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IAN STEWART

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A NOTE ON 2-STEP SUBIDEALS OF LIE ALGEBRAS

Ian Stewart

In this note we answer an implicit open question of de Ruiter [4].

We shall use the notation and terminology of [6, 7]. In particular $\mathfrak{N}_c$ is the class of nilpotent Lie algebras of nilpotency class $\leqq c$, and $\mathfrak{D}_n$ the class of Lie algebras in which every subalgebra is an n-step subideal. Using the methods of Roseblade [3] it was shown in [7] that there exists a function μ such that

$$\mathfrak{D}_n \leqq \mathfrak{N}_{\mu(n)}.$$

The proof was simplified by de Ruiter [4], yielding improved bounds for $\mu(n)$. Trivially we may take $\mu(1) = 1$. The general result of de Ruiter [4] yields $\mu(2) \leqq 7$. In section 4 of that paper a special argument is used to show that $\mu(2) = 2$ is best possible, except perhaps over fields of characteristic 3 where $2 \leqq \mu(2) \leqq 3$. It may therefore be worth noting that for characteristic 3 also we may take $\mu(2) = 2$, so that the classes $\mathfrak{D}_2$ and $\mathfrak{N}_2$ are equal. The proof arose in conversation with R. K. Amayo and B. Hartley, and follows from an analysis of the structure of the free 3-generator 2-Engel Lie algebra over a field of characteristic 3. We therefore introduce the class $\mathfrak{E}_n$ of n-Engel algebras, satisfying the identical relation $[x, {}_ny] = 0$. From Higgins [2] we know that $\mathfrak{E}_2 \leqq \mathfrak{N}_3$. Thus the free 3-generator 2-Engel algebra E is a homomorphic image of the free 3-generator $\mathfrak{N}_3$-algebra L. Let the generators be x, y, z. For convenience in computations write

$$x_1 \cdots x_n = [x_1, \cdots, x_n]$$

and left-norm all products. Then by basic commutator calculations (Hall [1]) we find that L has a basis consisting of

$$x, y, z, xy, xz, yz, xyx, xyy, xzx, xzy, xzz, yzx, yzy, yzz.$$

To obtain E we must quotient out the ideal I generated by all elements uvv $(u, v \in L)$. Certainly I contains

$$xyx, xyy, xzx, xzz, yzy, yzz,$$

and also the element

$$xyz + xzy \ (*)$$

since this equals

$$x(y+z)(y+z)-xyy-xzz.$$

Consider the subspace J spanned by these 7 elements. Then J is an ideal of L since it is central. To show that $I = J$ it is sufficient to show that L/J is a 2-Engel algebra.

Working modulo J we have, from (*),

$$(1) \qquad xyz+xzy = 0.$$

By the Jacobi identity,

$$xyz+yzx+zxy = 0,$$

so that

$$-xzy+yzx+zxy = 0,$$

or

$$(2) \qquad yzx+yxz = 0$$

since the characteristic is 3. Similarly

$$(3) \qquad zxy+zyx = 0.$$

Now let a, $b \in L/J$. We must show that $abb = 0$. Since $L \in \mathfrak{N}_3$ it suffices to prove this when a and b are linear combinations of the generators x, y, z (mod J). By linearity we may consider separately the cases $a = x, y$, or z. But now, if α, β, γ are scalars, and again working modulo J,

$$\begin{aligned} x(\alpha x+\beta y+\gamma z)(\alpha x+\beta y+\gamma z) \\ &= \beta\gamma(xyz+xzy) \\ &= 0, \text{ by } (1). \end{aligned}$$

Similarly we can deal with $a = y$ or $a = z$ using (2) or (3).

Hence $E = L/J$ is the free 3-generator 2-Engel algebra. We have dim $E = 7$, and $E^3 = \langle xyz \rangle \neq 0$; so that $E \notin \mathfrak{N}_2$. Further the centre of E is equal to E^3 so is of dimension 1.

Now let D be a Lie algebra of characteristic 3 belonging to $\mathfrak{D}_2$. We claim that $D \in \mathfrak{N}_2$. If every 3-generator subalgebra of D were in $\mathfrak{N}_2$ then so would D be. So if $D \notin \mathfrak{N}_2$ then D has a 3-generator subalgebra $H \notin \mathfrak{N}_2$. Now (as in de Ruiter [4] section 4) $\mathfrak{D}_2 \leqq \mathfrak{E}_2$, so H is a homomorphic image of E. But any proper ideal of E intersects the centre non-trivially (Schenkman [5] lemma 4), so contains E^3; and therefore the quotient is in $\mathfrak{N}_2$. It follows that $H \cong E$. But by considering the ideal closure series it is easy to see that $\langle xy \rangle$ is not a 2-step subideal of E. So $E \notin \mathfrak{D}_2$, so $H \notin \mathfrak{D}_2$, which is a contradiction.

Thus we have proved the:

THEOREM: *A Lie algebra is a $\mathfrak{D}_2$-algebra if and only if it is nilpotent of class 2.*

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Mathematics Institute
University of Warwick
Coventry
England