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ON THE WEAK*-BASIS THEOREM

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Suppose (E, τ) is a locally convex space; then a sequence (x_n) is called a basis of E if for every $x \in E$ there is a unique sequence of scalars (a_n) with

$$x = \sum_{n=1}^{\infty} a_n x_n$$

If, furthermore the coefficients a_n are given by

$$a_n = f_n(x)$$

where each f_n is a τ-continuous linear functional, we say that (x_n) is a Schauder basis of E.

The weak basis theorem of Mazur (see [2]) states that if X is a Banach space, then a basis of X in the weak topology is a Schauder basis of X in the strong topology; in particular it is a Schauder basis. This theorem has been extended to various classes of locally convex spaces. In particular it is natural to ask whether a basis (f_n) of X^* in the weak* topology $\sigma(X^*, X)$ is necessarily a Schauder basis; this is equivalent (see [8] p. 155) to asking whether there exists a basis (x_n) of X with (f_n) the corresponding coefficient functionals. Unfortunately Singer shows by example ([8] p. 153 or see [7]) that a weak* basis need not be Schauder.

However it is trivial that a weak*-basis of the dual of a reflexive Banach space is Schauder; in this paper we give another important class of spaces for which this theorem is true.

Let τ be an $\langle X, X^* \rangle$ polar topology on X^*, and let (f_k) be a τ-basis of X^*; suppose $(p_\lambda;\ \lambda \in \Lambda)$ is a collection of semi-norms defining the topology τ. We define

$$p_\lambda^*(x) = \sup_n p_\lambda\Big(\sum_{k=1}^{n} a_k x_k\Big)$$

where

$$x = \sum_{k=1}^{\infty} a_k x_k\ (\tau).$$

Then the collection of semi-norms $(p_\lambda^*;\ \lambda \in \Lambda)$ define a topology τ^* on X^*. We then have the following lemma (see McArthur [6] Lemma 2 or Bennett and Cooper [1] Lemma 1).

LEMMA 1: *(X^*, τ^*) is complete and (f_k) is a Schauder basis of (X^*, τ^*).*

PROOF: This is proved by a method similar to [1] Lemma 1 or [6] Lemma 3. It is only necessary to assume that whenever $\sum a_k f_k$ is a τ-Cauchy series then it converges; this follows from the sequential completeness of (X^*, τ).

LEMMA 2: *τ^* is weaker than the norm topology on X^*.*

PROOF: For

$$f = \sum_{k=1}^{\infty} a_k f_k \; (\tau),$$

the sequence

$$\sum_{k=1}^{n} a_k f_k$$

is τ-bounded and therefore norm bounded. Let

$$||f||^* = \sup_n || \sum_{k=1}^{n} a_k f_k||$$

Then the standard argument, used in [1] Lemma 1, shows that $(X^*, || \, ||^*)$ is a Banach space. As the identity map $(X^*, || \, ||^*) \to (X^*, || \, ||)$ is continuous, we obtain, by the open mapping theorem, a constant $K > 0$ such that

$$||f||^* \leqq K||f||$$

However as τ is weaker than the norm topology on X^*; then for each $\lambda \in \Lambda$ there exists K_λ with

$$p_\lambda(x) \leqq K_\lambda ||f|| \qquad (x \in E)$$

and so

$$\begin{aligned} p_\lambda^*(x) &\leqq K_\lambda ||f||^* \\ &\leqq K K_\lambda ||f|| \end{aligned}$$

and τ^* is weaker than the norm topology.

THEOREM: *Let μ be a (positive) measure on a set S; suppose X is a closed subspace of $L_1(\mu)$ and that τ is an $\langle X, X^* \rangle$ polar topology on X^*. Then any basis of (X^*, τ) is a Schauder basis.*

PROOF: Suppose $\{f_k\}$ is a basis of (X^*, τ); then $\{f_k\}$ is a Schauder basis of (X^*, τ^*), and so it is sufficient to show that every τ^*-continuous linear functional is also τ-continuous.

Let $J : X \to L_1(\mu)$ denote the inclusion map, and let B and C be the closed unit balls if X^* and $[L_1(\mu)]^*$ respectively; then we have $J^*(C) = B$. Let I be the identity map on X^*. The map $IJ^* : [L_1(\mu)]^* \to (X^*, \tau^*)$

is continuous by Lemma 2; furthermore, by Lemma 1, (X^*, τ^*) is a separable complete locally convex space.

We use the well-known result that $[L_1(\mu)]^*$ is isometrically isomorphic with the space $C(S)$ of continuous functions on a compact Stone space. This follows, in the case of μ σ-finite, from the remarks of Grothendieck [3] p. 167. More generally we may use the results of Kakutani ([4], [5]) to show that the real space $[L_1(\mu)]^*$ is an abstract M-space with unit, and therefore lattice isomorphic and isometric with a space $C(S)$ where S is compact and Hausdorff; as $[L_1(\mu)]^*$ is also clearly order-complete it follows that S is a Stone space.

Now, by a result of Grothendieck [3], p. 168, $IJ^* : [L_1(\mu)]^* \to (X^*, \tau^*)$ is weakly compact. Let σ^* denote the weak topology associated with τ^*; we have that $IJ^*(C) = B$ is σ^*-relatively compact. However B is τ-closed and therefore τ^*-closed; as B is convex we can deduce that B is σ^*-closed. Thus B is σ^*-compact; hence on B, σ^* coincides with the weaker Hausdorff topology $\sigma(X^*, X)$. If ϕ is a τ^*-continuous linear functional on X^*, then ϕ is σ^*-continuous and therefore $\sigma(X^*, X)$ continuous on B; it follows that ϕ is $\sigma(X^*, X)$-continuous and therefore τ-continuous. This completes the proof.

We conclude by remarking that if X satisfies the hypotheses of the theorem then X is weakly sequentially complete; conversely we may ask whether the theorem holds if X is weakly sequentially complete. This would seem very likely but we have been unable to prove it.

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