  $\alpha_{x_0}$  the *Albanese map* of  $V$  into  $\text{Alb}(V)$ . If we take another point  $x_1$  of  $V$  and construct the Albanese map  $\alpha_{x_1} : V \rightarrow \text{Alb}(V)$ , then we have

$$\alpha_{x_0}(x) = \alpha_{x_1}(x) + a,$$

where

$$a = \begin{pmatrix} \int_{x_1}^{x_0} \omega_1 \\ \int_{x_1}^{x_0} \omega_2 \\ \vdots \\ \int_{x_1}^{x_0} \omega_q \end{pmatrix}.$$

Hence the Albanese map is *unique up to translations*. In what follows we use the notation  $\alpha$  instead of  $\alpha_{x_0}$ , choosing suitably a point  $x_0$ . The Albanese map  $\alpha$  has the following universal property.

(2.4) Let  $g : V \rightarrow A$  be a morphism of  $V$  into a complex torus  $A$ . Then there exists the uniquely determined group homomorphism  $h : \text{Alb}(V) \rightarrow A$  such that

$$g = h \circ \alpha + b,$$

where  $b$  is a point of  $A$  uniquely determined by  $\alpha$  and  $g$ .

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & \text{Alb}(V) \\ & \searrow g & \swarrow h \\ & & A \end{array}$$

**REMARK 2.5:**

(1) When  $V$  is algebraic, the property (2.4) holds for any *rational map*  $g : V \rightarrow A$  of  $V$  into an *abelian variety*  $A$ . (See Lemma 2.6 and Lang [22] II Section 3.)

(2) For any compact complex manifold  $M$ , we can define the Albanese variety  $\text{Alb}(M)$ , which has the universal property (2.4). (Blanchard [1]). In this case  $\dim \text{Alb}(M) \leq q(M)$  and the equality does not necessarily hold. ([1] p 163–164).

**LEMMA 2.6:** *If two smooth algebraic varieties  $V_1$  and  $V_2$  are birationally equivalent, then  $\alpha : V_1 \rightarrow \text{Alb}(V_1)$  and  $\alpha : V_2 \rightarrow \text{Alb}(V_2)$  are birationally equivalent. Hence  $\text{Alb}(V_1)$  and  $\text{Alb}(V_2)$  are isomorphic.*

**PROOF:** Using the elimination of points of indeterminacy of rational maps [10], it is enough to consider the case that there exists a birational morphism  $f : V_1 \rightarrow V_2$ . Then by a succession of monoidal transformations of  $V_2$  with non-singular centers, we obtain a birational morphism  $g : V_2^* \rightarrow V_2$  such that  $h = f^{-1} \circ g : V_2^* \rightarrow V_1$  is a morphism [10]. By the universal property (2.4), if  $\alpha : V_2^* \rightarrow \text{Alb}(V_2^*)$  and  $\alpha : V_2 \rightarrow \text{Alb}(V_2)$  are birationally equivalent, then  $\alpha : V_1 \rightarrow \text{Alb}(V_1)$  and  $\alpha : V_2 \rightarrow \text{Alb}(V_2)$  are birationally equivalent. Hence we can assume that

$f: V_1 \rightarrow V_2$  is the inverse of a succession of monoidal transformations with non-singular centers. But in this case the lemma is obvious. q.e.d.

$$\begin{array}{ccc}
 V_2^* & \xrightarrow{\alpha} & \text{Alb}(V_2^*) \\
 \downarrow h & & \downarrow \\
 V_1 & \xrightarrow{\alpha} & \text{Alb}(V_1) \\
 \downarrow f & & \downarrow \\
 V_2 & \xrightarrow{\alpha} & \text{Alb}(V_2)
 \end{array}$$

$g$  (curved arrow from  $V_2^*$  to  $V_2$ )

**DEFINITION 2.7:** Let  $f: V \rightarrow A$  be a morphism of an algebraic variety (a compact complex manifold) into an abelian variety (a complex torus)  $A$ . We say  $(V, f)$  generates  $A$  if there exists an integer  $n$  such that the morphism

$$\begin{aligned}
 F: V \times V \times \cdots \times V &\longrightarrow A \\
 \underbrace{(z_1, z_2, \dots, z_n)}_{\cup} &\mapsto f(z_1) + f(z_2) + \cdots + f(z_n),
 \end{aligned}$$

is surjective. We say that a subvariety  $B$  of  $A$  generates  $A$  if  $(B, \iota)$  generates  $A$ , where  $\iota: B \rightarrow A$  is a natural injection.

The following Lemma is an easy consequence of the universal property (2.4).

**LEMMA 2.8:**  $(V, \alpha)$  generates  $\text{Alb}(V)$ .

**LEMMA 2.9:** Let  $\alpha: V \rightarrow \text{Alb}(V)$  be the Albanese map of an algebraic variety  $V$ . Then we have

$$q(\alpha(V)) = q(V).$$

**PROOF:** Let  $\tau: W \rightarrow \alpha(V)$  be a resolution of the singularities of  $\alpha(V)$ . Then by a succession of monoidal transformations of  $V$  with non-singular centers, we obtain a birational morphism  $\pi: V^* \rightarrow V$  such that  $h = \tau^{-1} \circ \alpha \circ \pi: V^* \rightarrow W$  is a morphism. Hence there exist group homomorphisms  $f: \text{Alb}(V^*) \rightarrow \text{Alb}(W)$  and  $g: \text{Alb}(W) \rightarrow \text{Alb}(V)$  such that the diagram (2.10) is commutative. By Lemma 2.8,  $f$  and  $g$  are surjective.

[2.10]

$$\begin{array}{ccc}
 V^* & \xrightarrow{\alpha} & \text{Alb}(V^*) \\
 \downarrow h & & \downarrow f \\
 W & \xrightarrow{\alpha} & \text{Alb}(W) \\
 \downarrow \tau & & \downarrow g \\
 \alpha(V) & \xrightarrow{i} & \text{Alb}(V)
 \end{array}$$

By Lemma 2.6,  $\text{Alb}(V)$  and  $\text{Alb}(V^*)$  are isomorphic. Hence  $\text{Alb}(V)$  and  $\text{Alb}(W)$  have the same dimensions. q.e.d.

LEMMA 2.11: *Let  $V$  be a smooth algebraic variety. If  $\dim \alpha(V) = 1$ , then the image  $\alpha(V)$  of the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  of  $V$  is a smooth curve and any fibre of the morphism  $\alpha$  is connected.*

PROOF: See Šafarevič [29a] Chapter IV, Theorem 3.

REMARK 2.12: If  $\dim \alpha(V) \geq 2$ , a fibre of the Albanese map of  $V$  is not necessarily connected. If a fibre is not connected, we use the Stein factorization  $\beta : V \rightarrow W, \gamma : W \rightarrow \alpha(V)$  of the morphism  $\alpha : V \rightarrow \alpha(V)$ , so that we have  $\alpha = \gamma \circ \beta$ , any fibre of  $\beta$  is connected and  $\gamma : W \rightarrow \alpha(V)$  is a finite morphism [3]. We call the fibre space  $\beta : V \rightarrow W$  the fibre space associated to the Albanese map. Note that in view of Prop. 1.9, we have

$$\kappa(W) \geq \kappa(\alpha(V)).$$

### 3. Subvarieties of abelian varieties

In this section we shall study structures of subvarieties of abelian varieties (complex tori).

Let  $A$  be an abelian variety (a complex torus) of dimension  $n$ . By *global coordinates* of  $A$ , we shall mean global coordinates of the universal covering  $C^n$  of  $A$  such that in these coordinates, covering transformations are represented by translations by elements of the lattice  $\Delta$ , where  $A = C^n/\Delta$ .

LEMMA 3.1: *Let  $B$  be an  $l$ -dimensional subvariety of an abelian variety (a complex torus)  $A$ . Then we have*

$$g_k^m(B) \geq \binom{l}{k} + m - 1, \quad k = 1, 2, \dots, l, \\ m = 1, 2, \dots$$

Hence a fortiori, we have

$$P_m(B) \geq 1, \quad \kappa(B) \geq 0, \quad q(B) \geq \dim B.$$

PROOF: Let  $\tau : B^* \rightarrow B$  be a desingularization of the variety  $B$ . There exist a point  $p \in B$  such that  $B$  is smooth at  $p$  and

$$z_1 - z_1^*, z_2 - z_2^*, \dots, z_l - z_l^*$$

induce local coordinates in  $B$  with a center  $p$ , where  $(z_1, z_2, \dots, z_l, \dots, z_n)$  is a system of global coordinates of  $A$  and  $(z_1^*, z_2^*, \dots, z_n^*)$  is a point of  $C^n$  lying over the point  $p$ . Let  $I_1, I_2, \dots, I_s, s = \binom{l}{k}$ , be all subsets

of  $\{1, 2, \dots, l\}$  consisting of  $k$  elements. Then it is clear that

$$\binom{\binom{l}{k} + m - 1}{m}$$

monomials of  $\tau^*(dz_{I_i}), I = 1, 2, \dots, s$ , of degree  $m$  with coefficients in  $\mathbb{C}$  are elements of  $H^0(B^*, S^m(\Omega_{B^*}^k))$  and are linearly independent. q.e.d.

REMARK 3.2: By Examples 1.12 (1), (2) and Lemma 3.1, we infer readily the well-known fact that *an abelian variety (a complex torus) does not contain ruled varieties and unirational varieties.*

Now we shall characterize a subvariety of an abelian variety (a complex torus) of Kodaira dimension zero.

THEOREM 3.3: *Let  $B$  be an  $l$ -dimensional subvariety of an abelian variety (a complex torus)  $A$ . Then the following conditions are equivalent.*

$$(1) \quad g_k^m(B) = \binom{\binom{l}{k} + m - 1}{m},$$

for a positive integer  $m$  and a positive integer  $k, 1 \leq k \leq l$ .

$$(2) \quad q(B) = l.$$

$$(3) \quad g_k(B) = \binom{l}{k},$$

for a positive integer  $k, 1 \leq k \leq l$ .

$$(4) \quad P_m(B) = 1,$$

for a positive integer  $m$ .

$$(5) \quad \kappa(B) = 0.$$

(6)  *$B$  is a translation of an abelian subvariety (a complex subtorus)  $A_1$  of  $A$  by an element  $a \in A$ .*

PROOF: It is clear that (6) implies (1), (2), (3), (4) and (5). (2), (3) and (4) are special cases of (1). And (5) implies (4), by virtue of Lemma 3.1. Hence it is enough to prove that (1) implies (6).

We can assume that the subvariety  $B$  contains the origin  $o$  of  $A$  and at the origin  $o$ ,  $B$  is smooth. Moreover we can choose a system of global coordinates  $(z_1, z_2, \dots, z_n)$  of  $A$  such that  $(z_1, z_2, \dots, z_l)$  gives a system of local coordinates in  $B$  with a center  $o$ . Hence in a neighborhood  $\mathfrak{U}$  of the origin  $o$  in  $A$ , the variety  $B$  is defined by the equations

$$[3.4] \quad \begin{aligned} z_k &= f_k(z_1, z_2, \dots, z_l), \\ k &= l+1, l+2, \dots, n, \end{aligned}$$

where  $f_k$ 's are holomorphic on  $\mathbb{U}$ .

We shall show that all  $f_k$ 's are linear functions of  $z_1, z_2, \dots, z_l$ .

Let  $\tau : B^* \rightarrow B$  be a desingularization of  $B$ . By our assumptions,  $m$ -tuple  $k$  form  $\tau^*((d\check{z}_{i_1} \wedge dz_{i_2} \wedge \dots \wedge dz_{i_k})^m)$  is written in the form

$$\sum_{\substack{m_1 + m_2 + \dots + m_s = m \\ m_i \geq 0}} a_{m_1, m_2, \dots, m_s} \cdot ((dz_{I_1})^{m_1}) \cdot ((dz_{I_2})^{m_2}) \cdot \dots \cdot ((dz_{I_s})^{m_s})$$

where  $a_{m_1, m_2, \dots, m_s}$  is a constant and  $I_1, I_2, \dots, I_s$  are the same as those in the proof of Lemma 3.1. Hence there exist constants  $c_{i,k}$  such that

$$\begin{aligned} \tau^*((dz_1 \wedge \dots \wedge d\check{z}_i \wedge \dots \wedge dz_l \wedge dz_k)^m) &= c_{i,k} \tau^*((dz_1 \wedge dz_2 \wedge \dots \wedge dz_l)^m), \\ i &= 1, 2, \dots, l, \\ k &= l+1, l+2, \dots, n. \end{aligned}$$

On the other hand, by (3.4), we have

$$\tau^*(dz_1 \wedge \dots \wedge d\check{z}_i \wedge \dots \wedge dz_l \wedge dz_k) = (-1)^{l-i} \frac{\partial f_k}{\partial z_i} \tau^*(dz_1 \wedge dz_2 \wedge \dots \wedge dz_l)$$

on  $\tau^{-1}(\mathbb{U} \cap B)$ . This implies that  $\partial f_k / \partial z_i$  is a constant for any  $i = 1, 2, \dots, l, k = l+1, l+2, \dots, n$ . Hence all  $f_k, k = l+1, l+2, \dots, n$ , are linear functions of  $z_1, z_2, \dots, z_l$ . Let  $L$  be a linear subspace of  $C^n$  defined by the equations (3.4). Then one of the irreducible components  $\tilde{B}$  of the inverse image of  $B$  in the universal covering  $C^n$  of  $A$  coincides  $L$  in a small neighborhood of the origin of  $C^n$ . As  $\tilde{B}$  and  $L$  are irreducible, we have  $\tilde{B} = L$ . This implies that  $B$  is an abelian subvariety (a complex subtorus) of  $A$ . q.e.d.

**COROLLARY 3.5:** *If a proper subvariety  $B$  of an abelian variety (a complex torus)  $A$  generates  $A$ , we have*

$$\kappa(B) > 0.$$

**PROOF:** Assume contrary. Then  $B$  is a translation of an abelian subvariety (a complex subtorus). Hence it is impossible that  $B$  generates  $A$ . q.e.d.

**COROLLARY 3.6:** *Let  $V$  be a smooth algebraic variety (a compact complex manifold which satisfies (2.2)) and let  $\alpha : V \rightarrow \text{Alb}(V)$  be the Albanese map. Then we have*

$$\kappa(\alpha(V)) \geq 0.$$

*Moreover  $\kappa(\alpha(V)) = 0$ , if and only if the Albanese map is surjective.*

PROOF: By Lemma 2.8,  $\alpha(V)$  generates  $A$ . Hence this is an easy consequence of Corollary 3.5. q.e.d.

The following Corollary is an affirmative answer to one of the questions raised by Iitaka [14].

COROLLARY 3.7: *Let  $B$  be a subvariety of an abelian variety (a complex torus)  $A$ . If  $q(B) > \dim B$ , then we have*

$$\kappa(B) > 0.$$

COROLLARY 3.8: *If Conjecture  $C_n$  is true, then Conjecture  $I_n$  is true.*

PROOF: Assume that  $q(V) > \dim V$ . Then the Albanese map is not surjective. Hence  $\kappa(\alpha(V)) > 0$ . Let  $\beta : V \rightarrow W$  be the fibre space associated to the Albanese map. By Proposition 1.9  $\kappa(W) \geq \kappa(\alpha(V)) > 0$ . Hence if  $C_n$  is true, we have  $\kappa(V) \geq \kappa(W) > 0$ , by virtue of Proposition 1.15. q.e.d.

LEMMA 3.9: *Let  $B$  be a subvariety of a complex torus  $T$ . If  $a(B) = \dim B$ , then  $B$  is a subvariety of an abelian variety  $A$ , which is a complex subtorus of  $T$ . Hence  $B$  is a projective variety.*

PROOF: For any positive integer  $m$ , we consider a morphism

$$g_m : \overbrace{B \times \cdots \times B}^{2m} \xrightarrow{\quad} T$$

$$(a_1, \dots, a_{2m}) \quad a_1 + \cdots + a_m - (a_{m+1} + \cdots + a_{2m}).$$

We set  $A_m = g_m(B \times \cdots \times B)$ . As  $g_m$  is a proper morphism,  $A_m$  is a subvariety of  $T$ . Moreover we have

$$A_1 \subset A_2 \subset \cdots \subset A_m \subset A_{m+1} \subset \cdots.$$

Hence there exists a positive integer  $m_0$  such that

$$A_{m_0} = A_{m_0+1} = \cdots = A.$$

By a theorem of Moišezon,  $a(A) = \dim A$ . (Moišezon [25] Chap. I Th. 2). On the other hand by our definition, we have  $o \in A$ ,  $x - y \in A$ , for any  $x, y \in A$ . Hence  $A$  is a complex subtorus of  $T$ . In view of  $a(A) = \dim A$ ,  $A$  is an abelian variety. (Weil [37], p 124 Théorème 3). For a point  $a \in B$ ,  $B \times a \times \cdots \times a$  is mapped biholomorphically into  $A$  by  $g_{m_0}$ . q.e.d.

THEOREM 3.10: *Let  $B$  be a subvariety of a complex torus  $A$ . Then there exist a complex subtorus  $A_1$  of  $A$  and a projective variety  $W$ , which is a subvariety of a complex torus, such that*

- (1)  $B$  is an analytic fibre bundle over  $W$ , whose fibre is  $A_1$ ;

$$(2) \quad \kappa(W) = \dim W = \kappa(B).$$

Moreover if  $B$  is an algebraic variety, then there exist finite unramified coverings  $\tilde{B}$  and  $\tilde{W}$  of  $B$  and  $W$ , respectively such that

$$\tilde{B} = A_1 \times \tilde{W}.$$

PROOF: By Theorem 1.14 there exist a desingularization  $\tau : B^* \rightarrow B$ , a smooth projective variety  $W^*$  and a morphism  $f : B^* \rightarrow W^*$ , which satisfies the conditions (1)~(5) of (1.14). Then there exists an open dense subset  $U^*$  of  $W^*$  such that  $B_w^* = f^{-1}(w)$  is smooth and  $\kappa(B_w^*) = 0$ , for all  $w \in U^*$ . We set  $B_w = \tau(B_w^*)$ . Then there exists an open dense subset  $\mathfrak{U}$  of  $U^*$  such that  $B_w$  is birationally equivalent to  $B_w$  for any point  $w \in \mathfrak{U}$ . As we have  $\kappa(B_w) = \kappa(B_w^*) = 0$  for  $w \in \mathfrak{U}$ ,  $B_w$  is a translation of a complex subtorus  $A_w$  of dimension  $l = \dim B - \kappa(B)$ , by Theorem 3.3. Since  $f^{-1}(\mathfrak{U}) \rightarrow \mathfrak{U}$  is a complex analytic family, all complex subtori  $A_w, w \in \mathfrak{U}$ , are isomorphic to a complex subtorus  $A_1$ , because the complex torus  $A$  contains only at most countably many complex subtori. We set  $A_2 = A/A_1$  and  $u : A \rightarrow A_2$  is the canonical quotient map. We set  $g = u \circ \tau$ .

$$\begin{array}{ccc}
 B^* & \xrightarrow{g} & A_2 \\
 \downarrow f & \searrow h & \uparrow pr_2 \\
 W^* & \xleftarrow{pr_1} & X
 \end{array}$$

Let  $X$  be the image of  $B^*$  in  $W^* \times A_2$  of the morphism  $(f, g) : B^* \rightarrow W^* \times A_2$  and let  $h : B^* \rightarrow X$  be the canonical morphism. Let  $pr_1 : X \rightarrow W^*$  and  $pr_2 : X \rightarrow A_2$  be the morphisms induced by the projections to the first factor and the second factor of  $W^* \times A_2$ , respectively. Since for any point  $w \in \mathfrak{U}$ ,  $g(f^{-1}(w))$  is a point,  $pr_1^{-1}(U)$  is isomorphic to  $\mathfrak{U}$ . Hence  $\dim X = \dim W^*$ . We set  $W = pr_2(X)$ .  $W$  is a subvariety of a complex torus  $A_2$ . We have  $\dim W \leq \dim W^*$ . We set  $S = g(f^{-1}(\mathfrak{U}))$ . As  $h^{-1} \circ pr_2^{-1}(S) = f^{-1}(\mathfrak{U})$  is open dense in  $B^*$ , we have

$$\dim W + \dim A_1 \geq \dim B^*.$$

Hence  $\dim W = \dim W^*$ . Hence  $\dim u^{-1}(W) = \dim B$ . As  $u^{-1}(W) \supset B$ , we conclude that  $u^{-1}(W) = B$ . Hence  $u : B \rightarrow W$  is an analytic fibre bundle over  $W$ , whose fibre is  $A_1$ .

Now we shall prove that  $\kappa(W) = \dim W = \kappa(B)$ . If  $\kappa(W) < \dim W$ , from the above arguments, we infer readily that there exists a morphism  $u' : W \rightarrow Y$ , which is an analytic fibre bundle over  $Y$  whose fibre is a complex torus. Hence  $u' \circ u : B \rightarrow Y$  is a fibre bundle whose fibre is a

torus bundle over a torus. In view of Example 1.12 (5) and Theorem 1.17, we have

$$\dim Y \geq \kappa(B) = \dim W.$$

This contradicts the assumption  $\dim Y = \kappa(W) < \dim W$ . Hence

$$\kappa(W) = \dim W.$$

Next assume that  $B$  is algebraic. Then by Lemma 3.9, we can assume that the complex torus  $A$  is an abelian variety. Then there exists a finite unramified covering  $\pi : \tilde{A} \rightarrow A$  such that  $\tilde{A} = A_1 \times A^*$ , where  $A^*$  is an abelian variety. Let  $\tilde{B}$  be one of the connected component of  $\pi^{-1}(B)$  and we set  $\tilde{W} = Pr_2(B)$ , where  $Pr_2 : A \rightarrow A^*$  is the projection to the second factor. Then by the above arguments,  $\tilde{B}$  and  $\tilde{W}$  are finite unramified covering manifolds of  $B$  and  $W$ , respectively. And we have  $\tilde{B} = Pr_2^{-1}(W) A_1 \times \tilde{W}$ . q.e.d

**LEMMA 3.11:** *Let  $B$  be a smooth subvariety of an abelian variety (a complex torus)  $A$ . Then for any positive integer  $m$  the sheaf  $\mathcal{O}(mK_B)$  is spanned by its global sections. Hence if  $\kappa(B) > 0$ , the complete linear system  $|mK_B|$  is free from base points and fixed components, for any positive integer  $m$ .*

**PROOF:** We have an exact sequence

$$0 \rightarrow T_B \rightarrow T_{A|B} \rightarrow N_B \rightarrow 0,$$

where  $T_A, T_B$  are tangent bundles of  $A$  and  $B$ , respectively and  $N_B$  is the normal bundle of  $B$  in  $A$ . As  $\mathcal{O}(T_A)$  is spanned by its global sections,  $\mathcal{O}(N_B)$  is spanned by its global sections. Hence  $\mathcal{O}(K_B) = \mathcal{O}(T_A \wedge N_B)$ ,  $l = \text{codim } B$ , is spanned by its global sections. Hence a fortiori  $\mathcal{O}(mK_B)$  is spanned by its global sections. q.e.d.

**COROLLARY 3.12:** *If  $B$  is a smooth subvariety of an abelian variety, the structure of the fibre bundle in Theorem 3.10 is birationally equivalent to the fibre space given by the  $m$ -th canonical map  $\Phi_{mK} : B \rightarrow W = \Phi_{mK}(B)$  for a sufficiently large  $m$ .*

#### 4. Basic elliptic threefolds

In this section we shall construct a certain elliptic threefolds, basic elliptic threefolds, after Kodaira [19] and Kawai [16]. Since we shall need later the explicit construction of these threefolds, we shall give some details. For the proof and more detailed arguments, see Kodaira [19] and Kawai [16].

DEFINITION 4.1: A fibre space  $f: V \rightarrow W$  is called an *elliptic fibre space* of dimension  $n$ , if

- (1)  $V$  is an  $n$ -dimensional compact complex space and  $W$  is an  $(n-1)$ -dimensional compact complex space;
- (2) general fibres of the morphism  $f$  are elliptic curves.

When  $n = 2$  we call such a  $f: V \rightarrow W$  an *elliptic surface*. When  $n = 3$  we call it an *elliptic threefold*.

First we shall assume that  $V$  and  $W$  are non-singular. Then any regular fibre<sup>1</sup> of  $f: V \rightarrow W$  is an elliptic curve. Let  $S$  be a proper analytic subset of  $W$  such that  $f$  is of maximal rank at any point of  $V - f^{-1}(S)$ . We will *not* assume that  $S$  is minimal in this property. We call  $S$  a *singular locus* of the elliptic fibre space  $f: V \rightarrow W$ . We set  $W' = W - S$  and  $V' = f^{-1}(W')$ .

For any point  $x \in W'$  there exists a neighborhood  $\mathfrak{U}$  of  $x$  in  $W'$ , a holomorphic 1 form  $\omega$  on  $f^{-1}(\mathfrak{U})$  and 3 cycles  $\alpha, \beta$  of  $H_3(f^{-1}(\mathfrak{U}), \mathbf{Z})$  such that

- (1)  $\omega$  induces a non-zero holomorphic 1 form  $\omega(y)$  on  $V_y = f^{-1}(y)$ ;
- (2)  $\{\alpha, \beta\}$  induces a symplectic basis  $\{\alpha(y), \beta(y)\}$  of  $H_1(V_y, \mathbf{Z})$ , that is,

$$\begin{aligned} \alpha(y) \cdot \alpha(y) &= 0, \beta(y) \cdot \beta(y) = 0, \\ \alpha(y) \cdot \beta(y) &= 1, \end{aligned}$$

for any  $y \in \mathfrak{U}$ . (See [26] § 2).

We set

$$T(y) = \int_{\alpha(y)} \omega(y) / \int_{\beta(y)} \omega(y).$$

Then  $T$  gives a morphism  $T: \mathfrak{U} \rightarrow H$  of  $\mathfrak{U}$  into the upper half plane  $H$ . Note that the morphism  $T$  does not depend on the choice of  $\omega$ , but depends on the choice of  $\{\alpha, \beta\}$ . In this way we can construct a *multivalued* holomorphic map  $T$  of  $W'$  into the upper half plane  $H$ . Hence we have a morphism  $\tilde{T}: \tilde{W}' \rightarrow H$  of the universal covering manifold  $\tilde{W}'$  of  $W'$  into  $H$  and a group homomorphism  $\Phi: \pi_1(W') \rightarrow \text{SL}(2, \mathbf{Z})$  such that

$$\begin{aligned} (4.2) \quad \tilde{T}(\gamma \cdot y) &= \frac{a\tilde{T}(\tilde{y}) + b}{c\tilde{T}(\tilde{y}) + d}, \quad \gamma \in \pi_1(W') \\ \Phi(\gamma) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbf{Z}). \end{aligned}$$

Hence for any elliptic fibre space  $f: V \rightarrow W$  and a singular locus  $S$  in  $W$  we can associate a pair  $(\tilde{T}, \Phi)$ , where  $\tilde{T}: \tilde{W}' \rightarrow H$  is a morphism of the universal covering manifold  $\tilde{W}'$  of  $W'$  into the upper half plane  $H$  and

<sup>1</sup> By a regular fibre  $V_w = f^{-1}(w)$ ,  $w \in W$ , we mean that  $f$  is of maximal rank at any point of  $V_w$ .

$\Phi : \pi_1(W') \rightarrow \text{SL}(2, \mathbf{Z})$  is a group representation, which satisfy the relation (4.2).

If two such pairs  $(\tilde{T}_1, \Phi_1)$  and  $(\tilde{T}_2, \Phi_2)$  are associated to  $f : V \rightarrow W$  and  $S$ , then there exists an element  $M \in \text{SL}(2, \mathbf{Z})$  such that

$$(4.3) \quad \begin{cases} \tilde{T}_1(\tilde{y}) = M \cdot \tilde{T}_2(\tilde{y}), \\ \Phi_1(\gamma) = M \cdot \Phi_2(\gamma)M^{-1}, \end{cases}$$

for any  $y \in W'$  and any  $\gamma \in \pi_1(W')$ . We call two pairs  $(\tilde{T}_1, \Phi_1)$  and  $(\tilde{T}_2, \Phi_2)$  are equivalent if the above condition (4.3) is satisfied.

Hence for any elliptic fibre space and its singular locus we can associate the equivalent class of a pair  $(\tilde{T}, \Phi)$ . We call it the characteristic pair. Note that  $J(y) = j(\tilde{T}(\tilde{y}))$  is a single valued meromorphic function on  $W$ , where  $j$  is the elliptic modular function.  $J(y)$  is uniquely determined by the characteristic pair.

After Kodaira and Kawai for any such characteristic pair  $(\tilde{T}, \Phi)$ , we shall associate the elliptic space  $\mu : B \rightarrow W$ , the basic elliptic fibre space, which is birationally characterized by the fact that  $\mu : B \rightarrow W$  has a global holomorphic section.

For that purpose we apply on  $W$  a succession of monoidal transformations with non-singular centers and we obtain a new manifold  $\pi : W^* \rightarrow W$  such that the total transform  $S^* = \pi^*(S)$  is a divisor with normally crossings and the meromorphic function  $J$  has no points of indeterminacy. As we have an isomorphism between  $W - S$  and  $W^* - S^*$  and we are interested in the birational geometry, we can assume that the singular locus  $S$  is a divisor with normally crossings and the meromorphic function  $J$  has no points of indeterminacy.

In what follows we only consider the case  $\dim W = 2$ . But the generalization to higher dimensional case is not difficult. Only notations become complicated.

For any element  $\beta \in \pi_1(W')$  and integers  $n_1, n_2$ , we let  $g(\beta; n_1, n_2)$  be an analytic automorphism of  $\tilde{W}' \times \mathbf{C}$  defined by

$$g(\beta; n_1, n_2) : (\tilde{y}, \zeta) \mapsto (\beta(\tilde{y}), f_\beta(\tilde{y}) \cdot (\zeta + n_1 \tilde{T}(\tilde{y}) + n_2)),$$

where

$$\begin{aligned} f_\beta(\tilde{y}) &= (c_\beta \tilde{T}(\tilde{y}) + d_\beta)^{-1} \\ \Phi(\beta) &= \begin{pmatrix} a_\beta & b_\beta \\ c_\beta & d_\beta \end{pmatrix}. \end{aligned}$$

The group  $\mathcal{G} = \{g(\beta; n_1, n_2) | \beta \in \pi_1(W'), (n_1, n_2) \in \mathbf{Z}^2\}$  acts on  $\tilde{W}' \times \mathbf{C}$  properly discontinuously and freely. Let  $B'$  be the quotient manifold  $\tilde{W}' \times \mathbf{C} / \mathcal{G}$ . Then the morphism

$$\begin{array}{ccc} \mu' : B' & \rightarrow & W' \\ & \Downarrow & \Downarrow \\ & [\tilde{y}, \zeta] & \mapsto \pi(\tilde{y}), \end{array}$$

gives a structure of a fibre space whose fibres are elliptic curves, where  $\pi : \tilde{W}' \rightarrow W'$  is the covering morphism.

We shall construct the basic elliptic threefold  $\mu : B \rightarrow W$  as an extension of the fibre space  $\mu' : B' \rightarrow W'$ . Note that the fibre space  $\mu' : B' \rightarrow W'$  has a holomorphic section  $o' : W' \rightarrow B'$  defined by  $y \mapsto [\tilde{y}, o]$ , where  $\tilde{y}$  is a point of  $\tilde{W}'$  lying over the point  $y$ .

We cover the singular locus by a finite number of sufficiently small coordinates neighborhoods  $D_v = \{(t_1, t_2) \mid |t_i| < \varepsilon_i\}$ ,  $v = 1, 2, \dots, n$ , in the manifold  $W$  such that  $S$  is defined in  $D_v$  by the equation

$$\begin{aligned} t_1 &= 0, \text{ for } v = 1, 2, \dots, h, \\ t_1 t_2 &= 0, \text{ for } v = h+1, h+2, \dots, n. \end{aligned}$$

We set

$$\begin{aligned} D_v &= D_{(1)} \times D_{(2)}, \quad D_{(i)} = \{t_i \mid |t_i| < \varepsilon_i\}, \\ D'_v &= \begin{cases} D_{(1)}^* \times D_{(2)}, & \text{for } v = 1, 2, \dots, h, \\ D_{(1)}^* \times D_{(2)}^*, & \text{for } v = h+1, h+2, \dots, n, \end{cases} \\ D_{(i)}^* &= \{t_i \mid 0 < |t_i| < \varepsilon_i\}, \\ U_{(i)} &= \left\{ l_i \in \mathbb{C} \mid \operatorname{Im}(l_i) > -\frac{1}{2\pi} \log \varepsilon_i \right\}. \end{aligned}$$

The universal covering  $U'_v$  of  $D'_v$  is given by

$$U'_v = \begin{cases} U_{(1)} \times D_{(2)} & \text{for } v = 1, 2, \dots, h, \\ U_{(1)} \times U_{(2)} & \text{for } v = h+1, h+2, \dots, n. \end{cases}$$

We denote  $\gamma_i$  a small circle in  $D_{(i)}^*$  rounding the origin once counterclockwise. There is a canonical group homomorphism  $\iota_v : \pi_1(D'_v) \rightarrow \pi_1(W')$  and a canonical morphism  $j_v : U'_v \rightarrow \tilde{W}'$ .  $\pi_1(D')$  is generated by  $\gamma_1$  for  $v = 1, 2, \dots, h$  and is generated by  $\gamma_1$  and  $\gamma_2$  for  $v = h+1, h+2, \dots, n$ . We set

$$(4.4) \quad \begin{cases} M = \Phi(\iota_v(\gamma_1)), \quad v = 1, 2, \dots, h, \\ M_i = \Phi(\iota_v(\gamma_i)), \quad v = h+1, h+2, \dots, n, \\ T(l_1, t_2) = \tilde{T}(j_v(l_1, t_2)), \quad v = 1, 2, \dots, h, \\ T(l_1, l_2) = \tilde{T}(j_v(l_1, l_2)), \quad v = h+1, h+2, \dots, n, \\ M^k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}, \quad (M_1)^{k_1} (M_2)^{k_2} = \begin{pmatrix} a_{k_1 k_2} & b_{k_1 k_2} \\ c_{k_1 k_2} & d_{k_1 k_2} \end{pmatrix}, \\ f_k(l_1, t_2) = (c_k T(l_1, t_2) + d_k)^{-1}, \\ f_{k_1 k_2}(l_1, l_2) = (c_{k_1 k_2} T(l_1, l_2) + d_{k_1 k_2})^{-1}, \end{cases}$$

where we omit the suffix  $\nu$  for simplicity.

Let  $\mathcal{G}_\nu$  be a group of analytic automorphisms of  $U'_\nu \times \mathbb{C}$  consisting of all automorphisms

$$g(k; n_1, n_2) : (l_1, t_2, \zeta) \mapsto (l_1 + k, t_2, f_k(l_1, t_2) \cdot (\zeta + n_1 T(l_1, t_2) + n_2)),$$

for  $\nu = 1, 2, \dots, h$ , and  $k, n_1, n_2 \in \mathbb{Z}$ ,

$$g(k_1, k_2; n_1, n_2) : (l_1, l_2, \zeta) \mapsto (l_1 + k_1, l_2 + k_2, f_{k_1 k_2}(l_1, l_2) \cdot (\zeta + n_1 T(l_1, l_2) + n_2)),$$

for  $\nu = h + 1, h + 2, \dots, n$ , and  $k_1, k_2, n_1, n_2 \in \mathbb{Z}$ .

Let  $B'_\nu = U'_\nu \times \mathbb{C} / \mathcal{G}_\nu$  be the quotient manifold. We denote a point on  $B'_\nu$  corresponding to a point  $(l_1, t_2, \zeta)$  (or  $(l_1, l_2, \zeta)$ ) by the symbol  $((l_1, t_2, \zeta))$  (or  $((l_1, l_2, \zeta))$ ). By our construction the fibre space  $\mu'_\nu : B' \rightarrow D'$  defined by

$$\begin{aligned} ((l_1, t_2, \zeta)) &\mapsto (e^{2\pi i l_1}, t_2), \nu = 1, 2, \dots, h, \\ ((l_1, l_2, \zeta)) &\mapsto (e^{2\pi i l_1}, e^{2\pi i l_2}), \nu = h + 1, h + 2, \dots, n, \end{aligned}$$

is isomorphic to  $B' | D'_\nu$ . The fibre space has a holomorphic section  $o'_\nu : D'_\nu \rightarrow B'_\nu$  defined by

$$\begin{aligned} (t_1, t_2) &\mapsto \left( \left( \frac{1}{2\pi i} \log t_1, t_2, 0 \right) \right), \nu = 1, 2, \dots, h, \\ (t_1, t_2) &\mapsto \left( \left( \frac{1}{2\pi i} \log t_1, \frac{1}{2\pi i} \log t_2, 0 \right) \right), \nu = h + 1, h + 2, \dots, n, \end{aligned}$$

which is the restriction of  $o'$  on  $D'_\nu$ .

We shall construct the fibre space  $\mu_\nu : B_\nu \rightarrow D_\nu$  which is an extension of  $\mu'_\nu : B'_\nu \rightarrow D'_\nu$  and the fibre space has a holomorphic section  $o_\nu : D_\nu \rightarrow B_\nu$  which is the extension of the section  $o'_\nu$ . The construction is done in such a way that we can patch together all  $B_\nu, \nu = 1, 2, \dots, n$  and  $B'_\nu$  so that we get the elliptic threefold  $\mu : B \rightarrow W$  which has a holomorphic section  $o : W \rightarrow B$ .

(4.5) *Case a:*  $\nu = 1, 2, \dots, h$ .  $M$  is of finite order, say  $m$ .

The matrix  $M$  is  $SL(2, \mathbb{Z})$ -conjugate to one of the matrices appeared in Table 6.7 below. Let  $U_\nu^\# = \{s_1 || s_1| < (\varepsilon_1)^{1/m}\} \times D_{(2)}$  be an  $m$  fold ramified covering of  $U_\nu$  defined by the morphism

$$(s_1, t_2) \mapsto ((s_1)^m, t_2) = (t_1, t_2).$$

We set

$$S(s_1, t_2) = T \left( \frac{m}{2\pi i} \log s_1, t_2 \right).$$

Then  $S$  is a single valued holomorphic map of  $U_\nu$  into  $H$ . For any pair of

integers  $(n_1, n_2)$ , we let  $g(n_1, n_2)$  be an analytic automorphism of  $U_v^* \times C$  defined by

$$g(n_1, n_2) : (s_1, t_2, \zeta) \mapsto (s_1, t_2, \zeta + n_1 S(s_1, t_2) + n_2).$$

Let  $F_v$  be the quotient manifold  $U_v \times C / \{g(n_1, n_2)\}$  and for any point  $(s_1, t_2, \zeta)$  we denote the corresponding point of  $F_v$  by the symbol  $[s_1, t_2, \zeta]$ . Let  $G$  be a cyclic group of order  $m$  of analytic automorphisms of  $F_v$  generated by

$$g : [s_1, t_2, \zeta] \mapsto [e_m s_1, t_2, g_1(s_1, t_2) \cdot \zeta],$$

where

$$g_1(s_1, t_2) = f_1 \left( \frac{m}{2\pi i} \log s_1, t_2 \right)$$

(see (4.4)) and  $e_m = \exp(2\pi i/m)$ . The quotient space  $B_v = F_v/G$  has the structure of a normal complex space. For any point  $[s_1, t_2, \zeta]$  we denote the corresponding point of  $B_v$  by the symbol  $[[s_1, t_2, \zeta]]$ . The morphism

$$\begin{array}{ccc} \mu_v : & B_v & \longrightarrow & D_v \\ & \cup & & \cup \\ & [[s_1, t_2, \zeta]] & \mapsto & ((s_1)^m, t_2) \end{array}$$

gives the structure of a fibre space which is an extension of  $\mu'_v : B'_v \rightarrow D'_v$ .  $\mu_v : B_v \rightarrow D_v$  has a holomorphic section

$$\begin{array}{ccc} o_v : & D_v & \longrightarrow & B_v \\ & \cup & & \cup \\ & (t_1, t_2) & \mapsto & [[(t_1)^{1/m}, t_2, 0]], \end{array}$$

which is the extension of  $o'_v : D'_v \rightarrow B'_v$ .

(4.6) *Case b:  $v = 1, 2, \dots, h$ .  $M$  is of infinite order.*

We can assume that  $T(l_1, t_2)$  has a form

$$T(l_1, t_2) = b \cdot l_1,$$

and  $M$  has a form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix},$$

where  $b$  is a positive integer.

First we shall consider the case

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

We have  $B'_v = U'_v \times C/\mathcal{G}$ , where  $\mathcal{G}$  consists of all automorphisms

$$g(k; n_1, n_2) : (l_1, t_2, \zeta) \mapsto (l_1 + k_1, t_2, \zeta + n_1 b l_1 + n_2), k, n_1, n_2 \in \mathbf{Z}$$

of  $U'_v \times \mathbf{C}$ .

Let  $\mathcal{N}$  be a normal subgroup of  $\mathcal{G}$  consisting of all  $g(k, 0, n_2)$ ,  $k, n_2 \in \mathbf{Z}$ .

There is an isomorphism

$$\begin{aligned} U'_v \times \mathbf{C} / \mathcal{N} &\simeq D'_v \times \mathbf{C}^* \\ \downarrow &\quad \downarrow \\ [(l_1, t_2, \zeta)] &\mapsto (e^{2\pi i t_1}, t_2, e^{2\pi i \zeta}) = (t_1, t_2, z). \end{aligned}$$

Hence the quotient group  $\mathcal{G} / \mathcal{N} = G$  operates on  $D'_v \times \mathbf{C}^*$ .  $G$  is generated by the automorphism

$$g : (t_1, t_2, z) \mapsto (t_1, t_2, z(t_1)^b),$$

of  $D'_v \times \mathbf{C}^*$ . Following Kodaira we can define the quotient manifold

$$F_v = D_v \times \mathbf{C}^* / G = B'_v \cup (0 \times D_{(2)} \times \mathbf{C}^*).$$

(See Kodaira [19] II p 597~598.) Now we set

$$\begin{aligned} x(t_1, t_2, z) &= -2 \sum_{n=1}^{\infty} \frac{n(t_1)^{bn}}{1-(t_1)^{bn}} + \sum_{n=-\infty}^{\infty} \frac{z(t_1)^{bn}}{(1-z(t_1)^{bn})^2} \\ y(t_1, t_2, z) &= \sum_{n=-\infty}^{+\infty} \frac{(1+z(t_1)^{bn})z(t_1)^{bn}}{(1-z(t_1)^{bn})^3}. \end{aligned}$$

Then we have

$$\begin{aligned} x(t_1, t_2, z(t_1)^b) &= x(t_1, t_2, z), \\ y(t_1, t_2, z(t_1)^b) &= y(t_1, t_2, z). \end{aligned}$$

The function  $x(t_1, t_2, z)$  and  $y(t_1, t_2, z)$  satisfy the equation

$$(4.7) \quad y^2 - 4x^3 - x^2 + g_2(t_1)x + g_3(t_1) = 0,$$

where

$$\begin{aligned} g_2(t_1) &= 20 \sum_{n=1}^{\infty} \frac{n^3(t_1)^{bn}}{1-(t_1)^{bn}} \\ g_3(t_1) &= \frac{1}{3} \sum_{n=1}^{\infty} \frac{(7n^5 + 5n^3)(t_1)^{bn}}{1-(t_1)^{bn}}. \end{aligned}$$

Let  $\mathbf{P}^2$  be a projective plane with non-homogenous coordinates  $(x, y)$  and let  $B_v$  be the subvariety in  $D_v \times \mathbf{P}^2$  defined by the above non-homogenous equation (4.7). Then the morphism  $\mu_v : B_v \rightarrow D_v$  defined by

$$\mu_v : (t_1, t_2, x, y) \mapsto (t_1, t_2)$$

is proper. Moreover the morphism

$$((t_1, t_2, z)) \mapsto (t_1, t_2, x(t_1, t_2, z), y(t_1, t_2, z))$$

gives the isomorphism between  $F_v$  and an open dense subset in  $B_v$ . Hence  $\mu_v : B_v \rightarrow D_v$  is an extension of  $\mu'_v : B'_v \rightarrow D'_v$ . The morphism

$$o_v : (t_1, t_2) \mapsto ((t_1, t_2, 1)) \in F_v,$$

gives a holomorphic section which is the extension of the holomorphic section  $o'_v : D'_v \rightarrow B'_v$ .

Next we shall consider the case

$$M = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}.$$

We have  $B'_v = U'_v \times C/\mathcal{G}$ , where  $\mathcal{G}$  is a group consisting of all analytic automorphisms

$$g(k; n_1, n_2) : (l_1, t_2, \zeta) \mapsto (l_1 + k, t_2, (-1)^k(\zeta + n_1 b l_1 + n_2)), k, n_1, n_2 \in \mathbf{Z}.$$

of  $U'_v \times C$ . Let  $\mathcal{L}$  be a subgroup of  $\mathcal{G}$  consisting of all  $g(k, n_1, n_2)$ ,  $k \equiv 0 \pmod{2}$ ,  $n_1, n_2 \in \mathbf{Z}$ . We set

$$E_{(1)} = \{s_1 \mid |s_1| < (\varepsilon_1)^{\frac{1}{2}}\}, \quad E'_{(1)} = E_{(1)} - \{0\}, \\ E_v = E_{(1)} \times D_{(2)}, \quad E'_v = E'_{(1)} \times D_{(2)}.$$

Then the quotient manifold  $F'_v = U'_v \times C/\mathcal{L}$  is isomorphic to the quotient manifold  $E'_v \times C^*/G$ , where  $G$  is an infinite cyclic group of analytic automorphisms of  $E'_v \times C^*$  generated by the automorphism

$$g : (s_1, t_2, z) \mapsto (s_1, t_2, z(s_1)^{2b}).$$

Now we shall construct a fibre space  $\tilde{\mu}_v : \tilde{B}_v \rightarrow E_v$  which is an extension of  $F'_v$  by the same method as above.  $\tilde{B}_v$  is defined in  $E \times P^2$  by the non-homogenous equation

$$y^2 - 4x^3 - x^2 + g_2((s_1)^2) + g_3((s_1)^2) = 0.$$

The quotient group  $H = \mathcal{G}/\mathcal{L}$  is a cyclic group of order two of analytic automorphisms of  $\tilde{B}_v$  generated by the automorphism

$$h : ((s_1, t_2, z)) \mapsto ((-s_1, t_2, z^{-1})) \text{ on } F'_v \\ (s_1, t_2, x, y) \mapsto (s_1, t_2, x, -y) \text{ on } \tilde{B}_v.$$

We set  $B_v = \tilde{B}_v/H$ . Then the morphism

$$\mu_v : \begin{array}{ccc} B_v & \longrightarrow & D_v \\ \cup & & \cup \\ (s_1, t_2, x, y) & \mapsto & ((s_1)^2, t_2), \end{array}$$

gives a structure of a fibre space which is a desired extension of  $B'_v$ . The morphism

$$o_v : (t_1, t_2) \mapsto (((t_1)^{\frac{1}{2}}, t_2, 1)) \in F_v/H$$

defines the holomorphic section which is the extension of  $o'_v$  where  $F_v = E_v \times C^*/G$ . (See Kodaira [19] II p 597~598.)

(4.8) *Case c:*  $v = h+1, h+2, \dots, n$ . The orders  $m_i$   $i = 1, 2$  of  $M_i$  are finite.

We consider  $E_v = \{(s_1, s_2) \mid |s_1| < (\varepsilon_1)^{1/m_1}, |s_2| < (\varepsilon_2)^{1/m_2}\}$  as a ramified covering of  $D_v$  by the morphism

$$(s_1, s_2) \mapsto ((s_1)^{m_1}, (s_2)^{m_2}).$$

We set

$$S(s_1, s_2) = T \left( \frac{m_1}{2\pi i} \log s_1, \frac{m_2}{2\pi i} \log s_2 \right).$$

Then  $S$  is a single valued holomorphic map of  $E_v$  into the upper half plane.

Let  $\mathcal{G}$  be a group of analytic automorphisms of  $E_v \times C$  consisting of all automorphisms

$$g(n_1, n_2) : (s_1, s_2, \zeta) \mapsto (s_1, s_2, \zeta + n_1 S(s_1, s_2) + n_2), \quad n_1, n_2 \in \mathbf{Z}.$$

We consider the automorphism  $g_{k_1 k_2}$  of the quotient space  $\tilde{B}_v = E_v \times C/\mathcal{G}$  defined by

$$g_{k_1 k_2} : [s_1, s_2, \zeta] \mapsto [e_{m_1}^{k_1} s_1, e_{m_2}^{k_2} s_2, h_{k_1 k_2}(s_1, s_2)],$$

where

$$h_{k_1 k_2}(s_1, s_2) = f_{k_1 k_2} \left( \frac{m_1}{2\pi i} \log s_1, \frac{m_2}{2\pi i} \log s_2 \right),$$

$$e_{m_i} = \exp(2\pi i/m_i).$$

Then the quotient space  $B_v = \tilde{B}_v/G$ ,  $G = \{g_{k_1 k_2}\}$ , is a normal complex space and the morphism

$$\mu_v : \begin{array}{ccc} B_v & \longrightarrow & D_v \\ \varpi & & \varpi \\ [[s_1, s_2, \zeta]] & & ((s_1)^{m_1}, (s_2)^{m_2}) \end{array}$$

gives an extension of the fibre space  $\mu'_v : B'_v \rightarrow D_v$ . The fibre space  $\mu_v : B_v \rightarrow D_v$  has the section

$$o_v : (t_1, t_2) \mapsto [[(t_1)^{1/m_1}, (t_2)^{1/m_2}, 0]],$$

which is the extension of  $o'_v : D'_v \rightarrow B'_v$ .

(4.9) *Case d:*  $v = h+1, h+2, \dots, n$ .  $M_1$  is of infinite order and  $M_2$  is of finite order.

By a suitable choice of coordinates we can assume that

$$T(l_1, l_2) = b \cdot l_1.$$

$$M_1 = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad M_2 = -I_2.$$

We consider  $E_v = \{(s_1, s_2) \mid |s_1| < (\varepsilon_1)^{1/\delta}, |s_2| < (\varepsilon_2)^{\frac{1}{2}}\}$  as a ramified covering of  $D_v$  by the morphism

$$(s_1, s_2) \mapsto ((s_1)^\delta, (s_2)^2),$$

where

$$\delta = \begin{cases} 1, & \text{if } M_1 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \\ 2, & \text{if } M_2 = -\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \end{cases}$$

By the same way as in (4.6), we shall construct a subvariety  $\tilde{B}_v$  in  $E_v \times \mathbb{P}^2$  defined by the non-homogenous equation

$$y^2 - 4x^3 - x^2 + g_2((s_1)^\delta) \cdot x + g_3((s_1)^\delta) = 0.$$

Let  $\tau_1, \tau_2$  be analytic automorphisms of  $\tilde{B}_v$  defined by

$$\tau_1 : (s_1, s_2, x, y) \mapsto ((-1)^{\delta-1} \cdot s_1, s_2, x, (-1)^{\delta-1} \cdot y),$$

$$\tau_2 : (s_1, s_2, x, y) \mapsto (s_1, -s_2, x, -y).$$

Let  $H$  be the group of analytic automorphisms of  $\tilde{B}_v$ , generated by  $\tau_1$  and  $\tau_2$ . Let  $B_v$  be the quotient space  $\tilde{B}_v/H$ . Then the morphism

$$\mu_v : \begin{array}{ccc} B_v & \longrightarrow & D_v \\ \cup & & \cup \\ [(s_1, s_2, x, y)] & \mapsto & ((s_1), (s_2)^2) \end{array}$$

gives a structure of a fibre space which is an extension of the fibre space  $\mu'_v : B'_v \rightarrow D'_v$ . The morphism

$$o_v : (t_1, t_2) \mapsto (((t_1)^{1/\delta}, (t_2)^{\frac{1}{2}}, 1)) \in F_v/H,$$

defines the holomorphic section which is the extension of  $o'$  where  $F_v = E_v \times \mathbb{C}^*/G$ . (See (4.5) and (4.6).

(4.10) *Case e:*  $v = h + 1, h + 2, \dots, n$ .  $M_1$  and  $M_2$  are of infinite order. We can assume that

$$T(l_1, l_2) = b_1 \cdot l_1 + b_2 \cdot l_2,$$

$$M_i = \pm \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2.$$

We set

$$E_v = \{(s_1, s_2) \mid |s_i| < (\varepsilon_i)^{1/\delta_i}\}, \quad i = 1, 2,$$

$$E'_v = \{(s_1, s_2) \mid 0 < |s_i| < (\varepsilon_i)^{1/\delta_i}\}, \quad i = 1, 2,$$

where

$$\delta_i = \begin{cases} 1, & \text{if } M_i = \begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix} \\ 2, & \text{if } M_i = -\begin{pmatrix} 1 & b_i \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2. \end{cases}$$

Let  $\mathcal{G}$  be a group of analytic automorphisms of  $U'_v \times \mathbb{C}$  consisting of all automorphisms

$$g(k_1, k_2; n_1, n_2) : (l_1, l_2, \zeta) \mapsto (l_1 + k_1, l_2 + k_2, (-1)^\beta (\zeta + n_1(\delta_1 b_1 l_1 + \delta_2 b_2 l_2) + n_2)),$$

$$\beta = k_1(\delta_1 - 1) + k_2(\delta_2 - 1), \quad k_1, k_2, n_1, n_2 \in \mathbb{Z}.$$

We set

$$\mathcal{L} = \{g(k_1, k_2; n_1, n_2) \in \mathcal{G} \mid k_i(\delta_i - 1) = 0, \quad i = 1, 2.$$

$$\mathcal{N} = \{g(k_1, k_2; 0, n_2) \in \mathcal{L}\}.$$

The group  $\mathcal{N}$  is a normal subgroup of  $\mathcal{L}$ . The quotient manifold  $U'_v \times \mathbb{C} / \mathcal{N}$  is isomorphic to  $E'_v \times \mathbb{C}^*$  by the morphism

$$((l_1, l_2, \zeta)) \mapsto (e^{(2\pi i l_1/\delta_1)}, e^{(2\pi i l_2/\delta_2)}, z).$$

We can consider the quotient group  $L = \mathcal{L} / \mathcal{N}$  as a group of analytic automorphisms of  $E'_v \times \mathbb{C}^*$  generated by the automorphism

$$\varphi_1 : (s_1, s_2, z) \mapsto (s_1, s_2, z(s_1)^{\delta_1 b_1}),$$

$$\varphi_2 : (s_1, s_2, z) \mapsto (s_1, s_2, z(s_2)^{\delta_2 b_2}).$$

Following Kodaira we can construct the quotient manifold  $F_v = E_v \times \mathbb{C}^* / L$  such that

$$F_v = (E'_v \times \mathbb{C}^* / L) \cup \{0 \times E_{(2)} \times \mathbb{C}^*\} \cup \{E_{(1)} \times 0 \times \mathbb{C}^*\},$$

where

$$E_{(i)} = \{s_i \mid |s_i| < (\varepsilon_i)^{1/\delta_i}\}.$$

(See Kodaira [19] II p 597 ~ 598.)

The meromorphic functions

$$x(s_1, s_2, z) = -2 \sum_{n=1}^{\infty} \frac{n((s_1)^{\delta_1 b_1} (s_2)^{\delta_2 b_2})^n}{1 - ((s_1)^{\delta_1 b_1} (s_2)^{\delta_2 b_2})^n} + \sum_{n=-\infty}^{+\infty} \frac{z((s_1)^{\delta_1 b_1} (s_2)^{\delta_2 b_2})^n}{\{1 - z((s_1)^{\delta_1 b_1} (s_2)^{\delta_2 b_2})^n\}^2}$$

$$y(s_1, s_2, z) = \sum_{n=-\infty}^{+\infty} \frac{(1 + z((s_1)^{\delta_1 b_1} (s_2)^{\delta_2 b_2})^n) z((s_1)^{\delta_1 b_1} (s_2)^{\delta_2 b_2})^n}{\{1 - z((s_1)^{\delta_1 b_1} (s_2)^{\delta_2 b_2})^n\}^3}$$

satisfy the equation

$$(4.11) \quad y^2 - 4x^3 - x^2 + g_2((s_1)^{\delta_1}, (s_2)^{\delta_2})x + g_3((s_1)^{\delta_1}, (s_2)^{\delta_2}) = 0,$$

where

$$g_2(u_1, u_2) = 20 \sum_{n=1}^{\infty} \frac{n^3((u_1)^{b_1}(u_2)^{b_2})^n}{1 - ((u_1)^{b_1}(u_2)^{b_2})^n},$$

$$g_3(u_1, u_2) = \frac{1}{3} \sum_{n=1}^{\infty} \frac{(7n^5 + 5n^3)((u_1)^{b_1}(u_2)^{b_2})^n}{1 - ((u_1)^{b_1}(u_2)^{b_2})^n}.$$

Let  $\tilde{B}_v$  be a subvariety in  $E_v \times \mathbb{P}^2$  defined by the non-homogenous equation (4.11). Then the group  $G = \mathcal{G}/\mathcal{L}$  can be considered as a group of analytic automorphisms of  $\tilde{B}_v$ , generated by two automorphisms

$$\tau_1 : (s_1, s_2, x, y) \mapsto ((-1)^{\delta_1-1} \cdot s_1, s_2, x, (-1)^{\delta_1-1} \cdot y),$$

$$\tau_2 : (s_1, s_2, x, y) \mapsto (s_1, (-1)^{\delta_2-1} \cdot s_2, x, (-1)^{\delta_2-1} \cdot y).$$

We set  $B_v = \tilde{B}_v/G$ . Then the morphism

$$\mu_v : \begin{array}{ccc} B_v & \longrightarrow & D_v \\ \cup & & \cup \\ ((s_1, s_2, x, y)) & \mapsto & ((s_1)^{\delta_1}, (s_2)^{\delta_2}) \end{array}$$

gives a structure of a fibre space which is an extension of  $\mu'_v : B'_v \rightarrow D'_v$ . The morphism

$$o_v : \begin{array}{ccc} D_v & \longrightarrow & F_v/G \\ \cup & & \cup \\ (t_1, t_2) & \mapsto & (((t_1)^{1/\delta_1}, (t_2)^{1/\delta_2}, 1)) \end{array}$$

gives a holomorphic section which is the extension of  $o'_v$ .

(4.12): Finally we patch together  $\mu' : B' \rightarrow W'$  and  $\mu_v : B_v \rightarrow D_v$ ,  $v = 1, 2, \dots, n$  and we obtain an elliptic threefold  $\mu : B \rightarrow W$ , which is an extension of  $\mu' : B' \rightarrow W'$ . As all sections  $o_v : D_v \rightarrow B_v$ ,  $v = 1, 2, \dots, n$  and  $o' : B' \rightarrow W'$  are compatible, we have a holomorphic section  $o : W \rightarrow B$ .

We call the elliptic fibre space  $\mu : B \rightarrow W$  thus obtained the *basic elliptic threefold* associated to a characteristic pair  $(T, \Phi)$ .

The following theorem is due to S. Kawai.

**THEOREM 4.13:** *The morphism  $\mu : B \rightarrow W$  of the elliptic threefold is a projective morphism. Hence if  $W$  is algebraic (projective),  $B$  is algebraic (projective).*

**PROOF:** See Kawai [16] p. 129–134.

Kawai shows that a principal ideal sheaf defined by the section  $o : W \rightarrow B$  is ample with respect to the morphism  $\mu$ .

### 5. Non-singular models of basic elliptic threefolds

We use the same notations as those in the previous section. In this section first we shall show how to resolve the singularities of a basic elliptic threefold  $\mu : B \rightarrow W$ . Using this desingularization we shall construct the commutative group manifold  $B_0^*$  over  $W_0 = W - \{p_{h+1}, p_{h+1}, \dots, p_n\}$ , where  $p_{h+1}, p_{h+2}, \dots, p_n$  are all points on  $W$  at which the singular locus  $S$  has ordinary double points. (That is, the origins of  $D_v$ ,  $v = h+1, h+2, \dots, n$ .) Then we shall define a cohomology group  $H_{\text{adm}}^1(W_0, \mathcal{O}(B_0^*))$  and for any element  $\eta \in H_{\text{adm}}^1(W_0, \mathcal{O}(B_0^*))$ , we shall construct a new elliptic threefold  $B^\eta$ .

(5.1) *Case a:* We set

$$g(s_1, t_2) = \sum_{k=0}^{m-1} \alpha^{-k} \cdot g_k(s_1, t_2),$$

$$w = g(s_1, t_2) \cdot \zeta,$$

where

$$\alpha = g_1(0, 0), \quad g_k(s_1, t_2) = f_k \left( \frac{m}{2\pi i} \log s_1, t_2 \right).$$

(See (4.4), (4.5).) In the coordinates  $(s_1, t_2, w)$ , the automorphism  $g$  of  $F_v$  ((4.5)) is written in the form

$$g : [s_1, t_2, w] \mapsto [e_m s_1, t_2, \alpha w].$$

Moreover  $\alpha$  is  $e_m$  or  $e_m^{-1}$ . (See (6.7) below.) Hence we can use the canonical resolutions due to Hirzebruch ([12], [29] Section 6 (C)) and obtain a non-singular model  $\hat{\mu}_v : \hat{B}_v \rightarrow D$  of  $\mu_v : B_v \rightarrow D_v$ . Note that when the matrix  $M$  is  $\text{SL}(2, \mathbf{Z})$ -conjugate to

$$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

exceptional varieties of the first kind appear. By Nakano [27] and Fujiki-Nakano [7], we can blow down these exceptional varieties. But because of the resolutions in the Case c (see (5.4) below) *we do not blow down these exceptional varieties*. The holomorphic section  $o_v : D_v \rightarrow B_v$  induces the holomorphic section  $\hat{o}_v : D_v \rightarrow \hat{B}_v$ . For any  $a \in \mathbf{C}$ ,  $|a| < \varepsilon_2$ , we set

$$(5.2) \quad \begin{cases} D_v(a) = \{(t_1, a) \mid |t_1| < \varepsilon_1\}, \\ \hat{B}_v(a) = \hat{\mu}_v^{-1}(D_v(a)). \end{cases}$$

Then  $\hat{\mu}_v(a) : \hat{B}_v(a) \rightarrow D_v(a)$  is a fibration of elliptic curves and has only one singular fibre over the point  $(0, a)$ , if  $M \neq I_2$ . The configuration of the singular fibres are independent of  $a$  but depends only on  $M$ . We call

the fibre of  $\hat{\mu}_v : \hat{B}_v \rightarrow D_v$  over  $S \cap D_v$ , regular or of type *Kod* (\*), if  $M$  is  $I_2$  or  $SL(2, \mathbf{Z})$ -conjugate to one of matrices in Table (6.7), which corresponds to a singular fibre of type (\*) of elliptic surfaces. (See (6.7).)

(5.3) *Case b*: The case

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

The singular locus of  $B_v$  lies over  $S \cap D_v$  and is isomorphic to  $S \cap D_v$ . By  $(b-1)$ -times successive blowing-ups along singular loci (which lie always over  $S \cap D_v$  and are isomorphic to  $S \cap D_v$ ), we obtain a non-singular model  $\hat{\mu}_v : \hat{B}_v \rightarrow D_v$ . Then  $\hat{\mu}_v(a) : \hat{B}_v(a) \rightarrow D_v(a)$  (see (5.2)) is a fibration of elliptic curves and has only one singular fibre of type  $(I_b)$ . (See Kodaira [19] II p. 597–600). We call the singular fibre over  $S \cap D_v$ , of type *Kod*  $(I_b)$ .

The case

$$M = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}.$$

First we resolve the singularities of  $\tilde{B}_v$  in the same way as above. Then the automorphism  $h$  of  $\tilde{B}_v$ , ((4.6)) can be lifted to this non-singular model  $\tilde{B}_v^*$  of  $\tilde{B}_v$ . We take the quotient  $\tilde{B}_v^*/H$  and resolve the singularities of this space in the same way as in (5.1). Thus we obtain a non-singular model  $\hat{\mu}_v : \hat{B}_v \rightarrow D_v$  of  $\mu_v : B_v \rightarrow D_v$ . Then the singular fibre of  $\hat{\mu}_v(a) : \hat{B}_v(a) \rightarrow D_v(a)$  ((5.2)) is of type  $I_b^*$ . (See Kodaira [19] II p. 600–602.) We call the singular fibre of  $\hat{\mu}_v : \hat{B}_v \rightarrow D_v$  over  $S \cap D_v$ , of type *Kod*  $(I_b^*)$ . In both cases the holomorphic section  $\sigma_v : D_v \rightarrow B_v$  induces the holomorphic section  $\hat{\sigma}_v : D_v \rightarrow \hat{B}_v$ .

(5.4) *Case c*: We set  $G_1 = \{g_{k_1, 0}\}$ ,  $G_2 = \{g_{0, k_2}\}$ . Then  $G = G_1 \times G_2$ . First we resolve the singularity of the quotient space  $\tilde{B}_v/G_1$  in the same way as in (5.1). Then the group  $G_2$  operates on this non-singular model  $B_v^{(1)}$ . We resolve the singularities of the quotient space  $B_v^{(1)}/G_2$  in the same way as in (5.1) and obtain a non-singular model  $\hat{\mu}_v : \hat{B}_v \rightarrow D_v$  of  $\mu_v : B_v \rightarrow D_v$ . The holomorphic section  $\sigma_v : D_v \rightarrow B_v$  can be lifted to the holomorphic section  $\hat{\sigma}_v : D_v \rightarrow \hat{B}_v$ .

(5.5) For readers' convenience we shall give the resolution in the simplest case,  $M_1 = M_2 = -I_2$ .

In this case  $g_{k_1 k_2}$  has a form

$$g_{k_1 k_2} : [s_1, s_2, \zeta] \mapsto [(-1)^{k_1} s_1, (-1)^{k_2} s_2, (-1)^{k_1+k_2} \cdot \zeta].$$

We set  $g_1 = g_{1, 0}$ ,  $g_2 = g_{0, 1}$ . Then  $g_1$  has four fixed subvarieties

$$D_{v_1, v_2} : s = 0, [\zeta] = \frac{v_1}{2} + \frac{v_2}{2} S(0, s_2),$$

$$v_1 = 0, 1, v_2 = 0, 1.$$

We set

$$w_{v_1, v_2} = \zeta - \frac{v_1}{2} - \frac{v_2}{2} S(0, s_2).$$

Then  $(s_1, s_2, w_{v_1, v_2})$  is a system of local coordinates of a neighborhood of  $D_{v_1, v_2}$  in  $\tilde{B}_v$ . In these coordinates  $g_1$  is represented in the form

$$(s_1, s_2, w_{v_1, v_2}) \mapsto (-s_1, s_2, -w_{v_1, v_2}).$$

Hence the quotient space  $\tilde{B}_v/G_1$  has four singular loci  $\mathcal{D}_{v_1, v_2}$  corresponding to  $D_{v_1, v_2}$ . Let  $U_{v_1, v_2}$  and  $V_{v_1, v_2}$  be open set in  $C^3$  defined by the inequalities:

$$U_{v_1, v_2} : |u_1| < \varepsilon_1, |s_2| < (\varepsilon_2)^{\frac{1}{2}}, |u_1(u_2)^2| < \varepsilon,$$

$$V_{v_1, v_2} : |(v_1)^2 v_2| < \varepsilon_1, |s_2| < (\varepsilon_2)^{\frac{1}{2}}, |v_2| < \varepsilon.$$

(We should write  $U_1^{(v_1, v_2)}, U_2^{(v_1, v_2)}$  etc. instead of  $U_1, U_2$ , etc. but we hope there is no confusion.) We patch together  $U_{v_1, v_2}$  and  $V_{v_1, v_2}$  by the relation

$$\begin{cases} v_1 = \frac{1}{u_2}, \\ v_2 = u_1(u_2)^2, \end{cases}$$

and we obtain a complex manifold  $M_{v_1, v_2}$ . The canonical resolution of the singularity  $\mathcal{D}_{v_1, v_2}$  is given by meromorphic maps

$$\begin{aligned} (s_1, s_2, w_{v_1, v_2}) &\mapsto \left( (s_1)^2, s_2, \frac{w_{v_1, v_2}}{s_1} \right) = (u_1, s_2, u_2) \in U_{v_1, v_2}, \\ &\searrow \\ &\left( \frac{s_1}{w_{v_1, v_2}}, s_2, (w_{v_1, v_2})^2 \right) = (v_1, s_2, v_2) \in V_{v_1, v_2}. \end{aligned}$$

(See also Appendix (8.16) below.) We patch together  $B_v^{(1)'} = B_v/G_1 - \cup \mathcal{D}_{v_1, v_2}$  and all  $M_{v_1, v_2}$ 's and obtain a complex manifold  $B_v^{(1)}$ . Then  $B_v^{(1)}$  is a fibre space of elliptic curves over  $E = \{(t_1, s_2) \mid |t_1| < \varepsilon_1, |s_2| < (\varepsilon_2)^{\frac{1}{2}}\}$  by the morphism

$$\begin{aligned} \mu_v^{(1)} : (u_1, s_2, u_2) &\mapsto (u_1, s_2), \\ (v_1, s_2, v_2) &\mapsto ((v_1)^2 v_2, s_2) \\ [[s_1, s_2, \zeta]] &\mapsto ((s_1)^2, s_2), \end{aligned}$$

where  $[[s_1, s_2, \zeta]]$  is the point on  $B_v^{(1)'}$  which is the image of the point  $[s_1, s_2, \zeta]$  on  $\tilde{B}_v$ . The fibre space  $\mu_v^{(1)} : B_v^{(1)} \rightarrow E$  has the singular fibre over any point  $(0, s_2)$ . The inverse image  $\mu_v^{(1)-1}\{(0, s_2) \mid |s_2| < (\varepsilon_2)^{\frac{1}{2}}\}$

consists of five surfaces  $R_{v_1, v_2}$ ,  $v_1 = 0, 1, v_2 = 0, 1$ , and  $R$ , defined by the equations;

$$\begin{aligned} R_{v_1, v_2} &: u_1 = 0, \text{ in } U_{v_1, v_2}, \\ &v_2 = 0, \text{ in } V_{v_1, v_2}, \\ R &: v_1 = 0, \text{ in } V_{v_1, v_2}, \\ &s_1 = 0, \text{ in } B_v^{(1)'} . \end{aligned}$$

$R_{v_1, v_2}$  is isomorphic to  $P^1 \times \{(0, s_2) \mid |s_2| < (\varepsilon_2)^{\frac{1}{2}}\}$  and  $R$  is a non-singular model of  $\bar{\mu}_v^{-1}(\{(0, s_2) \mid |s_2| < (\varepsilon_2)^{\frac{1}{2}}\})$ , which is also isomorphic to  $P^1 \times \{(0, s_2) \mid |s_2| < (\varepsilon_2)^{\frac{1}{2}}\}$ . Now the group  $G_2$  operates on  $B_v^{(1)}$  in the following way.

$$\begin{aligned} g_2 : U_{v_1, v_2} &\longrightarrow U_{v_1, v_2} \\ &(u_1, s_2, u_2) \mapsto (u_1, -s_2, -u_2), \\ g_2 : V_{v_1, v_2} &\longrightarrow V_{v_1, v_2} \\ &(v_1, s_2, v_2) \mapsto (-v_1, -s_2, v_2), \\ g_2 : B_v^{(1)'} &\longrightarrow B_v^{(1)'} \\ &(s_1, s_2, \zeta) \mapsto (s_1, -s_2, -\zeta). \end{aligned}$$

Hence  $g_2$  has five fixed manifolds  $F_{v_1, v_2}$ ,  $v_1 = 0, 1, v_2 = 0, 1$  and  $F$ , defined by the equations

$$\begin{aligned} F_{v_1, v_2} &: s_2 = 0, u_2 = 0, \text{ in } U_{v_1, v_2}, \\ F &: v_1 = 0, s_2 = 0, \text{ in } V_{v_1, v_2}, \\ &s_1 = 0, s_2 = 0, \text{ in } B_v^{(1)'} . \end{aligned}$$

The quotient space  $B_v^{(1)}/G_2$  has five singular loci  $\mathcal{F}_{v_1, v_2}$  and  $\mathcal{F}$  corresponding to  $F_{v_1, v_2}$  and  $F_v$ .

Let  $W_{v_1, v_2}, X_{v_1, v_2}$  be open sets in  $C^3$  defined by the inequalities:

$$\begin{aligned} W_{v_1, v_2} &: |u_1| < \varepsilon_1, |w_1| < \varepsilon_2, |u_1 w_1 (w_2)^2| < \varepsilon, \\ X_{v_1, v_2} &: |u_1| < \varepsilon_1, |(x_1)^2 x_2| < \varepsilon_2, |u_1 x_2| < \varepsilon. \end{aligned}$$

Patching together  $W_{v_1, v_2}$  and  $X_{v_1, v_2}$  by the relations

$$\begin{cases} x_1 = \frac{1}{w_2}, \\ x_2 = w_1 (w_2)^2, \end{cases}$$

we obtain a manifold  $M_{v_1, v_2}$ . Then the resolution of the singularity  $\mathcal{F}_{v_1, v_2}$  is given by the meromorphic maps

$$(u_1, s_2, u_2) \mapsto \left(u_1, (s_2)^2, \frac{u_2}{s_2}\right) = (u_1, w_1, w_2) \in W_{v_1, v_2},$$

$$(u_1, s_2, u_2) \mapsto \left(u_1, \frac{s_2}{u_2}, (u_2)^2\right) = (u_1, x_1, x_2) \in X_{v_1, v_2}.$$

Next we shall resolve the singularity  $\mathcal{F}$ . First we remark that on  $B_v^{(1)'}$  the automorphisms

$$(s_1, s_2, \zeta) \mapsto (s_1, -s_2, -\zeta)$$

and

$$(s_1, s_2, \zeta) \mapsto (-s_1, -s_2, \zeta)$$

of  $B_v^{(1)'}$  are the same. Hence we use the second form. Then the resolution of the singularity  $\mathcal{F}$  is given by the meromorphic maps

$$\begin{aligned} V_{v_1, v_2} \ni (v_1, s_2, v_2) &\mapsto \left((v_1)^2, \frac{s_2}{v_1}, v_2\right) = (y_1, y_2, v_2) \in Y_{v_1, v_2}, \\ &\searrow \\ &\left(\frac{v_1}{s_2}, (s_2)^2, v_2\right) = (z_1, z_2, v_2) \in Z_{v_1, v_2}, \end{aligned}$$

$$\begin{aligned} B_v^{(1)'} \ni (s_1, s_2, \zeta) &\mapsto \left((s_1)^2, \frac{s_2}{s_1}, \zeta\right) = (a_1, a_2, \zeta) \in A \\ &\searrow \\ &\left(\frac{s_1}{s_2}, (s_2)^2, \zeta\right) = (b_1, b_2, \zeta) \in B, \end{aligned}$$

where  $Y_{v_1, v_2}, Z_{v_1, v_2}, A, B$  are open sets in  $\mathbb{C}^3$  defined by similar inequalities as above. Patching together  $Y_{v_1, v_2}, Z_{v_1, v_2}, v_1 = 0, 1, v_2 = 0, 1, A$  and  $B$ , we obtain a complex manifold  $M$ . Then  $\hat{B}_v = (B_v^{(1)'}/G_2 - \cup_{v_1, v_2} \mathcal{F}_{v_1, v_2} \cup \mathcal{F}) \cup_{v_1, v_2} M_{v_1, v_2} \cup M$  is the desired non-singular model of  $B_v$ . The structure of the fibre space of elliptic curves is given by the morphism

$$\begin{aligned} \hat{\mu}_v : (u_1, w_1, w_2) &\mapsto (u_1, w_1), \\ (u_1, x_1, x_2) &\mapsto (u_1, (x_1)^2 \cdot x_2), \\ (y_1, y_2, v_2) &\mapsto (y_1 v_2, y_1 \cdot (y_2)^2), \\ (z_1, z_2, v_2) &\mapsto ((z_1)^2 \cdot z_2 v_2, z_2), \\ (a_1, a_2, \zeta) &\mapsto (a_1, a_1 (a_2)^2), \\ (b_1, b_2, \zeta) &\mapsto (b_1 (b_2)^2, b_2), \\ (s_1, s_2, \zeta) &\mapsto ((s_1)^2, (s_2)^2). \end{aligned}$$

$\hat{\mu}_v : \hat{B}_v \rightarrow D_v$  is of maximal rank at any point on  $\hat{\mu}_v^{-1}(D_v - D_v \cap S)$ . The inverse image  $\hat{\mu}_v^{-1}(D_v \cap S)$  consists of nine surfaces  $\mathfrak{R}_{v_1, v_2}, \mathfrak{S}_{v_1, v_2}, v_1 = 1, 2, v_2 = 1, 2, \mathfrak{R}_1, \mathfrak{R}_2, \mathfrak{S}$  defined by the equations

$$\begin{aligned} \mathfrak{R}_{v_1, v_2} : u_1 = 0, \text{ in } W_{v_1, v_2}, \\ v_2 = 0, \text{ in } B'_v = B_v^{(1)}/G_2 - \bigcup_{v_1, v_2} F_{v_1, v_2} \cup F, Y_{v_1, v_2}, Z_{v_1, v_2}. \\ \mathfrak{R}_1 : s_1 = 0, \text{ in } B'_v, \\ z_1 = 0, \text{ in } Z_{v_1, v_2}. \\ \mathfrak{R}_2 : x_1 = 0, \text{ in } Y_{v_1, v_2}, \\ s_2 = 0, \text{ in } B'_v. \\ \mathfrak{S}_{v_1, v_2} : w_1 = 0, \text{ in } X_{v_1, v_2}, \\ x_2 = 0, \text{ in } Y_{v_1, v_2}. \\ \mathfrak{S} : y_1 = 0, \text{ in } Y_{v_1, v_2}, \\ z_2 = 0, \text{ in } Z_{v_1, v_2}, \\ a_1 = 0, \text{ in } A, \\ b_2 = 0, \text{ in } B. \end{aligned}$$

Note that  $\mathfrak{R}_{v_1, v_2}$  and  $\mathfrak{R}_1$  are  $P^1$  bundle over  $0 \times D_{(2)}$  and  $\mathfrak{S}_{v_1, v_2}$  and  $\mathfrak{R}_2$  are  $P^1$  bundle over  $D_{(1)} \times 0$ .  $\mathfrak{R}_1$  and  $\mathfrak{R}_2$  are non-singular models of  $\hat{\mu}_v^{-1}(0 \times D_{(2)})$  and  $\hat{\mu}_v^{-1}(D_{(1)} \times 0)$ , respectively. The surface  $\mathfrak{S}$  has a structure of  $P^1$  bundle over  $P^1$ . Moreover over  $0 \times D'_{(2)}$  and  $D'_{(1)} \times 0$ , the singular fibres are of type  $\text{Kod}(I_0^*)$ . Because of the existence of the surface  $S$  the morphism  $\hat{\mu}_v : \hat{B}_v \rightarrow D_v$  is not flat.

(5.6) *Case d:* First we shall resolve the singularities of  $\tilde{B}_v$  in the same way as in (5.2). Then the automorphisms  $\tau_1$  and  $\tau_2$  operate on this non-singular model. Hence we can use the arguments in (5.5). The holomorphic section  $o_v : D_v \rightarrow B_v$  can be lifted to the holomorphic section  $\hat{o}_v : D_v \rightarrow \hat{B}_v$ .

(5.7) *Case e:* The singular loci of  $\tilde{B}_v$  are  $\mathcal{S}_1 = \{x = 0, y = 0, s_1 = 0\}$  and  $\mathcal{S}_2 = \{x = 0, y = 0, s_2 = 0\}$ . By a finite succession of monoidal transformations along the singularity  $\mathcal{S}_1$  we obtain a threefold  $B_v^{(1)}$  which is smooth outside  $\mathcal{S}_2$ . Next applying a finite number of monoidal transformations along the singularity  $\mathcal{S}_2$  we obtain a threefold  $B_v^{(2)}$ . Then  $B_v^{(2)}$  is smooth except a point over the origin  $s_1 = 0, s_2 = 0$ . Then we resolve this singularity in such a way that the automorphisms  $\tau_1$  and  $\tau_2$  operate on the non-singular model  $B_v^{(3)}$  thus obtained. Finally we can use the arguments in (5.5) and obtain a non-singular model  $\hat{B}_v$  of  $B_v$ . The holomorphic map section  $o_v : D_v \rightarrow B_v$  can be lifted to the holomorphic section  $\hat{o}_v : D_v \rightarrow \hat{B}_v$ .

Note that the fibre of  $\hat{\mu}_v : \hat{B}_v \rightarrow D_v$  over the origin may contain surfaces. Hence the morphism  $\hat{\mu}_v$  may not be flat.

(5.8) From the above considerations we can patch together  $B'$  and  $\hat{B}_v$ ,  $v = 1, 2, \dots, n$  and obtain a non-singular elliptic threefold  $\hat{\mu} : \hat{B} \rightarrow W$ .

Note that the fibres of  $\hat{\mu} : \hat{B} \rightarrow W$  over the points  $p_i, i = h+1, h+2, \dots, n$  may contain surfaces.

(5.9) Following Kodaira we can introduce a group manifold  $B_0^\#$  over  $W_0 = W - \{p_{h+1}, p_{h+2}, \dots, p_n\}$ , where every fibre of  $B_0^\#$  is a commutative complex Lie group and over  $W', B_0^\#|_{W'}$  is equal to  $B'$ . (See Kodaira [19] II, p 603 ~ 609.) Moreover  $B_0^\#$  operates on  $B_0^\# = B|_{W_0}$ . (See Kodaira [19] II p 609 ~ 613.) Hence for any open set  $\mathfrak{U}$  of  $W_0$ , if we have a holomorphic section  $\varphi \in H^0(\mathfrak{U}, \mathcal{O}(B_0^\#))$ , then the isomorphism

$$L^\#(\varphi) : B_{0|\mathfrak{U}}^\# \rightarrow B_{0|\mathfrak{U}}^\#$$

defined by the multiplication by  $\varphi$  can be extended to the isomorphism

$$L(\varphi) : \hat{B}|_{\mathfrak{U}} \rightarrow \hat{B}|_{\mathfrak{U}}.$$

We call an open covering  $\{\mathfrak{U}_i\}_{i \in I}$  of  $W_0$  an *admissible covering*, if  $\{\mathfrak{U}_i\}$  is locally finite and for any pair  $(i, j), i \neq j$ , the closure  $\overline{\mathfrak{U}_i} \cap \mathfrak{U}_j$  in  $W$  does not contain any  $p_i, i = h+1, h+2, \dots, n$ .

Let  $H_{\text{adm}}^1(W_0, \mathcal{O}(B_0^\#))$  be a first Cech cohomology group defined by all admissible covering of  $W_0$ . Then for any element  $\{\eta_{ij}\} = \eta \in H_{\text{adm}}^1(W_0, \mathcal{O}(B_0^\#))$ , we can define a fibre space  $\mu_\eta^0 : B_0^\# \rightarrow W_0$  by identifying  $\hat{B}|_{\mathfrak{U}_i}$  and  $\hat{B}|_{\mathfrak{U}_j}$  by the isomorphism  $L(\eta_{ij})$ . (See Kodaira [19] II p 613.) Since we only consider the admissible coverings, the fibre space  $\mu_\eta^0 : B_0^\# \rightarrow W_0$  can be naturally extended to the elliptic threefold  $\mu_\eta : B^\# \rightarrow W$ . That is, there exists an open neighborhood of  $p_i, i = h+1, h+2, \dots, n$  in  $W$  such that the restriction

$$\mu|_{\mathfrak{U}} : B|_{\mathfrak{U}} \rightarrow \mathfrak{U} \text{ and } \mu|_{\mathfrak{U}} : \hat{B}|_{\mathfrak{U}} \rightarrow \mathfrak{U}$$

are isomorphic.

DEFINITION 5.10:  $B^\#$  is called the elliptic threefold associated to an element  $\eta \in H_{\text{adm}}^1(W_0, \mathcal{O}(B_0^\#))$ .

### 6. The canonical bundle formula for elliptic threefolds

In this section we shall prove the following.

THEOREM 6.1: Let  $\mu_\eta : B^\# \rightarrow W$  be the elliptic threefold associated to an element  $\eta \in H_{\text{adm}}^1(W_0, \mathcal{O}(B_0^\#))$ . Then  $12K(B^\#)$  has the form

$$\mu_\eta^*(12K_W + [F]) + [G] + [H],$$

where  $F$  is an effective divisor on  $W, G$  is an effective divisor on  $B^\#$  whose components are contained in the fibres  $\mu_\eta^{-1}(P_v), v = h+1, \dots, n$ , and  $H$  is an effective divisor on  $B^\#$  whose components are contained in singular fibres of type Kod (III) and Kod (IV).

Moreover  $F$  is written in the form

$$(6.2) \quad \sum_b bS_{I_b} + \sum_b (6+b)S_{I_b^*} + 2S_{II} + 10S_{II^*} + 3S_{III} + 9S_{III^*} + 4S_{IV} + 8S_{IV^*},$$

where  $S_{(*)} = \sum S_j$  such that the singular fibre over  $S_j$  is of type Kod  $(*)$

COROLLARY 6.3: Conjecture  $C_3$  is true for the elliptic threefold  $\mu_\eta : B^n \rightarrow W$ .

For the proof we use the cusp form  $\Delta(z)$  of weight six with respect to the group  $SL(2, \mathbf{Z})$ . That is  $\Delta(z)$  has the form

$$\begin{aligned} \Delta(z) &= (2\pi)^{12} \left( \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} \right) \\ &= (2\pi)^{12} (e^{2\pi i z} - 24e^{4\pi i z} + 252e^{6\pi i z} + \dots), \end{aligned}$$

where  $\tau(n)$ 's are so called Ramanujan's numbers. (Gunning [8]).  $\Delta(z)$  is also written in the form

$$\Delta(z) = (2\pi)^{12} e^{2\pi i z} \left\{ \sum_{n=1}^{\infty} (1 - e^{2\pi i n z}) \right\}^{24}.$$

$\Delta(z)$  has the following properties which we use later.

(6.4)  $\Delta(z)$  is holomorphic on the upper half plane  $H$  and never vanishes at any point on  $H$ .

$$(6.5) \quad \begin{aligned} \Delta\left(\frac{az+b}{cz+d}\right) &= (cz+d)^{12} \cdot \Delta(z), \\ \text{for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\in SL(2, \mathbf{Z}). \end{aligned}$$

*Proof of Theorem 6.1:* We use freely the notations in Section 4 and Section 5.

(A) First we consider the case that  $W$  has a meromorphic 2 form  $\Omega$  and  $B^n = \hat{B}$ . We consider a 12-tuple meromorphic 3 form

$$\Xi' = \Delta(\tilde{T}(\tilde{y}))((\mu^*(\Omega) \wedge d\zeta))^{12} \text{ on } W' \times C.$$

Since we have

$$\begin{aligned} g(\beta; n_1, n_2)^*(\Xi') &= \Delta(\tilde{T}(\beta, \tilde{y}))(\mu'^*\Omega \wedge d\zeta)^{12} f_\beta(\tilde{y})^{12} \\ &= \Delta(\Phi(\beta) \cdot \tilde{T}(\tilde{u})) f_\beta(\tilde{y})^{12} (\mu'^*\Omega \wedge d\zeta)^{12} \\ &= \Delta(\tilde{T}(\tilde{y}))(\mu'^*\Omega \wedge d\zeta)^{12} \\ &= \Xi', \end{aligned}$$

we can consider  $\Xi'$  as 12-tuple meromorphic 3 form on  $B'$ . Note that

the divisor  $(\mathcal{E}')$  on  $B'$  defined by  $\mathcal{E}'$  on  $B'$  has the form  $\mu'^*((\Omega))$ , where  $(\Omega)$  is a 12-tuple canonical divisor on  $W'$  defined by  $\Omega$ .

In the following we shall prove that  $\mathcal{E}'$  can be extended to 12-tuple meromorphic 3 form  $\mathcal{E}$  on  $B$ .

(6.6) *Case a:* We set

$$\omega = \begin{cases} (s_1)^{m-2} ds_1 \wedge dt_2 \wedge g(s_1, t_2) d\zeta, & \text{if } \alpha = e_m, \\ ds_1 \wedge dt_2 \wedge g(s_1, t_2) d\zeta, & \text{if } \alpha = e_m^{-1}. \end{cases}$$

Then we have

$$g^*(\omega) = \omega.$$

(See (4.5)(5.1).) Hence  $\omega$  induces a holomorphic 3 form on  $B'_v$ . Using the canonical resolution (5.1) we can show that  $\omega$  induces a holomorphic 3 form on  $\hat{B}_v$ . Moreover if the singular fibre is not of type Kod (III) or Kod (IV) then  $\omega$  does not vanish at any point of  $\hat{B}_v$ . If the singular fibre is of type Kod (III) or Kod (IV), then  $\omega = 0$  defines the divisor whose components are contained in the singular fibre.

These divisors appear because of the existence of an exceptional surface of the first kind. (See (5.2).) On  $B'_v$ ,  $\mathcal{E}'$  is written in the form

$$(6.7) \quad \mathcal{E}'|_{B'_v} = \begin{cases} A(S(s_1, t_2))A(t_1, t_2)^{12}g(s_1, t_2)^{-12}(t_1)^{12/m}(\omega)^{12}, & \text{if } \alpha = e_m, \\ A(S(t_1, t_2))A(t_1, t_2)^{12}g(s_1, t_2)^{-12}(t_1)^{12(m-1)/m}(\omega)^{12}, & \text{if } \alpha = e_m^{-1} \end{cases}$$

where

$$\Omega|_{D_v} = A(t_1, t_2) dt_1 \wedge dt_2.$$

Hence  $\mathcal{E}'|_{B'_v}$  can be extended to 12-tuple 3 form  $\mathcal{E}|_{\hat{B}_v}$  on  $\hat{B}_v$  by (6.7) since  $\omega$  is holomorphic 3 form on  $\hat{B}_v$ . The divisor which corresponds to  $\mathcal{E}|_{\hat{B}_v}$  has the form

$$(12(\Omega|_{D_v}) + a(S \cap D_v)) + H$$

where

$$a = \begin{cases} \frac{12}{m} & \text{if } \alpha = e_m \\ \frac{12(m-1)}{m} & \text{if } \alpha = e_m^{-1}, \end{cases}$$

and  $H$  is the divisor whose components are contained in the singular fibres of type Kod (III) and Kod (IV).

Table (6.8)

Fibre	Monodromy $M$	$\alpha$	$a$
regular	$I_2$	1	0
Kod ( $I_0^*$ )	$-I_2$	-1	6
Kod (II)	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$e_6$	2
Kod (II*)	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$e_6^{-1}$	10
Kod (III)	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$e_4$	3
Kod (III*)	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$e_4^{-1}$	9
Kod (IV)	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$e_3$	4
Kod (IV*)	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$e_3^{-1}$	8

(6.9) *Case b*: the case

$$M = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

By the explicit resolution of singularities it is easy to see that a holomorphic 3 form  $dt_1 \wedge dt_2 \wedge dz/z$  on  $F_v$  can be naturally extended to the holomorphic 3 form  $\omega$  on  $\hat{B}_v$ .  $\omega$  does not vanish at any point on  $\hat{B}_v$ . Hence  $\mathcal{E}'|_{\hat{B}_v}$  can be extended to a meromorphic form by

$$\mathcal{E}|_{\hat{B}_v} = \Delta(T(l_1, t_2))A(t_1, t_2)^{12}(\omega)^{12},$$

where

$$\Omega|_{D_v} = A(t_1, t_2)dt_1 \wedge dt_2.$$

As we have

$$\Delta(T(l_1, t_2)) = (2\pi)^{12}((t_1)^b - 24(t_1)^{2b} + \dots),$$

the divisor  $(\mathcal{E}|_{\hat{B}_v})$  on  $\hat{B}_v$  which corresponds to  $\mathcal{E}|_{\hat{B}_v}$  has the form

$$(\mathcal{E}|_{\hat{B}_v}) = \mu_v^*((\Omega) + b(S \cap D_v)).$$

The case

$$M = \begin{pmatrix} -1 & -b \\ 0 & -1 \end{pmatrix}.$$

A holomorphic three form  $ds_1 \wedge dt_2 \wedge dz/z$  on  $\tilde{F}_v$  can be extended to a nowhere vanishing holomorphic form  $\tilde{\omega}$  on  $\tilde{B}_v$ . Then  $s_1 \cdot \omega$  induces a nowhere vanishing holomorphic form  $\omega$  on  $\hat{B}_v$ . Then  $\mathcal{E}'|_{\hat{B}_v}$  can be extended to a meromorphic form by

$$\Xi|_{\hat{B}_v} = \Delta(T(l_1, t_2))A(t_1, t_2)^{12}(t_1)^6(\omega)^{12},$$

where

$$\Omega|_{D_v} = A(t_1, t_2)dt_1 \wedge dt_2.$$

Hence we have

$$(\Xi|_{\hat{B}_v}) = \mu_v^*((\Omega|_{D_v}) + (6+b)(S \cap D_v)).$$

(6.10) *Case c:* We set

$$h(s_1, s_2) = \sum_{k_1=1, k_2=1}^{k_1=m_1, k_2=m_2} \alpha_1^{-k_1} \alpha_2^{-k_2} h_{k_1 k_2}(s_1, s_2)$$

where  $\alpha_1 = h_{1,0}(0, 0)$ ,  $\alpha_2 = h_{0,1}(0, 0)$ .

(See (4.8).) We set

$$\omega = (s_1)^{\varepsilon_1} (s_2)^{\varepsilon_2} ds_1 \wedge ds_2 \wedge h(s_1, s_2) d\zeta,$$

where

$$\varepsilon_i = \begin{cases} m_i - 2, & \text{if } \alpha_i = e_{m_i}, \\ 0, & \text{if } \alpha_i = e_{m_i}^{-1}. \end{cases}$$

Then in the same way as in (6.6) we can easily show that  $\omega$  can be extended to a holomorphic form  $\omega$  on  $\hat{B}_v$ . Hence  $\Xi'|_{B'_v}$  can be extended to a meromorphic form  $\Xi|_{\hat{B}_v}$  on  $\hat{B}_v$  by

$$\Delta(T(s_1, s_2))a(t_1, t_2)^{12}h(s_1, s_2)^{-12}(t_1)^{a_1}(t_2)^{a_2}(\omega)^{12}$$

where

$$a_i = \frac{12(m_i - \varepsilon_i - 1)}{m_i}, \quad i = 1, 2.$$

Hence we have

$$(\Xi|_{\hat{B}_v}) = \mu_v^*((\Omega|_{D_v}) + a_1 S_1 + a_2 S_2) + G + H,$$

where  $G$  is an effective divisor on  $B_v$  whose components are contained in the fibre over the origin and  $H$  is an effective divisor whose components are contained in the singular fibres over  $S_1 = 0 \times D_{(2)}$ , and  $S_2 = D_{(1)} \times 0$ . Note that if both singular fibres over  $S_1$  and  $S_2$  are neither of type Kod (III) nor Kod (IV), then  $H = 0$ .

(6.11) *Case d:* A holomorphic form  $ds_1 \wedge ds_2 \wedge dz/z$  on  $\tilde{F}_v$  can be extended to a nowhere vanishing holomorphic form on  $\tilde{B}_v$ . Then a holomorphic form  $(s_1)^{\delta/2} s_2 \cdot \tilde{\omega}$  (for  $\delta$  see (4.9)) can be extended to a holomorphic form  $\omega$  on  $B_v$ . Hence  $\Xi'|_{B'_v}$  can be extended to a meromorphic form  $\Xi|_{\hat{B}_v}$  on  $B_v$  by

$$\Xi|_{\hat{B}_v} = \Delta(T(l_1, s_1))A(t_1, t_2)^{12}(t_1)^{3\delta}(t_2)^6(\omega)^{12},$$

where

$$\Omega|_{D_v} = A(t_1, t_2)dt_1 \wedge dt_2.$$

As we have

$$\Delta(T(l_1, s_2)) = (2\pi)^{12}((t_1)^b - 24(t_1)^{2b} + \dots),$$

we have

$$(\mathcal{E}|_{\hat{B}_v}) = ((\Omega|_{D_v}) + (3\delta + b)S_1 + 6S_2) + G,$$

where  $G$  is an effective divisor whose components are contained in the fibre over the origin and  $S_1 = 0 \times D_{(2)}$ ,  $S_2 = D_{(1)} \times 0$ .

(6.12) *Case e:* By explicit calculations we can show that a holomorphic form  $ds_1 \wedge ds_2 \wedge dz/z$  can be extended to a holomorphic form  $\tilde{\omega}$  on  $\tilde{B}_v$ . Then a holomorphic form  $(s_1)^{\delta_1/2}(s_2)^{\delta_2/2} \cdot \tilde{\omega}$  can be extended to a holomorphic form  $\omega$  on  $\tilde{B}_v$ . Hence  $\mathcal{E}'|_{\tilde{B}_v}$  can be extended to a meromorphic form  $\mathcal{E}|_{\hat{B}_v}$  on  $\hat{B}_v$  by

$$\Delta(T(l_1, l_2))A(t_1, t_2)^{12}(t_1)^{3\delta_1}(t_2)^{3\delta_2}(\omega)^{12},$$

where

$$\Omega|_{D_v} = A(t_1, t_2)dt_1 \wedge dt_2.$$

As we have

$$\Delta(T(l_1, l_2)) = (2\pi)^{12}((t_1)^{b_1}(t_2)^{b_2} - 24(t_1)^{2b_1}(t_2)^{2b_2} + \dots),$$

we have

$$(\mathcal{E}|_{B_v}) = \mu_v^*((\Omega|_{D_v}) + (3\delta_1 + b_1)S_1 + (3\delta_2 + b_2)S_2) + G,$$

where

$$S_1 = 0 \times D_{(2)}, S_2 = D_{(1)} \times 0$$

and  $G$  is an effective divisor on  $B_v$  whose components are contained in the fibre over the origin.

By the above arguments we can extend  $\mathcal{E}'$  to a 12-tuple meromorphic three form  $\mathcal{E}$  on  $\hat{B}$ . Hence  $12K_{\hat{B}}$  has the desired form.

(B)  $W$  has a meromorphic form  $\Omega$  and a general elliptic threefold  $B^n$ .

From our construction of  $\mathcal{E}$  it is easy to show that

$$L(\eta_{ij})^*(\mathcal{E}|_{U_i \cap U_j}) = \mathcal{E}|_{U_i \cap U_j},$$

where  $\eta = \{\eta_{ij}\}$ . (See (5.9)). Hence  $\mathcal{E}$  can be considered as a 12-tuple three form on  $B^n$ . Hence  $12K(B^n)$  has the desired form.

(C) When  $W$  has no meromorphic forms. The above local argument shows that in this case  $12K(B^n)$  has the desired form. g.e.d.

REMARK 6.13: If we will only show that the above  $\omega$  on  $B'_v$  can be holomorphically extended to  $\omega$  on  $B_v$ , there is a much easier proof which also assures that conjecture  $C_3$  is valid for the elliptic threefold  $B^n$ . (See [36].)

But for the detailed study of elliptic threefolds, the explicit formula of the pluri-canonical bundle (6.1) is indispensable.

(2) The same argument as above gives another proof of the canonical bundle formula for elliptic surfaces (Compare (6.2) and Kodaira [19] III p. 14–15, especially (12.6) and (12.7)).

**PROPOSITION 6.14:** *If an elliptic threefold  $f: V \rightarrow W$  has a rational section then  $f: V \rightarrow W$  is birationally equivalent to the basic elliptic threefold.*

**PROOF:** We can assume  $V$  and  $W$  are smooth. Let  $S$  be an algebraic subset of  $W$  such that the rational section is regular at any point  $W' = W - S$  and  $V_w = f^{-1}(w)$  is an elliptic curve for any point  $w \in W'$ . We can assume that  $S$  is a divisor and has normal crossings. We set  $V' = f^{-1}(W')$ ,  $f' = f|_{V'}$ . Then by the rational section, the fibre space  $f': V' \rightarrow W'$  has a structure of an abelian scheme over  $W'$ . On the other hand the restriction  $\mu': B' \rightarrow W'$  of the basic elliptic threefold  $\mu: B \rightarrow W$  on  $W'$  has a structure of an abelian scheme.

Moreover by the construction of the basic elliptic threefold we infer readily that two fibre spaces  $f': V' \rightarrow W'$  and  $\mu': B' \rightarrow W'$  have the isomorphic analytic family of polarized Hodge structures. Hence by Deligne [5], Rappel (4.4.3), two abelian schemes are isomorphic. Hence two elliptic threefolds are birationally equivalent. q.e.d.

**REMARK 6.15:** We can generalize the arguments in Kodaira [19] II p 617–624, so that we can get the another proof of the above proposition. It will be discussed in [36].

## 7. Generalized Kummer manifolds

In this section we shall study structures of generalized Kummer manifolds and shows that Conjecture  $K_n$  is true for Kummer manifolds.

**DEFINITION 7.1:** A smooth algebraic variety  $V$  is called a *generalized Kummer manifold*, if there exist an abelian variety  $A$  and a generically surjective rational map  $g: A \rightarrow V$  of  $A$  onto  $V$ . (We do not assume  $\dim A = \dim V$ .)

A generalized Kummer manifold  $V$  is called a *Kummer manifold* if  $V$  is a non-singular model of a quotient variety  $A/G$  of an abelian variety  $A$  by a finite group  $G$  of analytic automorphisms of  $A$ .

**THEOREM 7.2:** *The Kodaira dimension of a generalized Kummer manifold  $V$  is 0 or  $-\infty$ .*

**PROOF:** Let  $g: A \rightarrow V$  be a generically surjective rational map of an

abelian variety  $A$  onto  $V$ . We can assume that  $g$  is holomorphic at the origin  $o$  of  $A$ . Then there exist global coordinates  $z_1, z_2, \dots, z_n$  of  $A$  such that  $z_1, z_2, \dots, z_l$  give local coordinates of a small neighborhood  $\mathfrak{B}$  of  $g(o)$  in  $V$  with center  $g(o)$ .

Let  $\varphi$  be an element of  $H^0(V, \mathcal{O}(mK_V))$ , represented as  $m$ -tuple  $l$  form. That is, on  $\mathfrak{B}$ ,  $\varphi$  is written in the form

$$(7.3) \quad \varphi = \Psi(z_1, z_2, \dots, z_l)(dz_1 \wedge \dots \wedge dz_l)^m,$$

where  $\Psi$  is holomorphic on  $\mathfrak{B}$ . Let  $\mathfrak{U}$  be a small neighborhood of the origin  $o$  in  $A$  such that  $g(\mathfrak{U}) \subset \mathfrak{B}$ . Then the pull back  $g^*(\varphi)$  is written in the same form (7.3) on  $\mathfrak{U}$ . By Proposition 1.2, the pull back  $g^*(\varphi)$  is an element of  $H^0(A, S^m(\Omega_A^1))$ . Since the sheaf  $\Omega_A^1$  is free,  $g^*(\varphi)$  is written, on  $A$  in the form

$$(7.4) \quad g^*(\varphi) = C \cdot (dz_1 \wedge dz_2 \wedge \dots \wedge dz_l)^m,$$

where  $C$  is a constant. Hence the holomorphic function  $\Psi(z_1, z_2, \dots, z_l)$  is constant. This implies that any two elements of  $H^0(V, \mathcal{O}(mK_V))$  are linearly dependent. q.e.d.

REMARK 7.5: Theorem 7.2 is also true for any compact complex manifold  $M$ , which is an image of a generically surjective meromorphic map  $g : T \rightarrow M$  of a complex torus  $T$  onto  $M$ . The proof is the same as above.

COROLLARY 7.6: *For any generalized Kummer manifold  $V$ , the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  is surjective. Hence a fortiori*

$$q(V) \leq \dim V.$$

Hence Conjecture  $I_n$  is true for generalized Kummer manifolds.

PROOF: A non-singular model of the image  $\alpha(V)$  of the Albanese map  $\alpha$  is also a generalized Kummer manifold. Hence we have  $\kappa(\alpha(V)) \leq 0$ . The corollary follows from Corollary 3.6. q.e.d.

LEMMA 7.7: *For any generalized Kummer manifold  $V$ , any fibre of the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  is connected. Moreover general fibres are generalized Kummer manifolds.*

PROOF: There exists a generically surjective rational map  $g : A \rightarrow V$  of an abelian variety  $A$  onto  $V$ . We can assume  $h = \alpha \circ g : A \rightarrow \text{Alb}(V)$  is a group homomorphism and  $g$  is holomorphic at the origin  $o$  of  $A$ . Let  $A_1$  be the connected component of  $h^{-1}(o)$  which contains the origin  $o$ .  $A_1$  is an abelian subvariety of  $A$ . There exists a finite unramified covering  $f : \tilde{A} \rightarrow A$  such that  $\tilde{A} = A_1 \times A_2$ , where  $A_2$  is an abelian variety. Hence we can assume  $\tilde{A} = A$ . As  $g$  is holomorphic at the origin, there

exists a neighborhood  $\mathfrak{U}$  of the origin  $o_2$  of  $A_2$  in  $A_2$  such that for any point  $x \in \mathfrak{U}$ , the composition  $g_x : A_1 \times x \hookrightarrow A \xrightarrow{g} V$  is a well defined rational map. As  $g_x(A_1 \times x)$  is a point of  $\text{Alb}(V)$ , general fibres of the Albanese map are generalized Kummer manifold, if fibres are connected.

Let

$$\begin{array}{ccc} V & \xrightarrow{\alpha} & \text{Alb}(V) \\ \beta \searrow & & \nearrow \gamma \\ & W & \end{array}$$

be the Stein factorization of the Albanese map. It is enough to prove that  $\gamma : W \rightarrow \text{Alb}(V)$  is a birational morphism. As we have

$$\begin{aligned} q(V) &\geq q(W) \geq q(\text{Alb}(V)) = q(V), \\ \dim W &= \dim V = q(V), \end{aligned}$$

we have

$$\dim W = q(W).$$

Let  $\tau : W^* \rightarrow W$  be a desingularization of  $W$ . First we shall prove that  $\gamma^* = \gamma \circ \tau : W^* \rightarrow \text{Alb}(V)$  is isomorphic to the Albanese map of  $W^*$ . There exist a birational morphism  $\pi : V^* \rightarrow V$  such that  $\beta^* = \beta \circ \pi : V^* \rightarrow W^*$  is the morphism. By Lemma 2.6,  $\text{Alb}(V)$  and  $\text{Alb}(V^*)$  are isomorphic. By the universal property of the Albanese map, there exist a group homomorphisms  $h_1 : \text{Alb}(V) \rightarrow \text{Alb}(W)$  and  $h_2 : \text{Alb}(W^*) \rightarrow \text{Alb}(V)$  such that the diagram (7.8) is commutative. This implies that  $h_1 \circ h_2$  is an isomorphism. As  $\dim \text{Alb}(V) = \dim(\text{Alb}(W^*))$ ,  $h_1$  and  $h_2$  are isomorphisms. Hence the morphism  $\gamma^* : W^* \rightarrow \text{Alb}(V)$  is isomorphic to the Albanese map of  $W^*$ . Now, because of the following Proposition, the lemma is proved.

$$\begin{array}{ccc} V^* & \xrightarrow{\alpha} & \text{Alb}(V) \\ \beta^* \downarrow & & \downarrow h_1 \\ W^* & \xrightarrow{\alpha} & \text{Alb}(W^*) \\ \gamma^* \downarrow & \nearrow h_2 & \\ & & \text{Alb}(V) \end{array}$$

**PROPOSITION 7.9:** *Let  $V$  be a generalized Kummer manifold. If  $q(V) = \dim V$ , then the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  is a birational morphism.*

**PROOF:** There is a generically surjective rational map  $g : A \rightarrow V$  of an abelian variety  $A$  onto  $V$  such that  $g$  is holomorphic at the origin  $o$  of  $A$ . Let  $z_1, z_2, \dots, z_l$  be global coordinates of  $\text{Alb}(V)$ . As the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  is surjective,  $\omega = \alpha^*(dz_1 \wedge dz_2 \wedge \dots \wedge dz_l)$  is a

non zero holomorphic  $l$  form on  $V$  and  $g^*(\omega)$  is non zero holomorphic  $l$  form on  $A$ . We set  $h = \alpha \circ g$ . Then  $h$  is a morphism and we can assume that  $h$  is a group homomorphism. The connected component  $A_1$  of  $h^{-1}(o)$ , which contain the origin of  $A$  is an abelian subvariety of  $A$ . If necessary we take a finite unramified covering of  $A$  and we can assume  $A = A_1 \times A_2$ , where  $A_2$  is an abelian subvariety of  $A$ . Then the restriction  $g'$  of  $g$  on the abelian subvariety  $A_2$  is well-defined rational map of  $A_2$  to  $V$ . As we have

$$h(A_2) = g'(A_2),$$

$$\dim A_2 = \dim V = \dim \text{Alb}(V),$$

$g'$  is generically surjective. Hence we can assume that  $A = A_2$ . Let  $g^* : A^* \rightarrow V$  be an elimination of the points of indeterminacy of the rational map  $g$ , where  $A^*$  is obtained by succession of monoidal transformations with non-singular centers. As  $g^*(\omega)$  is a nowhere vanishing form,  $g^{**}(\omega)$  vanishes only on the exceptional divisor  $D$  of  $A^*$  appeared by monoidal transformations. Hence the restriction  $\alpha' : V - g^*(D) \rightarrow \text{Alb}(V) - \alpha \circ g^*(D)$  of  $\alpha$  is a finite unramified covering. On the other hand, since every irreducible component of  $D$  is a ruled variety,  $\alpha \circ g^*(D)$  consists of algebraic subsets whose complex codimensions in  $\text{Alb}(V)$  are at least two. (See Remark 3.2.) Hence the fundamental group  $\pi_1(\text{Alb}(V))$  is isomorphic to the fundamental group  $\pi_1(\text{Alb}(V) - \alpha \circ g^*(D))$ . This implies that the unramified covering  $\alpha' : V - g^*(D) \rightarrow \text{Alb}(V) - \alpha \circ g^*(D)$  can be extended to the finite unramified covering  $\bar{\alpha} : \bar{V} \rightarrow \text{Alb}(V)$ . As  $\bar{V}$  is birationally equivalent to  $V$  and  $\bar{V}$  is an abelian variety,  $\alpha : V \rightarrow \text{Alb}(V)$  is a birational morphism. q.e.d.

**THEOREM 7.10:** *Let  $V$  be a generalized Kummer manifold of dimension  $l$ . If  $\kappa(V) = 0$ , then there exist an  $l$ -dimensional abelian variety  $A$  and a generically surjective rational map  $g : A \rightarrow V$ .*

**PROOF:** There exists a generically surjective rational map  $f : A \rightarrow V$  of an  $n$ -dimensional abelian variety  $A$  onto  $V$ . Let  $S$  be an algebraic subset of  $A$  such that  $f$  is not holomorphic at any point of  $S$  and holomorphic at any point of  $A - S$ . The codimension of  $S$  in  $A$  is at least two. We can assume that the origin  $o \in A - S$ . We set  $p = f(o)$ . Let  $f^* : A^* \rightarrow V$  be an elimination of points of indeterminacy of the rational map  $f$ , where  $\tau : A^* \rightarrow A$  is the inverse of successive monoidal transformations with non-singular centers and  $f^* = f \circ \tau$ . Let  $S^*$  be the exceptional divisor in  $A^*$  appeared by the monoidal transformations. We have  $\tau(S^*) = S$ . We set  $A' = A - S$  and  $V' = V - f^*(S^*)$ .

Let  $\varphi \in H^0(V, \mathcal{O}(mK_V))$  be a non-zero  $m$ -tuple  $l$  form on  $V$ . We can find global coordinates  $z_1, z_2, \dots, z_n$  of  $A$  such that the pull back  $f^*(\varphi)$

is represented in the form

$$f^*(\varphi) = C \cdot (dz_1 \wedge dz_2 \wedge \cdots \wedge dz_l)^m,$$

where  $C$  is a non-zero constant. (See (7.4).) Hence we can find a locally finite open covering  $\{\mathfrak{B}_j\}_{j \in J}$  of  $A'$  such that

$$\begin{aligned} w_i^1 &= z_1 - a_i^1, w_i^2 = z_2 - a_i^2, \cdots, w_i^n = z_n - a_i^n, \\ (a_i^1, a_i^2, \cdots, a_i^n) &\in \mathbf{C}^n, \end{aligned}$$

are local coordinates of  $\mathfrak{B}_i$  and the morphism  $f|_{\mathfrak{B}_i}$  is written in these coordinates as the projection

$$(w_i^1, w_i^2, \cdots, w_i^n) \mapsto (w_i^1, w_i^2, \cdots, w_i^l).$$

Now we consider the linear subspace  $L$  in  $\mathbf{C}^n$  defined by the equations

$$z_1 = 0, z_2 = 0, \cdots, z_l = 0.$$

Let  $\pi : \mathbf{C}^n \rightarrow A$  be the morphism of the universal covering of  $A$ . Then  $L' = L - \pi^{-1}(S)$  is connected, since  $\pi^{-1}(S)$  is an analytic subset and  $L \not\subset \pi^{-1}(A)$ . Let  $\{\mathfrak{U}_i\}_{i \in I}$  be an open covering of  $L$  consisting of countably many open sets  $\mathfrak{U}_i$  such that  $\pi(\mathfrak{U}_i) \subset \mathfrak{B}_j$  for some  $j \in J$ . Then  $\pi(U_i)$  is contained in an analytic subset in  $V_j$  defined by the equations

$$\begin{aligned} w_j^1 &= b_j^1, w_j^2 = b_j^2, \cdots, w_j^l = b_j^l, \\ (b_j^1, b_j^2, \cdots, b_j^l) &\in \mathbf{C}^n. \end{aligned}$$

Hence  $f \circ \pi(U_i)$  is a point. Hence  $f \circ \pi(L')$  consists of at most countably many points. As  $f \circ \pi(L')$  is connected,  $f \circ \pi(L')$  is the point  $p = f(o)$ . Let  $A_1^\#$  be the irreducible component of  $f^{\#-1}(o)$ , which contains  $(f|_{A'})^{-1}(p)$ . We set  $A_1 = \tau(A_1^\#)$ . Then  $\dim A_1^\# = \dim A_1 = n - l$ , since on  $A'$ ,  $A_1$  is equal to  $(f|_{A'})^{-1}(o)$ . We shall show that  $A_1$  is an abelian subvariety of  $A$ . As  $A_1 \supset \pi(L')$  and  $L'$  is dense in  $L$ , we have  $A_1 \supset \pi(L)$ . On the other hand  $\pi(L)$  is a complex Lie subgroup of  $A$  of complex dimension  $n - l$ . As  $A_1 \supset \pi(L)$  and  $\dim A_1 = n - l$ ,  $\pi(L)$  is a closed Lie subgroup. Hence  $\pi(L)$  is an abelian subvariety of  $\dim n - l$  and  $A_1 = \pi(L)$ .

If necessary, we take a finite unramified covering of  $A$  and we can assume that  $A = A_1 \times A_2$ , where  $A_2$  is an abelian subvariety of  $A$ . Then the restriction  $f_2 : A_2 \rightarrow V$  of the rational map  $f$  to  $A_2$  is well defined and gives a generically surjective rational map. As  $\dim A_2 = l$ , the theorem is proved.

**COROLLARY 7.11:** *Let  $V$  be a generalized Kummer manifold of Kodaira dimension zero. For any effective  $m$ -th canonical divisor  $D = \sum_{i=1}^N n_i \cdot D_i \in |mK_V|$  for some  $m$ , we have  $\kappa(D_i) = -\infty$ , where  $D_i$  is an irreducible component of  $D$ .*

PROOF: There exists a generically surjective rational map  $g : A \rightarrow V$  of an abelian variety  $A$  onto  $V$ , where  $\dim A = \dim V$ . Let  $g^* : A^* \rightarrow V$  be an elimination of points of indeterminacy of the rational map  $g$ , where  $\tau : A^* \rightarrow A$  is the inverse of a succession of monoidal transformations with non-singular centers and  $g = g^* \circ \tau^{-1}$ . Let  $\varphi$  be an element of  $H^0(V, \mathcal{O}(mK_V))$  such that  $\varphi = 0$  is the equation of the divisor  $D$ . The pull back  $g^{**}(\varphi)$  has only zeros on the exceptional varieties of  $A^*$  appeared by the above monoidal transformations. Hence  $D_i$  is the image of an irreducible component of exceptional varieties, which is a ruled variety. Hence we have  $\kappa(D_i) = \infty$ . q.e.d.

REMARK 7.12: *Theorem 7.10 supports Conjecture  $C_n$ .* Suppose we have a fibre space  $g : A^* \rightarrow V$  such that  $A^*$  is birationally equivalent to a simple abelian variety and  $\kappa(V) = 0$ . If  $\dim A > \dim V$ , the Kodaira dimensions of general fibres of  $g$  are positive by Theorem 3.3. Hence  $C_n$  implies that  $\dim A = \dim V$ . And this is true because of Theorem 7.10.

THEOREM 7.13: *Let  $V$  be a Kummer manifold. Then  $\kappa(V) = 0$ , if and only if there exists a smooth algebraic variety  $V^*$  such that*

- (1)  $V^*$  is birationally equivalent to  $V$ ,
- (2) the Albanese map  $\alpha : V^* \rightarrow \text{Alb}(V^*)$  gives the structure of an analytic fibre bundle whose fibre is a Kummer manifold of Kodaira dimension zero.

PROOF: If part is a consequence of Main theorem of Nakamura – Ueno [26].

We shall prove only if part. By Definition 7.1, there exist an abelian variety  $A$  of dimension  $l = \dim V$  and a finite group  $G$  such that  $V$  is a non-singular model of the quotient space  $A/G$ . Moreover we can assume that  $A$  is a product of simple abelian varieties.

Let  $g : \tilde{A} \rightarrow A$  be a finite unramified covering of  $A$  such that  $\tilde{A}$  is a product of simple abelian varieties. Let  $S$  be a singular locus of  $A/G$ . Then  $f^{-1}(S)$  is of codimension at least two in  $A$ . Hence  $\tilde{S} = g^{-1}(f^{-1}(S))$  is of codimension at least two in  $\tilde{A}$ . We set  $\tilde{A}' = \tilde{A} - \tilde{S}$  and  $V' = A/G - S$ . By the proof of Theorem 7.2 we infer that  $f \circ g_{\tilde{A}'} : \tilde{A}' \rightarrow V'$  is a finite unramified covering. Hence the fundamental group  $\pi_1(A')$  is a subgroup of the fundamental group  $\pi_1(V')$  of finite index. Then there exists a subgroup  $H$  of  $\pi_1(A')$  such that  $H$  is a normal subgroup of  $\pi_1(V')$  of finite index. As the algebraic set  $\tilde{S}$  is at least of codimension two, we have  $\pi_1(\tilde{A}) = \pi_1(\tilde{A}')$ . Hence there exists a finite unramified covering  $u : \tilde{\tilde{A}} \rightarrow \tilde{A}$  which corresponds to the subgroup  $H \subset \pi_1(A)$ . As  $\tilde{A}$  is a product of simple abelian varieties,  $\tilde{\tilde{A}}$  is a product of simple abelian varieties.

We set  $\tilde{A}' = u^{-1}(\tilde{A}')$ . Then  $f \circ g \circ u|_{\tilde{A}'}: \tilde{A}' \rightarrow V'$  is a finite Galois covering. Hence the field extension  $C(\tilde{A})/C(V)$  induced by this morphism is Galois. The Galois group  $\tilde{G}$  of this field extension is a group of birational transformations of  $\tilde{A}$ . As an abelian variety is an absolutely minimal model,  $\tilde{G}$  is a group of analytic automorphisms of  $\tilde{A}$ . (Lang [22] II Section 1. Theorem 2.) As we have  $C(\tilde{A})^{\tilde{G}} = C(V)$ ,  $V$  is a non-singular model of a quotient variety  $\tilde{A}/\tilde{G}$ .

Hence we can assume that  $A$  itself is already a product of simple abelian varieties. Let  $f: A \rightarrow V$  be a generically surjective rational map induced by the canonical morphism  $A \rightarrow A/G$ . By Proposition 1.10 we have

$$q(V) = \dim H^0(A, \Omega_A^1)^G.$$

Hence there exist global coordinates  $z_1, z_2, \dots, z_l$  of  $A$  such that  $dz_{l-q+1}, dz_{l-q+2}, \dots, dz_l$  are invariant under the action of  $G$ . We can write  $A = A_1 \times A_2$ , where  $A_1$  and  $A_2$  are abelian subvariety of  $A$  such that  $A_2$  is a finite unramified covering of  $\text{Alb}(V)$  by the morphism  $\alpha \circ f$ . (We can assume that  $\alpha \circ f$  is a group homomorphism.) Then we can assume that  $A_1$  has global coordinates  $z_1, z_2, \dots, z_{l-q}$  and  $A_2$  has global coordinates  $z_{l-q+1}, z_{l-q+2}, \dots, z_l$ .

By these coordinate an analytic automorphism  $g \in G$  is written in the form

$$g : \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_l \end{pmatrix} \mapsto A(g) \cdot \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_l \end{pmatrix} + \begin{pmatrix} a_1(g) \\ a_2(g) \end{pmatrix}.$$

where

$$A(g) = \left( \begin{array}{cc} \overset{1-q}{\widetilde{A_1}(g)} & \overset{q}{\widetilde{B}(g)} \\ 0 & I_q \end{array} \right) \}_{l-q}^q \in GL(l, \mathbf{C}),$$

$$a_1(g) = \begin{pmatrix} a_1(g) \\ a_2(g) \\ \vdots \\ a_{l-q}(g) \end{pmatrix} \in \mathbf{C}^{l-q}, \quad a_2(g) = \begin{pmatrix} a_{l-q+1}(g) \\ a_{l-q+2}(g) \\ \vdots \\ a_l(g) \end{pmatrix} \in \mathbf{C}^q.$$

We set  $H = \{g \in G | a_2(g) = 0\}$ . Then  $H$  is a normal subgroup of  $G$ . We set

$$C = - \frac{1}{|H|} \sum_{h \in H} B(h).$$

Since we have

$$B(g \circ h) = A_1(g) \cdot B(h) + B(g),$$

we have

$$(7.14) \quad A_1(g) \cdot C - C = B(g), \quad g \in H.$$

Let  $w_1, w_2, \dots, w_{l-q}, z_{l-q+1}, \dots, z_l$  be new global coordinates of  $A$  defined by

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{l-q} \end{bmatrix} = (I_{l-q}, C) \cdot \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_l \end{bmatrix}.$$

In view of (7.14), by these new coordinates, any automorphism  $h \in H$  is written in the form

$$(7.15) \quad \begin{pmatrix} w \\ z \end{pmatrix} \mapsto \begin{pmatrix} A_1(h) & 0 \\ 0 & I_q \end{pmatrix} \cdot \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} a_1(h) \\ 0 \end{pmatrix},$$

where

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{l-q} \end{bmatrix}, \quad z = \begin{bmatrix} z_{l-q+1} \\ z_{l-q+2} \\ \vdots \\ z_l \end{bmatrix}.$$

Note that for any fixed  $(z)$ ,  $w_1, w_2, \dots, w_{l-q}$  are global coordinates of  $A_1$ .

Let  $H_1$  be a group of analytic automorphisms of the abelian variety  $A_1$  consisting of all automorphisms

$$h_1 : \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{l-q} \end{bmatrix} \mapsto A_1(h) \cdot \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_{l-q} \end{bmatrix} + a_1(h), \quad h \in H.$$

We set  $F = A_1/H_1$ . Then the natural morphism  $\tau : X = A/H \rightarrow A_2$ , induced by the projection map  $A = A_1 \times A_2 \rightarrow A_2$  gives a structure of a fibre bundle whose fibre is the quotient space  $F = A_1/H_1$ , in view of (7.15). As  $H$  is a normal subgroup of  $G$ , the quotient group  $\bar{G} = G/H$  operates on  $X$ . Any element  $\bar{g} \neq \text{id}$  of  $\bar{G}$  gives a fibre preserving analytic automorphism of the fibre space  $\tau : X \rightarrow A_2$  and has no fixed points since  $a_2(g) \neq 0$ . Moreover  $\bar{g}$  operates on  $A_2$  by

$$\begin{pmatrix} z_{l-q+1} \\ z_{l-q+2} \\ \vdots \\ z_l \end{pmatrix} \mapsto \begin{pmatrix} z_{l-q+1} \\ z_{l-q+2} \\ \vdots \\ z_l \end{pmatrix} + a_2(g),$$

where  $g \in G$  is a representative of  $\bar{g}$ . Hence the morphism  $\tau : X \rightarrow A_2$  is  $\bar{G}$  equivariant.

By Hironaka [11], there exists a non-singular model  $F^*$  of  $F$  such that  $\text{Aut}(F)$  can be lifted to a group of analytic automorphisms of  $F^*$ . By replacing the fibre  $F$  of the fibre bundle  $\tau : X \rightarrow A_2$  by  $F^*$ , we obtain the associated fibre bundle  $\tau^* : X^* \rightarrow A_2$ . Then  $\bar{G}$  operates on  $X^*$  and the morphism  $\tau^*$  is  $\bar{G}$  equivariant. Moreover any element  $\bar{g} \neq \text{id}$  operates on  $X^*$  without fixed points. Hence the quotient  $\pi : V^* = X^*/\bar{G} \rightarrow A_2/\bar{G}$  is a fibre bundle over  $A_2/\bar{G}$  whose fibre is  $F^*$ .  $V^*$  is birationally equivalent to  $V$ . Because of our construction it is easy to show that

$$A_2/\bar{G} = \text{Alb}(V^*) = \text{Alb}(V),$$

and the morphism  $\pi : V^* \rightarrow A_2/\bar{G}$  is the Albanese map. q.e.d.

REMARK 7.16: The above fibre bundle  $\alpha : V^* \rightarrow \text{Alb}(V^*)$  is not only an analytic fibre bundle but also a fibre bundle in the étale topology. The proof is as follows.

We use the same notations as above. By the above proof it is enough to show that there exists a finite unramified covering  $\delta : \tilde{A}_2 \rightarrow A_2$  such that the pull back  $\tilde{\tau} : \tilde{X} \rightarrow A_2$  of  $\tau : X \rightarrow A_2$  is a trivial fibre bundle.

Let  $\Omega_1$  and  $\Omega_2$  be period matrices of abelian varieties  $A_1$  and  $A_2$  with respect to global coordinates  $(z_1, z_2, \dots, z_{l-q}), (z_{l-q+1}, z_{l-q+2}, \dots, z_l)$ , respectively. Let  $\tilde{A}_2$  be an abelian variety with a period matrix  $\varepsilon\Omega_2$  with respect to global coordinates  $(z_{l-q+1}, z_{l-q+2}, \dots, z_l)$ , where  $\varepsilon = |H|$ . Then by the natural morphism

$$\delta : \begin{matrix} \tilde{A}_2 \\ \wr \\ \begin{pmatrix} z_{l-q+1} \\ z_{l-q+2} \\ \vdots \\ z_l \end{pmatrix} \end{matrix} \rightarrow \begin{matrix} A_2 \\ \wr \\ \begin{pmatrix} z_{l-q+1} \\ z_{l-q+2} \\ \vdots \\ z_l \end{pmatrix} \end{matrix},$$

$\tilde{A}_2$  is a finite unramified covering of  $A_2$ . We set  $\tilde{A} = A_1 \times \tilde{A}_2$ . Then  $w_1, w_2, \dots, w_{l-q}, z_{l-q+1}, \dots, z_l$  are global coordinates of  $A$  and with respect to these coordinates a period matrix of  $A$  has a form

$$\begin{pmatrix} \Omega_1, & \varepsilon C \Omega_2 \\ 0, & \varepsilon \Omega_2 \end{pmatrix}.$$

As  $B(h), h \in H$  gives a homomorphism of  $A_2$  into  $A_1$ , by the very definition of  $C$ , there exists an  $(l - q) \times q$  integral matrix  $M$  such that

$$\varepsilon C \Omega_2 = \Omega_1 \cdot M.$$

Hence  $A$  has a period matrix

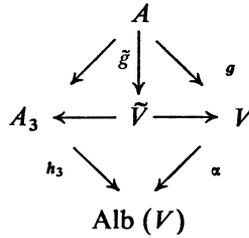
$$\begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix}$$

with respect to the global coordinates  $w_1, w_2, \dots, w_{l-q}, z_{l-q+1}, \dots, z_l$  and these global coordinates give the splitting of  $\tilde{A}$  into  $A_1 \times \tilde{A}_2$ . The group  $H$  operates on  $\tilde{A}$  by the same form as (7.15). Then it is clear that  $\tilde{X} = \tilde{A}/H \rightarrow \tilde{A}_2$  is the pull back of  $X \rightarrow A_2$  by  $\delta$  and  $\tilde{X} = F \times \tilde{A}_2$ . q.e.d.

**THEOREM 7.17:** *Let  $V$  be a generalized Kummer manifold. Suppose  $q(V) = \dim V - 1$ . Then ①  $\kappa(V) = -\infty$  if and only if general fibres of the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  are  $\mathbf{P}^1$ , ②  $\kappa(V) = 0$  if and only if the Albanese map is birationally equivalent to a fibre bundle over  $\text{Alb}(V)$  whose fibre is an elliptic curve.*

**PROOF:** If part is a consequence of Corollary 1.16 and Theorem 7.13. We shall prove only if part. There exist an abelian variety  $A$  and a generically surjective rational map  $g : A \rightarrow V$  of  $A$  onto  $V$ . We can assume that  $g$  is holomorphic at the origin of the abelian variety  $A$  and  $h = \alpha \circ g : A \rightarrow \text{Alb}(V)$  is a surjective group homomorphism. By Lemma 7.7 general fibres of the Albanese map  $\alpha : V \rightarrow \text{Alb}(V)$  are elliptic curves or  $\mathbf{P}^1$ . Assume that general fibres are elliptic curves. Then the all elliptic curves appeared in the general fibres are the same elliptic curve  $E$ , because for every such an elliptic curve is an image of a translation of an abelian subvariety of  $A$  by the rational map  $g$  and the abelian variety has only finitely many abelian subvarieties. Let  $A_1$  be the irreducible component of  $h^{-1}(0)$ , which contains the origin of  $A$ . Then  $A_1$  is an abelian subvariety of  $A$ . If necessary we shall take a finite unramified covering of  $A_1$  and we can assume that  $A_1 = E_1 \times A_2$ , where  $A_2$  is an abelian subvariety of  $A_1$  and  $E_1$  is an elliptic curve such that a translation of  $E_1$  is mapped onto a general fibre of the Albanese map  $\alpha$  by  $g$ . Moreover we can assume that  $A = E_1 \times A_2 \times A_3$ , where  $A_3$  is an abelian subvariety of  $A$ . The restriction  $h_3 : A_3 \rightarrow \text{Alb}(V)$  of a group homomorphism  $h$  on  $A_3$  is a finite unramified covering morphism. Let  $\tilde{V}$  be the fibre product of  $V$  and  $A_3$  over  $\text{Alb}(V)$ . As  $h_3$  is finite unramified,  $\tilde{V}$  is a finite unramified covering of  $V$ . Hence we have  $\kappa(V) = \kappa(\tilde{V})$ . On the other

hand there exists a generically surjective rational map  $\tilde{g} : A \rightarrow \tilde{V}$ .



Moreover the composition  $A_3 \rightarrow A \xrightarrow{\tilde{g}} \tilde{V}$  gives a rational section of the elliptic fibre space  $\tilde{\alpha} : V \rightarrow A_3$ . Hence by Proposition 6.14 the elliptic fibre space  $\tilde{\alpha} : V \rightarrow A_3$  is birationally equivalent to the basic elliptic fibre space, which is constructed by the similar way as in Section 4. Let  $S$  be a singular locus of  $\tilde{\alpha} : \tilde{V} \rightarrow A_3$ . (See Section 4, p 299.) As all regular fibres are the same elliptic curve  $E$ , the image of  $\Phi : \pi_1(A_3 - S) \rightarrow \text{SL}(2, \mathbf{Z})$  is a finite group. (See Section 4, p. 299.) Then there exists a finite unramified covering  $u' : B' \rightarrow A_3 - S$  whose covering transformation group is isomorphic to  $G = \Phi(\pi_1(A_3 - S))$ . Then we can extend the finite ramified covering  $u : B \rightarrow A_3$  such that  $B$  is a normal complex space and the group  $G$  operates on  $B$ . Moreover  $u : B \rightarrow A_3$  is unramified if and only if every component of  $S$  is of codimension at least two. By Hironaka [11] there exists a non-singular model  $\tilde{B}$  of  $B$  such that  $G$  operates on  $\tilde{B}$ . For any element  $g \in G$ ,  $g$  operates on  $\tilde{B} \times E$  by

$$\begin{array}{ccc}
 \tilde{B} \times E & \rightarrow & \tilde{B} \times E \\
 \cup & & \cup \\
 (x, \zeta) & \mapsto & (g(x), (c\tau + d)^{-1}\zeta)
 \end{array}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbf{Z})$$

and  $(1, \tau)$  is fundamental periods of the elliptic curve  $E$ . Then  $\alpha : \tilde{V} \rightarrow A_3$  is birationally equivalent to  $\tilde{B} \times E/G \rightarrow \tilde{B}/G$ . Let  $\tilde{\alpha} : \tilde{V} \rightarrow \tilde{A}$  be a non-singular model of  $\tilde{B} \times E/G \rightarrow \tilde{B}/G$ . Then there is a birational morphism  $\pi : \tilde{A} \rightarrow A_3$ . By the arguments in Section 6 we infer readily that  $12K_{\tilde{V}}$  is linearly equivalent to  $\tilde{\alpha}^*(12K_{\tilde{A}} + F) + G + H$ , where  $F$  is an effective divisor on  $\tilde{A}$  and  $G$  and  $H$  are effective divisors on  $\tilde{V}$ . We write  $F = \sum_{v=1}^m n_v C_v$  where  $C_v$ 's are prime divisors on  $\tilde{W}$ . We can assume  $\pi(C_1), \dots, \pi(C_r)$  are of codimension one in  $A_3$  and  $\pi(C_{r+1}), \pi(C_{r+2}), \dots, \pi(C_m)$  is of codimension at least two. As  $\tilde{V}$  and  $V$  are birationally equivalent  $12K_V$  is linearly equivalent to an effective divisor

$$\tilde{\alpha}^*\left(\sum_{v=1}^r n_v \pi(C_v)\right) + F' + G'$$

where  $F'$  and  $G'$  are effective divisors on  $\tilde{V}$ . As we have  $\kappa(\tilde{V}) = 0$ , for any positive integer  $l$

$$\begin{aligned} 1 &\geq \dim_{\mathbb{C}} H^0(\tilde{V}, \mathcal{O}(\tilde{\alpha}^*(\sum_{v=1}^r l n_v \pi(C_v)) + lF' + lG')) \\ &\geq \dim_{\mathbb{C}} H^0(A_3, \mathcal{O}(\sum_{v=1}^r l n_v \pi(C_v))). \end{aligned}$$

Since  $A_3$  is an abelian variety this means

$$\sum_{v=1}^r n_v \pi(C_v) = 0.$$

Hence again using the arguments in Section 6 we infer readily that  $\tilde{\alpha} : \tilde{V} \rightarrow A_3$  is bimeromorphically equivalent to an analytic fibre bundle over  $A_3$  whose fibre is the elliptic curve  $E$ . Hence  $\alpha : V \rightarrow \text{Alb}(V)$  is also bimeromorphically equivalent to an analytic fibre bundle over  $\text{Alb}(V)$  whose fibre is the elliptic curve  $E$ . q.e.d.

### 8. Examples

In this section we shall construct certain Kummer manifolds of Kodaira dimension zero and calculate dimensions of their infinitesimal deformation spaces.

**EXAMPLE 8.1:** Let  $A$  be a complex torus of dimension  $l$  and let  $z_1, z_2, \dots, z_l$  be its global coordinates. Then the automorphism  $(z_1, z_2, \dots, z_l) \mapsto (-z_1, -z_2, \dots, -z_l)$  of  $\mathbb{C}^n$  induces the involution  $g : A \rightarrow A$ .  $g$  has  $2^{2l}$  fixed points. Hence if  $l \geq 2$ , the quotient space  $A/G$  has  $2^{2l}$  singular points, which correspond to fixed points, where  $G$  is a cyclic group of analytic automorphisms of  $A$  generated by  $g$ . Each singular point has a neighborhood which is isomorphic to a neighborhood of the singular point  $p$  in  $Q_2^1$  in Appendix (8.16). By the canonical resolution of its singularities given in (8.16), we obtain a non-singular model  $M^{(l)}$  of  $A/G$ . The complex manifold  $M^{(l)}$  is usually called a Kummer manifold.

If  $l = 1$ ,  $A/G$  is analytically isomorphic to  $P^1$ .

**LEMMA 8.2:** For  $l \geq 2$ , we have

$$\begin{aligned} g_k(M^{(l)}) &= \begin{cases} \binom{l}{k}, & k \equiv 0 \pmod{2}, \\ 0, & k \not\equiv 0 \pmod{2}, \end{cases} \quad k = 1, 2, \dots, l, \\ P_m(M^{(l)}) &= \begin{cases} 1, & ml \equiv 0 \pmod{2}, \\ 0, & ml \not\equiv 0 \pmod{2}, \end{cases} \quad m = 1, 2, \dots. \end{aligned}$$

Hence  $\kappa(M^{(l)}) = 0$ .

Moreover for  $ml = 0 \ (2)$ , the effective  $m$ -th canonical divisor has the form

$$\sum_{i=1}^{2^{2l}} \left( \frac{ml}{2} - 1 \right) E_i,$$

where  $E_i$  is an exceptional divisor appeared by the canonical resolution. (See (8.16).)

PROOF: As we have

$$\dim H^0(A, \Omega_A^l)^G = \begin{cases} \binom{l}{k}, & k \equiv 0 \ (2), \\ 0, & k \equiv 1 \ (2), \end{cases}$$

the first part holds in view of Proposition 1.10. On the other hand we have

$$\dim H^0(A, \mathcal{O}(mK_A))^G = \begin{cases} 1, & mk \equiv 0 \ (2), \\ 0, & mk \equiv 1 \ (2). \end{cases}$$

We use the notation of Appendix 8.16. For  $mk \equiv 0 \ (2)$ , we have

$$(dz_1 \wedge \cdots \wedge dz_l)^m = \frac{(w_i^i)^{m(l-2)/2}}{2^m} (dw_i^1 \wedge \cdots \wedge dw_i^l)^m.$$

This implies that  $G$ -invariant  $m$ -tuple  $l$  form  $(dz_1 \wedge \cdots \wedge dz_l)^m$  induces a holomorphic  $m$ -tuple  $l$  form on  $M^{(l)}$ . Hence by Remark 1.11 and Appendix 8.16 we have desired results. q.e.d.

LEMMA 8.3:  $M^{(l)}$  is simply connected.

PROOF: See Spanier [32]. Another proof. Since  $M^{(l)}$  is a deformation of a Kummer manifold which is obtained from a product of  $l$  elliptic curves (See (8.9)), we can apply a similar argument as one in the proof of Lemma 8.12. q.e.d.

COROLLARY 8.4: If  $l \equiv 1 \ (2)$ , there does not exist a complex manifold  $V$  such that  $V$  is bimeromorphically equivalent to  $M^{(l)}$  and  $2K_V$  is analytically trivial.

PROOF: Since  $P_g(M^{(l)}) = 0$ , if such  $V$  exists, then  $\pi_1(V) \neq 1$  by Kodaira [20] II, Theorem 33. q.e.d.

LEMMA 8.5:

$$\dim H^0(M^{(l)}, \Theta) = \begin{cases} 20, & l = 2, \\ l^2, & l \geq 3. \end{cases}$$

PROOF: When  $l = 2$ ,  $M^{(2)}$  is a  $K3$  surface and the result is well known. (See for example Kodaira [20] I p 782.) Assume  $l \geq 3$ . We set  $\mathfrak{C} = \cup E_i$ .

We have a long exact sequence of local cohomology groups

$$(8.6) \quad \rightarrow H_{\mathbb{C}}^k(M^{(l)}, \Theta) \rightarrow H^k(M^{(l)}, \Theta) \rightarrow \\ \rightarrow H^k(M^{(l)} - \mathbb{C}, \Theta) \rightarrow H_{\mathbb{C}}^{k+1}(M^{(l)}, \Theta) \rightarrow.$$

On the other hand  $A - S$  is a covering manifold of  $M^{(l)} - \mathbb{C}$ , where  $S$  is the set of all fixed points of  $g$ . Hence we have a spectral sequence

$$(8.7) \quad H^p(G, H^q(A - S, \Theta)) \Rightarrow H^{p+q}(M^{(l)} - \mathbb{C}, \Theta),$$

which degenerates and we have a canonical isomorphism

$$H^q(A - S, \Theta)^G \simeq H^q(M^{(l)} - \mathbb{C}, \Theta).$$

If  $q \leq l - 2$ , by the Riemann extension theorem of cohomology groups (Scheja [30]), we have an isomorphism

$$H^q(A - S, \Theta)^G \simeq H^q(A, \Theta).$$

Hence we have an isomorphism

$$(8.8) \quad H^q(M^{(l)} - \mathbb{C}, \Theta) \simeq H^q(A, \Theta)^G.$$

In our case it is easy to show that  $H^1(A, \Theta)^G = H^1(A, \Theta)$ . By the excision theorem of local cohomology groups, we have

$$H_{\mathbb{C}}^q(M^{(l)}, \Theta) = \bigoplus_{i=1}^{2^{2l}} H_{E_i}^q(M^{(l)}, \Theta),$$

and

$$H_{E_i}^q(M^{(l)}, \Theta) = H_E^q(M, \Theta),$$

where  $E$  and  $M$  are the same as those in (8.16). Hence by Lemma 8.18 we have

$$H_{\mathbb{C}}^1(M^{(l)}, \Theta) = 0.$$

Hence by (8.6),  $H^1(M^{(l)}, \Theta)$  is a sub space of  $H^1(M^{(l)} - \mathbb{C}, \Theta) \simeq H^1(A, \Theta)$ . Now we consider the analytic family  $\hat{\mu}: \mathfrak{M} \rightarrow \mathfrak{U}$  constructed in Remark 8.9 below. There exists a point  $p \in \mathfrak{U}$  such that  $\hat{\mu}^{-1}(P) = M^{(l)}$ . Let  $\rho: T_p(\mathfrak{U}) \rightarrow H^1(M^{(l)}, \Theta)$  be the Kodaira-Spencer map for the family  $\hat{\mu}: \mathfrak{M} \rightarrow \mathfrak{U}$ . Then it is easy to show that the composition  $\hat{\rho}: T_p(\mathfrak{U}) \rightarrow H^1(M^{(l)}, \Theta) \rightarrow H^1(M^{(l)} - \mathbb{C}, \Theta) \rightarrow H^1(A, \Theta)$  is nothing but the Kodaira-Spencer map of the analytic family  $\mu: \mathfrak{U} \rightarrow \mathfrak{U}$  of the complex tori. As  $\hat{\rho}$  is isomorphic (Kodaira-Spencer [21] Chap. VI Th. 14.3), we have

$$\dim_{\mathbb{C}} H^1(M^{(l)}, \Theta) = l^2,$$

and

$$H^1(M^{(l)}, \Theta) \simeq H^1(A, \Theta). \qquad \text{q.e.d.}$$

When  $A$  is an abelian variety, the above lemma is proved by Schlesinger [31].

REMARK 8.9: Let  $\mathfrak{U}$  be a set of all  $l \times l$  complex matrices  $X$  such that

$$\det \begin{pmatrix} I_l & X \\ I_l & \bar{X} \end{pmatrix} \neq 0.$$

$\mathfrak{U}$  is an open set in  $\mathbf{C}^{2l}$ . Let  $\mathcal{G}$  be a group of analytic automorphisms of  $\mathfrak{U} \times \mathbf{C}^l$  consisting of all automorphisms

$$g(v) : \mathfrak{U} \times \mathbf{C}^l \rightarrow \mathfrak{U} \times \mathbf{C}^l \\ \begin{matrix} \cup \\ (X, z) \end{matrix} \mapsto \begin{pmatrix} \cup \\ X, z + v \cdot \begin{pmatrix} I_l \\ X \end{pmatrix} \end{pmatrix}, \quad v \in \mathbf{Z}^{2l}.$$

$\mathcal{G}$  acts properly discontinuously and freely on  $\mathfrak{U} \times \mathbf{C}^l$ . Let  $\mathfrak{A}$  be the quotient manifold  $\mathfrak{U} \times \mathbf{C}^l / \mathcal{G}$  and by  $(X, [z])$ , we denote the point in  $A$  which corresponds to the point  $(X, z) \in \mathfrak{U} \times \mathbf{C}^l$ . Then the projection

$$\mu : \begin{matrix} \mathfrak{A} & \longrightarrow & \mathfrak{U} \\ \cup & & \cup \\ (X, [z]) & \longrightarrow & X \end{matrix}$$

gives a complete effective analytic family of complex tori.

Let  $G$  be a cyclic group of order two of analytic automorphisms of  $\mathfrak{A}$  generated by the automorphism

$$g : \begin{matrix} \mathfrak{A} & \longrightarrow & \mathfrak{A} \\ \cup & & \cup \\ (X, [z]) & \mapsto & (X, [-z]). \end{matrix}$$

Then  $g$  has  $2^{2l}$  fixed manifolds  $D_i, i = 1, 2, \dots, 2^{2l}$  and the quotient space  $\mathfrak{A}/G$  has  $2^{2l}$  singular loci  $\mathcal{D}_i$  corresponding to  $D_i$ . For any point  $p \in \mathcal{D}_i$ , there exists an open set  $\mathfrak{B}$  in  $\mathfrak{A}/G$  such that  $\mathfrak{B}$  is isomorphic to  $\mathfrak{B} \times W$ , where  $\mathfrak{B}$  is an open set in  $\mathfrak{U}$  and  $W$  is a neighborhood of the singular point  $p$  in  $Q_2^1$ . (See (8.16) below.) Hence we can resolve the singularities of  $\mathfrak{A}/G$ , by using the canonical resolution of the singularity of  $Q_2^1$  and obtain a complex manifold  $\mathfrak{M}$ . Moreover the natural projection  $\tilde{\mu} : \mathfrak{A}/G \rightarrow \mathfrak{U}$  can be lifted to the morphism  $\hat{\mu} : \mathfrak{M} \rightarrow \mathfrak{U}$ .  $\hat{\mu} : \mathfrak{M} \rightarrow \mathfrak{U}$  is an analytic family of Kummer manifolds. If  $l \geq 3$ , from the isomorphism  $H^1(M^{(l)}, \Theta) \simeq H^1(A, \Theta)$ , we infer readily that the family  $\hat{\mu} : \mathfrak{M} \rightarrow \mathfrak{U}$  is complete and effectly parametrized.

EXAMPLE 8.10: Let  $E_\rho$  be the elliptic curve with a period matrix  $(1, \rho)$ , where  $\rho = \exp(2\pi i/3)$ . We set  $E^{(l)} = \underbrace{E \times E \times \dots \times E}_l$ . Let  $G$  be a cyclic group of order three of analytic automorphisms of  $E^{(l)}$  generated by the automorphism

$$g : \begin{array}{ccc} E^{(l)} & \longrightarrow & E^{(l)} \\ \cup & & \cup \\ (z_1, z_2, \dots, z_l) & \mapsto & (\rho z_1, \rho z_2, \dots, \rho z_l). \end{array}$$

Then  $g$  has  $3^l$  fixed points  $p_i, i = 1, 2, \dots, 3^l$ . Hence if  $l \geq 2$ , the quotient space  $E^{(l)}/G$  has  $3^l$  singular points  $p_i, i = 1, 2, \dots, 3^l$ , corresponding to the fixed point  $p_i$ . Each singular point has a neighborhood in  $E^{(l)}/G'$  which is analytically isomorphic to a neighborhood of the singular point  $p$  in  $Q_3^1$ . By the canonical resolution of singularities, we obtain a complex manifold  $N^{(l)}$ .

LEMMA 8.11:

$$g_k(N^{(l)}) = \begin{cases} \binom{l}{k}, & k \equiv 0(3), \quad k = 1, 2, \dots, l, \\ 0, & k \not\equiv 0(3), \end{cases}$$

$$P_m(N^{(l)}) = \begin{cases} 1, & l \geq 3, \quad lm \equiv 0(3), \\ 0, & l \geq 3 \quad lm \not\equiv 0(3) \\ \text{or } l = 1, 2, \quad m = 1, 2, \dots. \end{cases}$$

Hence  $\kappa(N^{(l)}) = 0$ , if  $l \geq 3$ . Moreover for  $l \geq 3, lm = 0(3)$ , the effective  $m$ -th canonical divisor has the form

$$\sum_{i=1}^{3^l} \left( \frac{lm}{3} - 1 \right) E_i,$$

where  $E_i$  is an exceptional divisor appeared by the canonical resolution.

The proof is similar as that of Lemma 8.2.

LEMMA 8.12:  $N^{(l)}$  is simply connected.

PROOF: We use the fibration  $f: N^{(l)} \rightarrow N^{(1)} = \mathbf{P}^1$  induced by the projection

$$\begin{array}{ccc} E^{(l)} & \longrightarrow & E \\ \cup & & \cup \\ (z_1, \dots, z_l) & \mapsto & (z_1). \end{array}$$

The morphism  $f$  is of maximal rank at any point over

$$\Delta' = N^{(1)} - \{[0], [\frac{1}{3} + \frac{2}{3}\rho], [\frac{2}{3} + \frac{1}{3}\rho]\}$$

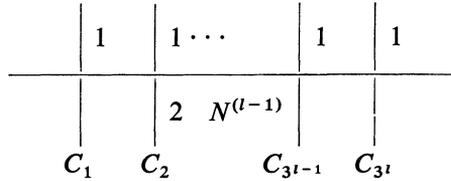
and  $f: N^{(l)} \rightarrow N^{(1)} = \mathbf{P}^1$  is a fibre space of principally polarized abelian varieties. It is easy to generalize the argument of Ueno [34] I p. 86–87 and we infer readily that the singular fibres over these three points

$$q_1 = [0], q_2 = [\frac{1}{3} + \frac{2}{3}\rho], q_3 = [\frac{2}{3} + \frac{1}{3}\rho]$$

have the form

$$2N^{(l-1)} + \sum_{i=1}^{3^{l-1}} C_i,$$

where  $C_i$  is one of the exceptional divisors  $E_k, k = 1, 2, \dots, 3^l$ . (See (8.11).)



$C_i$  is analytically isomorphic to  $\mathbf{P}^{l-1}$ .  $C_i$  and  $N^{(l-1)}$  intersect transversally.

Now we shall prove the lemma by induction on  $l$ .  $N^{(1)}$  is isomorphic to  $\mathbf{P}^1$ , hence it is simply connected. Assume that  $N^{(l-1)}$  is simply connected. Then the above singular fibres  $\mathcal{F}_j, j = 1, 2, 3$  over  $q_j$  are *simply connected*. As  $\mathcal{F}_j$  is a compact algebraic set,  $\mathcal{F}_j$  has a tubular neighborhood  $\mathfrak{B}_j$  in  $N^{(l)}$ , which has a retraction to  $\mathcal{F}_j$ . Hence  $\mathfrak{B}_j$  is *simply connected*. We set  $N' = N^{(l)} - \bigcup_{j=1}^3 \mathcal{F}_j, \mathfrak{B} = \bigcup_{j=1}^3 \mathfrak{B}_j$ . Then  $N'$  is a torus bundle over  $\Delta'$ . As  $N^{(1)}$  is isomorphic to  $\mathbf{P}^1, \pi_1(\Delta')$  is a free group generated by the homotopy classes of loops  $\tau_1, \tau_2$  which are small circles around  $q_j, j = 1, 2$  in  $\Delta'$ , respectively. Let  $\gamma_1, \gamma_2, \dots, \gamma_{2(l-1)}$  be generators of the fundamental group of a general fibre  $f^{-1}(u), u \in \Delta'$ . Then  $\pi_1(N')$  is generated by  $\gamma_1, \gamma_2, \dots, \gamma_{2(l-1)}, \delta_1, \delta_2$ , where  $\delta_1, \delta_2$  lie over  $\tau_1, \tau_2$ , respectively. We can choose  $\tau_i$  so small that there exists a neighborhood  $\mathfrak{B}_i$  of  $q_i$  in  $N^{(1)}$  such that  $\tau_i \subset \mathfrak{B}_i$  and  $\mathfrak{B}_i \supset f^{-1}(\mathfrak{B}_i)$ . Then by Van Kampen's theorem, we infer readily that  $\pi_1(N^{(l)})$  is trivial. q.e.d.

**COROLLARY 8.13:** *If  $l > 3, l \not\equiv 0 \pmod{3}$ , then  $N^{(l)}$  does not have a birational model  $V$  such that  $3K_V$  is analytically trivial.*

The proof is similar as that of Corollary 8.4.

**LEMMA 8.14:** *For  $l \geq 3$ , we have*

$$H^1(N^{(l)}, \Theta) = 0.$$

**PROOF:** By similar arguments as those of the proof of Lemma 8.5, we infer readily that

$$\dim_{\mathbf{C}} H^1(N^{(l)}, \Theta) \leq \dim_{\mathbf{C}} H^1(E^{(l)}, \Theta)^G = 0. \quad \text{q.e.d.}$$

**EXAMPLE 8.15:**  $E^{(l)}$  is the same as above. Let  $G$  be a cyclic group of order three of analytic automorphism of  $E^{(l)}$  generated by the automorphism

$$g : \begin{array}{ccc} E^{(l)} & \xrightarrow{\quad} & E^{(l)} \\ \cup & & \cup \\ (z_1, z_2, \dots, z_l) & \mapsto & (\rho z_1, \rho^2 z_2, \dots, \rho^2 z_l). \end{array}$$

Then  $g$  has  $3^l$  fixed points and the quotient space has  $3^l$  singular points. The resolution of these singularities is obtained by generalizing the process given in [34] I p 56 and we obtain a non-singular model  $L^{(l)}$  of the quotient space  $E^{(l)}/G$ . Then  $L^{(l)}$  is simply connected. Moreover  $\kappa(L^{(l)}) = 0$ . If  $l \geq 3$ ,  $mK_L(l)$  is not analytically trivial for any positive integer  $m$ . If  $l = 2$ ,  $L^{(2)}$  is a  $K3$  surface.

*Appendix. The canonical resolution of certain quotient singularities*

In this appendix we shall give the canonical resolution of the singular point  $\mathfrak{p}$  of  $Q_m^l$  (see (8.16) below) and calculate local cohomology groups associated to the resolution.

(8.16) Let  $G$  be a cyclic group of analytic automorphisms of  $C^l$  generated by

$$g : (z_1, z_2, \dots, z_l) \mapsto (e_m z_1, e_m z_2, \dots, e_m z_l)$$

where  $e_m = \exp(2\pi i/m)$ . The quotient space  $Q_m^l = C^l/G$  has only one singular point  $\mathfrak{p}$ , which corresponds to the origin of  $C^l$ . We shall resolve this singularity.

Let  $U_i, i = 1, 2, \dots, l$  be  $l$  copies of  $C^l$ , whose coordinates are  $(w_i^1, w_i^2, \dots, w_i^l)$ , respectively. We shall construct a complex manifold  $M = \bigcup_{i=1}^l U_i$  by identifying open subsets of  $U_{i-1}$  and  $U_i, i = 2, 3, \dots, l$ , through the following relations.

$$(8.17) \quad \left\{ \begin{array}{l} w_i^k = \frac{w_{i-1}^k}{w_{i-1}^i}, \quad k \neq i-1, i, \\ w_i^{i-1} = \frac{1}{w_{i-1}^i}, \\ w_i^i = (w_{i-1}^i)^m w_{i-1}^{i-1}. \end{array} \right.$$

Meromorphic maps

$$T_{U_i} : \begin{array}{ccc} C^l & \xrightarrow{\quad} & U_i \\ \cup & & \cup \\ (z_1, z_2, \dots, z_l) & \mapsto & \left( \frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, (z_i)^m, \frac{z_{i+1}}{z_i}, \dots, \frac{z_l}{z_i} \right) \end{array}$$

induce a meromorphic map  $T : Q_m^l \rightarrow M$ . Let  $E$  be a submanifold in  $M$  defined by the equations

$$w_i^i = 0, \text{ in } U_i, i = 1, 2, \dots, l.$$

Then  $E$  is analytically isomorphic to  $(l-1)$ -dimensional complex projective space  $P^{l-1}$ .

The meromorphic map  $T: Q_m^l \rightarrow M$  induces an isomorphism between  $Q_m^l - \mathfrak{p}$  and  $M - E$ . Hence  $M$  is a non-singular model of  $Q_m^l$ .

We set  $V = \{(z_1, \dots, z_l) \mid |z_i| < (\varepsilon)^{1/m}\}$ . Then the group  $G$  operates on  $V$  and the quotient space  $V/G$  has only one singular point  $\mathfrak{p}$ . Let  $\tilde{M}$  be an open set in  $M$  defined by the inequalities:

$$\begin{aligned} |(w_i^k)^m w_i^j| &< \varepsilon, \quad k \neq i, \\ |w_i^i| &< \varepsilon, \quad \text{in } U_i, \quad i = 1, 2, \dots, l. \end{aligned}$$

Then  $E \subset \tilde{M}$  and the above meromorphic map  $T$  induces an isomorphism between  $V/G - \mathfrak{p}$  and  $\tilde{M} - E$ . Hence  $\tilde{M}$  is a non-singular model of the quotient space  $V/G$ .

The procedure of resolving singularities will be called the *canonical resolution*.

LEMMA 8.18: *If  $l \geq 3$  we have*

$$H_E^1(M, \Theta) = 0$$

PROOF: We have a long exact sequence

$$\begin{aligned} 0 \rightarrow H_E^0(M, \Theta) \rightarrow H^0(M, \Theta) \rightarrow H^0(M-E, \Theta) \\ \rightarrow H_E^1(M, \Theta) \rightarrow H^1(M, \Theta) \rightarrow H^1(M-E, \Theta). \end{aligned}$$

It is easily shown that

$$H_E^0(M, \Theta) = 0,$$

and

$$H^0(M, \Theta) \simeq H^0(M-E, \Theta).$$

On the other hand by the same method as in the proof of Lemma 8.5 we have an isomorphism

$$H^1(M-E, \Theta) \simeq H^1(C^l - \{0\}, \Theta)^G.$$

As  $l \geq 3$ , we have

$$H^1(C^l - \{0\}, \Theta) = 0.$$

Hence we have an isomorphism

$$H_E^1(M, \Theta) \simeq H^1(M, \Theta).$$

Using the Čech cohomology group with respect to the Stein covering  $\{U_{ij}\}$ , it is easily shown that

$$H^1(M, \Theta) = 0. \qquad \text{q.e.d.}$$

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