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N. J. KALTON

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## ON THE WEAK\*-BASIS THEOREM

N. J. Kalton

Suppose  $(E, \tau)$  is a locally convex space; then a sequence  $(x_n)$  is called a basis of  $E$  if for every  $x \in E$  there is a unique sequence of scalars  $(a_n)$  with

$$x = \sum_{n=1}^{\infty} a_n x_n$$

If, furthermore the coefficients  $a_n$  are given by

$$a_n = f_n(x)$$

where each  $f_n$  is a  $\tau$ -continuous linear functional, we say that  $(x_n)$  is a Schauder basis of  $E$ .

The weak basis theorem of Mazur (see [2]) states that if  $X$  is a Banach space, then a basis of  $X$  in the weak topology is a Schauder basis of  $X$  in the strong topology; in particular it is a Schauder basis. This theorem has been extended to various classes of locally convex spaces. In particular it is natural to ask whether a basis  $(f_n)$  of  $X^*$  in the weak\* topology  $\sigma(X^*, X)$  is necessarily a Schauder basis; this is equivalent (see [8] p. 155) to asking whether there exists a basis  $(x_n)$  of  $X$  with  $(f_n)$  the corresponding coefficient functionals. Unfortunately Singer shows by example ([8] p. 153 or see [7]) that a weak\* basis need not be Schauder.

However it is trivial that a weak\*-basis of the dual of a reflexive Banach space is Schauder; in this paper we give another important class of spaces for which this theorem is true.

Let  $\tau$  be an  $\langle X, X^* \rangle$  polar topology on  $X^*$ , and let  $(f_k)$  be a  $\tau$ -basis of  $X^*$ ; suppose  $(p_\lambda; \lambda \in \Lambda)$  is a collection of semi-norms defining the topology  $\tau$ . We define

$$p_\lambda^*(x) = \sup_n p_\lambda \left( \sum_{k=1}^n a_k x_k \right)$$

where

$$x = \sum_{k=1}^{\infty} a_k x_k(\tau).$$

Then the collection of semi-norms  $(p_\lambda^*; \lambda \in \Lambda)$  define a topology  $\tau^*$  on  $X^*$ . We then have the following lemma (see McArthur [6] Lemma 2 or Bennett and Cooper [1] Lemma 1).

LEMMA 1:  $(X^*, \tau^*)$  is complete and  $(f_k)$  is a Schauder basis of  $(X^*, \tau^*)$ .

PROOF: This is proved by a method similar to [1] Lemma 1 or [6] Lemma 3. It is only necessary to assume that whenever  $\sum a_k f_k$  is a  $\tau$ -Cauchy series then it converges; this follows from the sequential completeness of  $(X^*, \tau)$ .

LEMMA 2:  $\tau^*$  is weaker than the norm topology on  $X^*$ .

PROOF: For

$$f = \sum_{k=1}^{\infty} a_k f_k(\tau),$$

the sequence

$$\sum_{k=1}^n a_k f_k$$

is  $\tau$ -bounded and therefore norm bounded. Let

$$\|f\|^* = \sup_n \left\| \sum_{k=1}^n a_k f_k \right\|$$

Then the standard argument, used in [1] Lemma 1, shows that  $(X^*, \|\cdot\|^*)$  is a Banach space. As the identity map  $(X^*, \|\cdot\|^*) \rightarrow (X^*, \|\cdot\|)$  is continuous, we obtain, by the open mapping theorem, a constant  $K > 0$  such that

$$\|f\|^* \leq K\|f\|$$

However as  $\tau$  is weaker than the norm topology on  $X^*$ ; then for each  $\lambda \in A$  there exists  $K_\lambda$  with

$$p_\lambda(x) \leq K_\lambda \|f\| \quad (x \in E)$$

and so

$$\begin{aligned} p_\lambda^*(x) &\leq K_\lambda \|f\|^* \\ &\leq KK_\lambda \|f\| \end{aligned}$$

and  $\tau^*$  is weaker than the norm topology.

THEOREM: Let  $\mu$  be a (positive) measure on a set  $S$ ; suppose  $X$  is a closed subspace of  $L_1(\mu)$  and that  $\tau$  is an  $\langle X, X^* \rangle$  polar topology on  $X^*$ . Then any basis of  $(X^*, \tau)$  is a Schauder basis.

PROOF: Suppose  $\{f_k\}$  is a basis of  $(X^*, \tau)$ ; then  $\{f_k\}$  is a Schauder basis of  $(X^*, \tau^*)$ , and so it is sufficient to show that every  $\tau^*$ -continuous linear functional is also  $\tau$ -continuous.

Let  $J: X \rightarrow L_1(\mu)$  denote the inclusion map, and let  $B$  and  $C$  be the closed unit balls of  $X^*$  and  $[L_1(\mu)]^*$  respectively; then we have  $J^*(C) = B$ . Let  $I$  be the identity map on  $X^*$ . The map  $IJ^*: [L_1(\mu)]^* \rightarrow (X^*, \tau^*)$

is continuous by Lemma 2; furthermore, by Lemma 1,  $(X^*, \tau^*)$  is a separable complete locally convex space.

We use the well-known result that  $[L_1(\mu)]^*$  is isometrically isomorphic with the space  $C(S)$  of continuous functions on a compact Stone space. This follows, in the case of  $\mu$   $\sigma$ -finite, from the remarks of Grothendieck [3] p. 167. More generally we may use the results of Kakutani ([4], [5]) to show that the real space  $[L_1(\mu)]^*$  is an abstract  $M$ -space with unit, and therefore lattice isomorphic and isometric with a space  $C(S)$  where  $S$  is compact and Hausdorff; as  $[L_1(\mu)]^*$  is also clearly order-complete it follows that  $S$  is a Stone space.

Now, by a result of Grothendieck [3], p. 168,  $IJ^* : [L_1(\mu)]^* \rightarrow (X^*, \tau^*)$  is weakly compact. Let  $\sigma^*$  denote the weak topology associated with  $\tau^*$ ; we have that  $IJ^*(C) = B$  is  $\sigma^*$ -relatively compact. However  $B$  is  $\tau$ -closed and therefore  $\tau^*$ -closed; as  $B$  is convex we can deduce that  $B$  is  $\sigma^*$ -closed. Thus  $B$  is  $\sigma^*$ -compact; hence on  $B$ ,  $\sigma^*$  coincides with the weaker Hausdorff topology  $\sigma(X^*, X)$ . If  $\phi$  is a  $\tau^*$ -continuous linear functional on  $X^*$ , then  $\phi$  is  $\sigma^*$ -continuous and therefore  $\sigma(X^*, X)$  continuous on  $B$ ; it follows that  $\phi$  is  $\sigma(X^*, X)$ -continuous and therefore  $\tau$ -continuous. This completes the proof.

We conclude by remarking that if  $X$  satisfies the hypotheses of the theorem then  $X$  is weakly sequentially complete; conversely we may ask whether the theorem holds if  $X$  is weakly sequentially complete. This would seem very likely but we have been unable to prove it.

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Department of Pure Mathematics  
Singleton Park  
Swansea, Wales