COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 27, nº 2 (1973), p. 135-140

<http://www.numdam.org/item?id=CM_1973__27_2_135_0>

COMPOSITIO MATHEMATICA, Vol. 27, Fasc. 2, 1973, pag. 135-140
Noordhoff International Publishing
Printed in the Netherlands

HOMOTOPIC HOMEOMORPHISMS OF INFINITE-DIMENSIONAL MANIFOLDS

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1. Introduction and preliminaries

We use F to represent any Fréchet space which is homeomorphic ($\cong$) to its own countable infinite product F^{ω}, and by a *Fréchet manifold* (or *F-manifold*) we mean a paracompact manifold modelled on F. The condition $F \cong F^{\omega}$ is known to be satisfied by any infinite-dimensional separable Fréchet space (as they are all homeomorphic [1]), by any infinite-dimensional Hilbert or reflexive Banach space [2], and by the space of bounded sequences, l_{∞} [3]. More generally there is no known example of an infinite-dimensional Fréchet space for which this condition is not satisfied.

In this note we establish the following two results on homotopic homeomorphisms of F-manifolds. (All our homeomorphisms will be onto.)

THEOREM 1: *Let M be an F-manifold and let $f, g : M \to M$ be homotopic homeomorphisms. Then f is ambient isotopic to g (i.e. each level of the isotopy is onto).*

We remark that previously this result was known for separable l^2-manifolds, where l^2 is separable infinite-dimensional real Hilbert space. The first proof was given by Burghelea and Henderson, with an argument that mixed differential and point-set techniques [4]. Later Wong gave another proof of the separable case (and some substantial generalizations) by using some recent results on fiber-preserving homeomorphisms on product bundles, with base space a locally compact polyhedrom and fiber a separable l^2-manifold [11]. We give here a short proof of Theorem 1 that uses pointset techniques and a simple modification of the Alexander trick, [7, p. 321].

THEOREM 2: *Let M, N be F-manifolds and let $f, g : M \to N$ be homo-*

[1] Supported in part by NSF Grant GP 14429.

topic open embeddings. Then f is isotopic to g, with each level of the isotopy being an open embedding.

(We will call an isotopy for which each level is an open embedding an *open isotopy*.)

Previously this result was not known even for l^2-manifolds and it answers affirmatively a question that was raised at a problem seminar held in January (1969) at Cornell University.

There is one special definition that we will need. A closed subset K of a space X is said to be a *Z-set* in X provided that given any non-null, homotopically trivial open set U in X, $U \backslash K$ is non-null and homotopically trivial. There are several places we will need the following homeomorphism extension theorem of [6], and we henceforth refer to it as the HET: *Let K_1, K_2 be Z-sets in an F-manifold M and let $h : K_1 \to K_2$ be a homeomorphism which is homotopic to id_{K_1} (the identity on K_1). There is an invertible ambient isotopy $G : M \times I \to M$ such that $G_0 = id_M$ and $G_1|K_1 = h$.*
(Here $I = [0, 1]$ and an ambient isotopy $H : X \times I \to X$ is said to be *invertible* provided that $H^* : X \times I \to X$, defined by $H^*(x, t) = H_t^{-1}(x)$, is continuous.)

2. Proof of Theorem 1

Without loss of generality we may assume that $g = id_M$ and we may replace M by $M \times (0, 1]$. (See [6] for references to papers which show, among other things, that $M \cong M \times [0, 1] = M \times (0, 1] \cong M \times (0, 1)$.) Thus we can reduce the problem to showing that a homeomorphism $f: M \times (0, 1] \to M \times (0, 1]$ which is homotopic to id is ambient isotopic to id (we suppress the subscript on id when the meaning is clear).

Note that $M \times \{1\}$ is a Z-set in $M \times (0, 1]$ and $f|M \times \{1\}$ is a homeomorphism of $M \times \{1\}$ onto a Z-set in $M \times (0, 1]$ which is homotopic to id. Using the HET we see that f is ambient isotopic to a homeomorphism $f' : M \times (0, 1] \to M \times (0, 1]$ which satisfies $f'|M \times \{1\} = \text{id}$.

Using the Alexander trick let $G : (M \times (0, 1]) \times I \to M \times (0, 1]$ be defined by

$$G_t(x, s) = \begin{cases} (x, s), & \text{for } t \leqq s \\ tf'(x, s/t), & \text{for } s < t \end{cases}$$

(where $t(x, u) = (x, tu)$). Then G is an ambient isotopy satisfying $G_0 = \text{id}$ and $G_1 = f'$.

3. Some technical lemmas

Since the proof of Theorem 2 is a bit more involved it will be convenient to describe some of the apparatus needed there.

LEMMA 3.1: *Let $M \subset F$ be a connected F-manifold which is a Z-set in F and let U be an open subset of F containing M. Then there exists a closed embedding $h : M \times I \to F$ such that $h(x, 0) = x$, for all $x \in M$, $h(M \times I) \subset U$, and $h(M \times \{1\}) = Bd_F(h(M \times I))$($Bd_F$ is the topological boundary operator in F).*

PROOF. Let $f : M \to F$ be an open embedding (which exists by [8]) and let $f' : M \times (-1, 1) \to F \times (-1, 1)$ be defined by $f'(x, t) = (f(x), t)$. By using motions only in the $(-1, 1)$ - direction we can easily construct a homeomorphism $\alpha : M \times (-1, 1) \to M \times (-1, 1)$ so that $f' \circ \alpha(M \times [0, 1))$ is closed in $F \times (-1, 1)$, thus $Bd_{F \times (-1, 1)}(f' \circ \alpha(M \times [0, 1))) = f' \circ \alpha(M \times \{0\})$. Let $\beta : F \times (-1, 1) \to F$ be a homeomorphism and let $\gamma : M \times [0, 1) \to M \times [0, 1]$ be a homeomorphism which satisfies $\gamma(M \times \{0\}) = M \times \{1\}$. (To see how to construct γ let $\gamma_1 : M \to M \times [0, 1)$ be a homeomorphism and let $\gamma_2 : [0, 1) \times [0, 1) \to [0, 1) \times [0, 1]$ be a homeomorphism satisfying $\gamma_2([0, 1) \times \{0\}) = [0, 1) \times \{1\}$. Then let $\gamma(x, t) = (\gamma_1^{-1}(p_1 \circ \gamma_1(x),\ p_1 \circ \gamma_2(p_2 \circ \gamma_1(x), t)),\ p_2 \circ \gamma_2(p_2 \circ \gamma_1(x), t))$, where we adopt the convention that p_i represents projection onto the i^{th} factor, for $i = 1, 2$.) Then $\beta \circ f' \circ \alpha \circ \gamma^{-1} : M \times [0, 1] \to F$ is a closed embedding satisfying $Bd_F(\beta \circ f' \circ \alpha \circ \gamma^{-1}(M \times [0, 1])) = \beta \circ f' \circ \alpha \circ \gamma^{-1}(M \times \{1\})$. Since $\beta \circ f' \circ \alpha \circ \gamma^{-1}(M \times \{0\})$ is a Z-set in F we can use the HET to get a homeomorphism $\delta : F \to F$ so that $\delta \circ \beta \circ f' \circ \alpha \circ \gamma^{-1}(x, 0) = x$, for all $x \in M$. Put $g = \delta \circ \beta \circ f' \circ \alpha \circ \gamma^{-1}$.
Now $g^{-1}(U)$ is an open subset of $M \times [0, 1]$ containing $M \times \{0\}$, and we can clearly construct a closed embedding $\phi : M \times [0, 1] \to M \times [0, 1]$ such that $\phi(x, 0) = (x, 0)$, for all $x \in M$, $\phi(M \times [0, 1]) \subset g^{-1}(U)$, and $\mathrm{Bd}(\phi(M \times [0, 1])) = \phi(M \times \{1\})$. Then $h = g \circ \phi$ fulfills our requirements.

LEMMA 3.2: *Let U be an open subset of a connected F-manifold M and let $K \subset U$ be a Z-set in U. Then there exists an open embedding $f : U \to M$ such that $f(K)$ is a Z-set in M and f is openly isotopic to* id_U.

PROOF: We may once more use [8] to assume that M is an open subset of F. Theorem 2.2 of [10] shows that $U \times F \cong U$, and it is clear that the proof given there can be modified slightly to get a homotopy $G : U \times F \times I \to U$ such that $G_0 = \pi_U$(projection onto U), $G_t : U \times F \to U$ is a homeomorphism for all $t \in (0, 1]$, and $\lim_{t \to 0} d(G_t, \pi_U) = 0$, where $d(G_t, \pi_U) = \sup \{d(G_t(x), \pi_U(x)) | x \in U \times F\}$ and d is a suitably chosen metric for F. Similarly let $H : M\times F\times I \to M$ be a homotopy such that $H_0 = \pi_M$, $H_t : M\times F \to M$ is a homeomorphism, for all $t \in (0, 1]$, and $\lim_{t\to 0} d(H_t, \pi_M) = 0$. Then define $f' = H_1 \circ G_1^{-1}$, which is an open embedding of U in M. We now show that f' can be modified to fulfill our requirements.

Observe that $H_t \circ G_t^{-1}$, $0 < t \leqq 1$, gives an isotopy of open embeddings of U in M. It is also clear that $\lim_{t\to 0} d(H_t \circ G_t^{-1}, \mathrm{id}_U) = 0$. Thus f' is openly isotopic to id_U. Using Theorem 1 of [6] there exists a homeomorphism $\alpha : U\times F \to U\times F$ so that $\alpha \circ G_1^{-1}(K) \subset U\times\{0\}$. It is also clear from the construction given in [6] that α may be chosen to be homotopic to id, thus by our Theorem 1 ambient isotopic to id. We can use motions only in the F-direction to construct a homeomorphism $\beta : U\times F \to U\times F$ so that $\beta\circ\alpha\circ G_1^{-1}(K)$ is a Z-set in $M\times F$ and β is ambient isotopic to id. Thus $f = H_1\circ\beta\circ\alpha\circ G_1^{-1}$ gives an open embedding of U in M which is openly isotopic to f' and for which $f(K)$ is a Z-set.

LEMMA 3.3: *Let M be an F-manifold and let $f, g : M \to M$ be closed bicollared embeddings of M into itself which are homotopy equivalences. Then there is a homeomorphism $h : M \to M$ such that $h|f(M) = g\circ f^{-1}$.* (A set A in a space X is *bicollared* provided that there exists an open embedding $\phi : A\times(-1, 1) \to X$ such that $\phi(a, 0) = a$, for all $a \in A$.)

PROOF: Since M is an ANR [9, page 3] we can use Lemma 11.3 of [5] to conclude that $M\backslash f(M) = M_1 \cup M_2$ and $M\backslash g(M) = N_1 \cup N_2$, where M_1, M_2 and N_1, N_2 are disjoint pairs of subsets of M such that $Cl(M_1) \cong Cl(M_2) \cong Cl(N_1) \cong Cl(N_2)$, $f(M)$ is a strong deformation retract of each of $Cl(M_1)$ and $Cl(M_2)$, $g(M)$ is a strong deformation retract of each of $Cl(N_1)$ and $Cl(N_2)$, $f(M)$ is collared in each of $Cl(M_1)$ and $Cl(M_2)$, and $g(M)$ is collared in each of $Cl(N_1)$ and $Cl(N_2)$.

To finish the proof we can imitate the proof of Theorem 9 of [5] provided that we use the HET and Theorem 6 of [8], which implies that every homotopy equivalence between F-manifolds is homotopic to a homeomorphism.

4. Proof of Theorem 2

Since each component of N is open we may assume, without loss of generality, that N is connected. We may also replace M by $M\times[0, 1)$. Using Lemma 3.2 we can additionally assume that $f(M\times\{0\})$ and $g(M\times\{0\})$ are Z-sets in N. Thus $f|M\times\{0\}$ and $g|M\times\{0\}$ are homotopic homeomorphisms of $M\times\{0\}$ onto Z-sets in N. We can use the HET to obtain an ambient invertible isotopy $G : N\times I \to N$ such that $G_0 = \mathrm{id}$

and $G_1 \circ f = g$ (on $M \times \{0\}$). Thus $f' = G_1 \circ f : M \times [0, 1) \to N$ is an open embedding which satisfies $f'|M \times \{0\} = g|M \times \{0\}$ and f' is openly isotopic to f. Then $(f')^{-1}(g(M \times [0, 1)))$ is an open subset of $M \times [0, 1)$ containing $M \times \{0\}$, and we can find an open embedding $\theta : M \times [0, 1) \to M \times [0, 1)$ such that $\theta(M \times [0, 1)) \subset (f')^{-1}(g(M \times [0, 1)))$, $\theta|M \times \{0\} =$ id, and θ is openly isotopic to id. Thus $f_1 = f' \circ \theta : M \times [0, 1) \to g(M \times [0, 1))$ is an open embedding which satisfies $f_1|M \times \{0\} = g|M \times \{0\}$ and for which f_1 is openly isotopic to f.

Using Lemma 3.1 and the fact that N can be embedded as an open subset of F, there is a closed embedding $h : M \times [0, 1] \to g(M \times [0, 1))$ such that $h(M \times [0, 1]) \subset f_1(M \times [0, 1))$, $h(x, 0) = g(x, 0)$, for all $x \in M$, and $Bd_N(h(M \times [0, 1])) = h(M \times \{1\})$. Using Lemma 11.3 of [5] as applied to our Lemma 3.3, it follows that for $A = g(M \times [0, 1)) \backslash h(M \times [0, \frac{1}{2}))$, $h(M \times \{\frac{1}{2}\})$ is collared in A and $h(M \times \{\frac{1}{2}\})$ is a strong deformation retract of A. Using the HET we can construct a homeomorphism $\alpha : M \times [0, 1) \to g(M \times [0, 1))$ such that $\alpha|M \times [0, \frac{1}{2}] = h|M \times [0, \frac{1}{2}]$. Note that there is a homeomorphism $\beta : M \times [0, \frac{1}{2}) \to M \times [0, 1)$ such that $\beta(x, 0) = (x, 0)$, for all $x \in M$ and β is openly isotopic to id.

Then $\alpha \circ \beta \circ \alpha^{-1} : h(M \times [0, \frac{1}{2})) \to g(M \times [0, 1))$ is a homeomorphism such that $\alpha \circ \beta \circ \alpha^{-1} \circ h(x, 0) = g(x, 0)$, for all $x \in M$, and $\alpha \circ \beta \circ \alpha^{-1}$ is openly isotopic to id. Let $\gamma = \alpha \circ \beta \circ \alpha^{-1}$ and similarly let $\delta : f_1(M \times [0, 1)) \to h(M \times [0, \frac{1}{2}))$ be a homeomorphism which satisfies $\delta \circ f_1(x, 0) = h(x, 0)$, for all $x \in M$, and δ is openly isotopic to id.

Then $g^{-1} \circ \gamma \circ \delta \circ f_1 : M \times [0, 1) \to M \times [0, 1)$ is a homeomorphism which satisfies $g^{-1} \circ \gamma \circ \delta \circ f_1(x, 0) = (x, 0)$, for all $x \in M$, and it is therefore homotopic to id. Using Theorem 1 there exists an ambient isotopy $\Phi : (M \times [0, 1)) \times I \to M \times [0, 1)$ such that $\Phi_0 = g^{-1} \circ \gamma \circ \delta \circ f_1$ and $\Phi_1 =$ id. Then $g \circ \Phi_t : M \times [0, 1) \to g(M \times [0, 1))$ is an isotopy of open embeddings such that $g \circ \Phi_0 = \gamma \circ \delta \circ f_1$ and $g \circ \Phi_1 = g$. Since γ and δ were constructed to be openly isotopic to id, it follows that f_1 is openly isotopic to g. As f_1 is openly isotopic to f we are done.

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(Oblatum 2–III–1971 & 7–V–1973)

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