COMPOSITIO MATHEMATICA

RALPH K. AMAYO

Locally coalescent classes of Lie algebras

Compositio Mathematica, tome 27, nº 2 (1973), p. 107-117
<http://www.numdam.org/item?id=CM_1973__27_2_107_0>

COMPOSITIO MATHEMATICA, Vol. 27, Fasc. 2, 1973, pag. 107-117
Noordhoff International Publishing
Printed in the Netherlands

LOCALLY COALESCENT CLASSES OF LIE ALGEBRAS

Ralph K. Amayo

1. Introduction

A result of Roseblade and Stonehewer [5] states that every subjunctive class of groups is locally coalescent. Whether the corresponding result for Lie algebras is true remains an open question. In this paper we will prove that certain 'nice' subjunctive classes of Lie algebras are locally coalescent. These include the classes $\mathfrak{N}$, $L\mathfrak{N}$, and $L\mathfrak{F}$ of nilpotent, locally nilpotent and locally finite Lie algebras respectively.

All Lie algebras considered in this paper will be of finite or infinite dimension over a fixed but arbitrary field k of characteristic zero, unless otherwise specified.

Notation and terminology for Lie algebras will be the same as in Stewart [6] p. 80–81 and Hartley [4] p. 260–261.

A class $\mathfrak{X}$ of Lie algebras is *coalescent* if and only if in any Lie algebra the join of a pair of $\mathfrak{X}$-subideals is always an $\mathfrak{X}$-subideal. We say that $\mathfrak{X}$ is *locally coalescent* if and only if whenever H and K are $\mathfrak{X}$-subideals of a Lie algebra L, then every finitely generated subalgebra C of $J = \langle H, K \rangle$ is contained in some $\mathfrak{X}$-subideal X of L with $C \leqq X \leqq J$.

Evidently if $\mathfrak{X}$ is a locally coalescent class, then $\mathfrak{G} \cap \mathfrak{X}$ is coalescent, where $\mathfrak{G}$ is the class of finitely generated Lie algebras.

A class $\mathfrak{X}$ is said to be *subjunctive* if and only if (i) subideals of $\mathfrak{X}$-algebras are again $\mathfrak{X}$-algebras and (ii) in any Lie algebra the join of any pair of $\mathfrak{X}$-ideals is in $\mathfrak{X}$.

The classes $\mathfrak{N}$, $\mathrm{L}\mathfrak{N}$, and $\mathrm{L}\mathfrak{F}$ all satisfy (i), and by lemmas 1 and 7 and the remark at the end of p. 265 of Hartley [4], they all satisfy (ii).

2. Preliminary results

Let A, B be (vector) subspaces of a Lie algebra L. We define $[A, B]$ to be the subspace spanned by Lie products of the form $[a, b]$ where $a \in A$ and $b \in B$. Inductively we define $[A, {}_nB] = [[A, {}_{n-1}B], B]$ if $n > 0$, and put $A = [A, {}_0B]$. We let A^B denote the smallest subspace containing A

and invariant under Lie multiplication by the elements of B. Evidently

$$(1) \qquad A^B = \sum_{i=0}^{\infty} [A, {}_iB]$$

We say that A and B are *permutable* if and only if

$$[A, B] \subseteq A+B.$$

Let $A_1 = \langle A \rangle$ be the subalgebra generated by A and suppose that A and B are permutable. Now A_1 is spanned (qua vector space) by elements of the form $[a_1, \cdots, a_n]$ where $a_1, \cdots, a_n \in A$ and $n > 0$. If $b \in B$, then by the Jacobi identity

$$[[a_1, \cdots, a_n], b] = [a_1, \cdots, a_{n-1}, b, a_n] + [[a_1, \cdots, a_{n-1}], [a_n, b]].$$

From this and a straight forward induction on n we get $[a_1, \cdots, a_n, b] \in A_1+B$ for all $a_1, \cdots, a_n \in A$ and $b \in B$. Thus $[A_1, B] \subseteq A_1+B$. Similarly if $B_1 = \langle B \rangle$, then $[A_1, B_1] \subseteq A_1+B_1$. Thus A_1 and B_1 are permutable.

This together with lemma 4 of Amayo [1] yields

LEMMA 1: (*Over any field*). *Let $\mathfrak{X}$ be a subjunctive class and let H, K be X-subideals of a Lie algebra L. If A, B are permutable subspaces of H and K respectively, then there is an $\mathfrak{X}$-subideal X of L such that*

$$\langle A, B \rangle \leqq X \leqq H+K.$$

(Here $H+K$ denotes the subspace of L spanned by H and K; it may not be a subalgebra of L.)

We recall briefly a construction of Hartley [4] p. 265–266. Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic zero. Let $\mathfrak{k}_0 = \mathfrak{k}\langle t \rangle$ be the field of formal power series in the indeterminate t with coefficients in $\mathfrak{k}$; a typical element α of k_0 is of the form

$$\alpha = \sum_{r=n}^{\infty} \alpha_r t^r$$

$\alpha_r \in \mathfrak{k}$, $n = n(\alpha)$ is an integer; addition and multiplication is defined in the usual way. We denote by $L^{\dagger}$, the set of all formal power series

$$x = \sum_{r=n}^{\infty} x_r t^r$$

with coefficients $x_r \in L$ and where n is an integer depending on x. Let $y = \sum y_r t^r$ be another element of $L^{\dagger}$ and define addition, multiplication. and multiplication of elements of $L^{\dagger}$ by scalars from $\mathfrak{k}_0$ according to the rules

$$x+y = \sum (x_r + y_r)t^r,$$

$$[x, y] = \sum z_r t^r, \quad \text{where } z_r = \sum_{i+j=r} [x_i, y_j]$$

$$\alpha x = \sum u_r t^r, \quad \text{where } u_r = \sum_{i+j=r} \alpha_i x_j .$$

This makes $L^{\uparrow}$ into a Lie algebra over $\mathfrak{k}_0$ and L is contained in $L^{\uparrow}$ as a subset.

Now let A be any subspace of L and let $A^{\uparrow}$ be the set of elements $x = \sum x_r t^r$ of $L^{\uparrow}$ such that every $x_r \in A$. This clearly makes $A^{\uparrow}$ a ($\mathfrak{k}_0$-) subspace of $L^{\uparrow}$. If also A is a subalgebra of L, then it is easy to see that $A^{\uparrow}$ is a ($\mathfrak{k}_0$-) subalgebra of $L^{\uparrow}$.

From the definitions above we obtain

I
(a) *If* $A \lhd H \leqq L$, *then* $A^{\uparrow} \lhd H^{\uparrow} \leqq L^{\uparrow}$,
(b) *If* $H \leqq L$, *then for any* $n > 0$, $(H^{\uparrow})^n \leqq (H^n)^{\uparrow}$ *and* $(H^{\uparrow})^{(n)} \leqq (H^{(n)})^{\uparrow}$,
(c) *If* A *is finite dimensional over* $\mathfrak{k}$, *then* $A^{\uparrow}$ *is finite dimensional over* $\mathfrak{k}_0$.

Evidently any non-zero element x of $L^{\uparrow}$ may be expressed uniquely in the form

$$x = t^n \sum_{r=0}^{\infty} x_r t^r,$$

where $x_0 \neq 0$. We shall refer to x_0 as the *first coefficient* of x. Now let $y = t^m \sum_{r=0}^{\infty} y_r t^r$, where $y_0 \neq 0$, and let $\alpha, \beta \in k$. Then

$$\alpha x + t^{n-m}\beta y = t^n \sum_{r=0}^{\infty} (\alpha x_r + \beta y_r) t^r$$

and $[x, y] = t^{n+m}[x_0, y_0]+$terms involving higher powers of t.

Let M be a subspace (i.e. $\mathfrak{k}_0$-vector subspace) of $L^{\uparrow}$ and denote by $M^{\downarrow}$, the collection of all first coefficients of elements of M together with the zero element. Then the equations above show that

II
(a) *If* M *is a subspace* (*resp. subalgebra*) *of* $L^{\uparrow}$, *then* $M^{\downarrow}$ *is a subspace* (*resp. subalgebra*) *of* L,
(b) *If* $M \lhd N \leqq L^{\uparrow}$, *then* $M^{\downarrow} \lhd N^{\downarrow} \leqq L$,
(c) *If* $M \leqq L^{\uparrow}$, *then for each* $n > 0$, $(M^n)^{\downarrow} = (M^{\downarrow})^n$ and $(M^{(n)})^{\downarrow} = (M^{\downarrow})^{(n)}$,
(d) *If* A *is a subspace of* L, *then* $(A^{\uparrow})^{\downarrow} = A$.

Let d be a derivation of L. We define a mapping exp (td) of $L^{\uparrow}$ by the following; for $x = \Sigma x_r t^r \in L^{\uparrow}$,

$$x^{\exp(td)} = \sum u_r t^r, \quad \text{where } u_r = \sum_{i+j=r} x_i d^j / j!.$$

Then it is easy to show that $\exp(td)$ is a Lie automorphism of L. Now let $d_1, \cdots, d_s$ be derivations of L and let $e = \exp(td)$, $e_i = \exp(td_i)$, $1 \leqq i \leqq s$.

If z is any element of L, then as z can be considered as an element of $L^{\uparrow}$, we have

$$z^e - z = \sum_{r=1}^{\infty} (zd^r/r!)t^r \in L^{\uparrow}. \tag{2}$$

Thus

$$zd = 0 \text{ or } 1^{\text{st}} \text{ coefficient of } z^e - z. \tag{3}$$

Put

$$w_s = \sum_{\substack{1 \leqq i_1 < \cdots < i_j \leqq s \\ 0 \leqq j \leqq s}} (-1)^{s-j} z^{(e_{i_1} \cdots e_{i_j})}. \tag{4}$$

Furthermore if A is a subspace of L, define

$$A^* = A \otimes_{\mathfrak{k}} \mathfrak{k}_0.$$

Then A^* can be considered as naturally embedded in $L^{\uparrow}$, and consists of elements $\Sigma\, x_r t^r$ such that all the coefficients x_r lie in some finite dimensional subspace of A; it is obviously a subspace of $L^{\uparrow}$ and $(A^*)^{\downarrow} = A$.

LEMMA 2:
(a) *$zd_1 \cdots d_s = 0$ or 1^{st} coefficient of w_s;*
(b) *If A is a subspace of L then there exists a finite number $\alpha_1, \cdots, \alpha_k$ ($k \leqq 2^s$) of automorphisms of $L^{\uparrow}$ such that if $M = A^{*\alpha_1} + \cdots + A^{*\alpha_k}$, then $Ad_1 \cdots d_s \subseteq M^{\downarrow}$.*

PROOF:
(a) For $s = 1$, this follows from (3). Let $s > 1$ and suppose that the result is true for $s-1$. Let $w = zd_1 \cdots d_{s-1}$. If $w \neq 0$, then w is the first coefficient of w_{s-1}. Thus by (3), $zd_1 \cdots d_s = wd_s = 0$ or 1^{st} coefficient of $w^{e_s} - w$. But it is easy to see that

$$w_{s-1}{}^{e_s} - w_{s-1} = w_s$$

and so the result follows for s.

(b) Let $\{\alpha_1, \cdots, \alpha_k\} = \{e_{i_1} \cdots e_{i_j}, 0 \leqq j \leqq s \text{ and } 1 \leqq i_1 < \cdots < i_j \leqq s\}$. Then (4) and (a) above show that for every element z of A, $zd_1 \cdots d_s$ is either zero, or the first coefficient of some element of M. Hence the required result follows.

It is not very hard to see that if $1 \leqq i_1 < \cdots < i_j \leqq s$, then

$$Ad_{i_1} \cdots d_{i_j} \subseteq M^{\downarrow}, \quad \text{for } 0 \leqq j \leqq s.$$

Let A, B be subspaces of L and let $\exp(B)$ = group of automorphisms of

L generated by all $\exp\ (t \operatorname{ad}(b))$ with $b \in B$ and $\operatorname{ad}(b)$ denoting the adjoint map of L defined by b.
Define

$$N = \sum_{\alpha \in \exp(B)} A^{*\alpha}.$$

It follows from (1) and lemma 2 that

$$A^B = N^{\downarrow} \tag{5}$$

(for clearly $N \subseteq (A^B)^{\uparrow}$ and by II(d) $((A^B)^{\uparrow})^{\downarrow} = A^B$)

We note that if $L^* = L \otimes_k k_0$, then L^* is a subalgebra of $L^{\uparrow}$. Furthermore if in I we replace '$\uparrow$' by '$*$' then the results of I remain true; indeed the inequalities of I(b) may be replaced by equality signs.

DEFINITION: A class $\mathfrak{X}$ of Lie algebras is said to be *complete* if and only if (i) if $L \in \mathfrak{X}$, and L is defined over a field $\mathfrak{k}$ (of characteristic zero), then $L^* \in \mathfrak{X}$, as a Lie algebra over $\mathfrak{k}_0$ and (ii) if L is a Lie algebra over $\mathfrak{k}$ and $M \leqq L^*$ and $M \in \mathfrak{X}$, then $M^{\downarrow} \in \mathfrak{X}$ as a Lie algebra over $\mathfrak{k}$.

REMARK: Previously by a class $\mathfrak{X}$ of Lie algebras, we have meant a collection of Lie algebras all defined over a fixed field $\mathfrak{k}$, together with their isomorphic copies and the zero dimensional algebra. In the definition above we have extended a class to include not only algebras over $\mathfrak{k}$, but others which are defined over $\mathfrak{k}_0$ (and so over $\mathfrak{k}$) and are members of the class only as algebras over $\mathfrak{k}_0$. Thus in a complete class $\mathfrak{X}$, a member over $\mathfrak{k}$, gives rise to one over $\mathfrak{k}_0$; and one over over $\mathfrak{k}_0$ yields by restriction another member over $\mathfrak{k}$.

It is clear from I and II that the classes $\mathfrak{N}$ and $\mathrm{E}\mathfrak{A}$ of nilpotent and soluble Lie algebras are complete.

Suppose that L is defined over $\mathfrak{k}$ and $L \in \mathfrak{F}$ $\mathfrak{F}$ is the class of finite dimensional Lie algebras). Then by II, $L^{\uparrow} \in \mathfrak{F}$ and so $L^* \in \mathfrak{F}$. Conversely suppose that $M \leqq H^*$ and $M \in \mathfrak{F}$. Let $u_1, \cdots, u_r$ be a basis for M over $\mathfrak{k}_0$. By the remarks just before lemma 2, each each u_i can be written in the form $u_i = \Sigma\, u_{ij} t^j$, where all the u_{ij} lie in some finite dimensional subspace A_i of H. Thus $u_i \in A_i \otimes_{\mathfrak{k}} \mathfrak{k}_0 = A_i{}^*$. Hence if $A = \Sigma\, A_i$, then $M \leqq A^*$, so $M^{\downarrow} \leqq (A^*)^{\downarrow} = A$ and therefore $M^{\downarrow} \in \mathfrak{F}$. Hence $\mathfrak{F}$ is complete. From this it is trivial to show that $\mathrm{L}\mathfrak{F}$ is also complete.

If $L \in \mathrm{L}\mathfrak{N}$, then as above for any finite subset S of L^* we can find a finitely generated subalgebra A of L such that $S \subseteq A^*$; but $A \in \mathfrak{N}$, since $L \in \mathrm{L}\mathfrak{N}$, and so by *I*, $A^* \in \mathfrak{N}$. Conversely if $M \in \mathrm{L}\mathfrak{N}$, and $M \leqq H^*$ for some H, then for any finite subset T of $M^{\downarrow}$ we can find a finitely generated subalgebra N of M such that $T \subseteq N^{\downarrow}$; but $N \in \mathfrak{N}$, implies by II that $N^{\downarrow} \in \mathfrak{N}$. Hence $M^{\downarrow} \in \mathrm{L}\mathfrak{N}$. Therefore $\mathrm{L}\mathfrak{N}$ is a complete class.

REMARK: In general if $M \leqq L^{\uparrow}$, for some L and $M \in \mathrm{L}\mathfrak{N}$, then $M^{\downarrow} \in \mathrm{L}\mathfrak{N}$. However it is easy to see that if $L \in \mathrm{L}\mathfrak{N}$, then $L^{\uparrow}$ need not be locally nilpotent.

Evidently if M and A are (respectively) ascending unions of $\mathfrak{k}_0$ and $\mathfrak{k}$-vector spaces, $M = \cup_i M_i$ and $A = \cup_i A_i$, then

$$M^{\downarrow} = \bigcup_i M_i^{\downarrow} \quad \text{and} \quad A^* = \bigcup_i A_i^*.$$

We recall also (see for instance Stewart [6]) that a Lie algebra is said to be hypercentral if and only if it has an ascending central series. We denote by $\mathfrak{Z}$, the class of all such algebras. Then by the definitions and the remarks above it follows that $\mathfrak{Z}$ is complete. It is well known to be subjunctive.

Thus we have

LEMMA 3: *The classes* $\mathfrak{F}$, $\mathfrak{N}$, $\mathrm{E}\mathfrak{A}$, $\mathfrak{Z}$, $\mathrm{L}\mathfrak{N}$, $\mathrm{L}\mathfrak{F}$ *are all complete and subjunctive.*

Let L be a Lie algebra over $\mathfrak{k}$ (a field of characteristic zero) and d a derivation of L. Let $e = \exp(td)$ be the automorphism of L induced by d. In general L^* is not invariant under e. However if we make the

DEFINITION: A *derivation* d of a Lie algebra L is said to be *locally finite* if and only if every finite subset of L is contained in a finite dimensional d-invariant subspace of L.
(Thus a nil derivation is necessarily locally finite) Then we have

LEMMA 4: *Suppose that L is a Lie algebra and H a* $\mathrm{L}\mathfrak{F}$*-subideal of L. Then every element of H induces a locally finite derivation of L.*

Also from Stewart [6] p. 85 we have

LEMMA 5: *Let L be a Lie algebra over a field of characteristic zero, H a subspace of L and d a locally finite derivation of L which leaves H invariant. Then if* $e = \exp(td)$,

$$H^{*e} = H^*.$$

We recall that if B is a subspace of L then $\exp(B)$ is the group of automorphisms of $L^{\uparrow}$ generated by all $\exp(t \operatorname{ad}(b))$ with $b \in B$.

LEMMA 6: *Let K be a* $\mathrm{L}\mathfrak{F}$*-subideal of a Lie algebra L, B a finite dimensional subspace of K, and A any finite dimensional subspace of L. Then there exists a finite number* $\alpha_1, \cdots, \alpha_r$ *of elements of* $\exp(B)$ *such that if* $D = A^{*\alpha_1} + \cdots + A^{*\alpha_r}$, *then*

$$A^B = D^{\downarrow}.$$

PROOF: Suppose that $K \lhd^m L$. Then $[A,_m B] \subseteq K$ and so as $K \in \mathrm{L}\mathfrak{F}$

and $[A,_m B]$ is finite dimensional, we have $\langle [A,_m B], B\rangle \in \mathfrak{F}$. Hence $\Sigma_{i=m}^{\infty}[A,_i B]$ is a finite dimensional space. But $A^B = A+[A,B]+\cdots+[A,_{m-1}B]+\Sigma_{i=m}^{\infty}[A,_i B]$ and so A^B is finite dimensional. Hence so is $(A^B)^*$. By lemmas 4 and 5, $(A^B)^*$ is invariant under exp (B) and so contains $N = \Sigma_{\alpha\in\exp(B)}A^{*\alpha}$; thus N is finite dimensional. Hence for some $\alpha_1,\cdots,\alpha_r \in \exp(B)$, $N = D = A^{*\alpha_1}+\cdots+A^{*\alpha_r}$. But by (5), $A^B = N^{\downarrow}$ and the result follows.

3. The Main results

THEOREM (A): *(Over fields of characteristic zero). If $\mathfrak{X}$ is a complete and subjunctive class of Lie algebras and $\mathfrak{X} \leqq$ L$\mathfrak{F}$, then $\mathfrak{X}$ is locally coalescent. In particular $\mathfrak{G} \cap \mathfrak{X}$ is a coalescent class.*

PROOF: Let H, K be $\mathfrak{X}$-subideals of a Lie algebra L and $J = \langle H, K\rangle$. Clearly every finitely generated subalgebra of J is contained in one of the form $C = \langle A, B\rangle$, where A, B are finite dimensional subspaces of H, K respectively. So it is enough to show that we can find an $\mathfrak{X}$-subideal X of L such that $C \leqq X \leqq J$.

Now $H, K \in$ L$\mathfrak{F}$, and so if D is the subspace of L^* defined in lemma 6, then

$$A^B+B = D^{\downarrow}+B$$

so $D^{\downarrow}$ and B are permutable. Let $M_i = \langle H^{*\alpha_1},\cdots, H^{*\alpha_i}\rangle$ for $1 \leqq i \leqq r$. Then $D \subseteq M_r$.

Suppose that $H \lhd^m L$. We induct on m to show that we can find an X with the required properties. If $m = 1$, then $H \lhd L$, so $J = H+K$ and J si L, by lemma 5 of Hartley [4]. Furthermore H and K are permutable and so by lemma 1, $J \in \mathfrak{X}$. Thus we may take J for X. Suppose $M > 1$, and the result true for $m-1$. Let $H_1 = \langle H^L\rangle$. By lemma 5, H_1^* is invariant under exp (B). Hence as $H \lhd^{m-1} H_1$ it follows by I that

$$H^{*\alpha_i} \lhd^{m-1} H_1^* \quad \text{for } 1 \leqq i \leqq r.$$

Furthermore $\mathfrak{X}$ is a complete class, so $H^* \in \mathfrak{X}$ and hence

$$H^{*\alpha_i} \in \mathfrak{X} \quad \text{for } 1 \leqq i \leqq r.$$

Therefore by the inductive hypothesis on $m-1$ and a second simple induction on i, it follows that given any $\mathfrak{G}$-subalgebra C_i of M_i, there exists an $\mathfrak{X}$-subideal X_i of H_1^* such that

$$C_i \leqq X_i \leqq M_i.$$

In particular we can find an $\mathfrak{X}$-subideal X_r of H_1^* such that $D \subseteq X_r \leqq M_r$. Now $X_r^{\downarrow} \in \mathfrak{X}$ (since $\mathfrak{X}$ is complete) and $X_r^{\downarrow}$ si $H_1 \lhd L$. Finally $D^{\downarrow}$ and B

are permutable. Thus applying lemma 1 to $X_r^{\downarrow}$ and K, there exists an $\mathfrak{X}$-subideal X of L such that

$$D^{\downarrow}+B \subseteq X \leqq X_r^{\downarrow}+K.$$

But $X_r \leqq M_r \leqq (H^B)^*$, so $X_r^{\downarrow}+K \subseteq J$; and $C = \langle A,B\rangle = \langle A^B\rangle + B \subseteq \langle D^{\downarrow}, B\rangle$ so $C \leqq X$. This completes our induction on m and with it the proof of theorem A.

It is well known that every finitely generated and nilpotent Lie algebra is finite dimensional. Furthermore $\mathfrak{N} \leqq \mathfrak{Z} \leqq \mathrm{L}\mathfrak{N} \leqq \mathrm{L}\mathfrak{F}$. Thus by lemma 3 we have

THEOREM (B): (Over fields of characteristic zero) *The classes* $\mathfrak{N}$, $\mathfrak{Z}$, $\mathrm{L}\mathfrak{N}$, *and* $\mathrm{L}\mathfrak{F}$ *are all locally coalescent. In particular the classes* $\mathfrak{F} = \mathfrak{G} \cap \mathrm{L}\mathfrak{F}$ *and* $\mathfrak{G} \cap \mathfrak{N} = \mathfrak{F} \cap \mathfrak{N}$ *are coalescent.*

REMARK: The result that $\mathfrak{F}$ and $\mathfrak{G} \cap \mathfrak{N}$ are coalescent has been proved by Hartley [4], using different methods.

Following [5] we may define for each class $\mathfrak{X}$, the classes

$$\mathrm{J}_s\mathfrak{X},\ \mathrm{L}_s\mathfrak{X}$$

by $L \in \mathrm{J}_s\mathfrak{X}$ if and only if L can be generated by its $\mathfrak{X}$-subideals; $L \in \mathrm{L}_s\mathfrak{X}$ if and only if every finitely generated subalgebra of L is contained in some $\mathfrak{X}$-subideal of L. Thus for any $\mathfrak{X}$, $\mathrm{L}_s\mathfrak{X} \leqq \mathrm{J}_s\mathfrak{X}$ and if $\mathfrak{X}$ is locally coalescent then it is not very hard to show that $\mathrm{J}_s\mathfrak{X} = \mathrm{L}_s\mathfrak{X}$, and that $\mathrm{J}_s\mathfrak{X}$ is locally coalescent.

So we have

COROLLARY (C): *If* $\mathfrak{X}$ *is a complete and subjunctive class and* $\mathfrak{X} \leqq \mathrm{L}\mathfrak{F}$, *then* $\mathrm{J}_s\mathfrak{X} = \mathrm{L}_s\mathfrak{X}$ *and* $\mathrm{J}_s\mathfrak{X}$ *is locally coalescent.*

However $\mathfrak{N} < \mathrm{J}_s\mathfrak{N}$ and $\mathfrak{F} < \mathrm{J}_s\mathfrak{F}$. But we have

$$\mathrm{L}\mathfrak{N} = \mathrm{J}_s\mathrm{L}\mathfrak{N} \text{ and } \mathrm{L}\mathfrak{F} = \mathrm{J}_s\mathrm{L}\mathfrak{F}.$$

DEFINITION: (Hartley (unpublished)). Let L be a Lie algebra and define the 'locally nilpotent radical' $\beta^*(L)$ by

$$\beta^*(L) = \langle H \mid H \in L\mathfrak{N} \text{ and } H \text{ si } L\rangle.$$

Then it follows from above that $\beta^*(L) = \cup \{H \mid H \in \mathrm{L}\mathfrak{N} \text{ and } H \text{ si } L\}$. Hence $\beta^*(L) \in \mathrm{L}\mathfrak{N}$ (a result also obtained by Hartley in an unpublished paper).

PROPOSITION (D): *Let L be a Lie algebra over a field of characteristic zero. Then $\beta^*(L)$ is invariant under every locally finite derivation of L. In particular if L is locally finite, then $\beta^*(L) \lhd L$.*

PROOF: Let H be a $\mathrm{L}\mathfrak{N}$-subideal of L and d a locally finite derivation of L and $e = \exp(td)$. By lemma 5 $L^{*^e} = L^*$. By lemma 3, $H^* \in \mathrm{L}\mathfrak{N}$; hence H^* and H^{*^e} are $\mathrm{L}\mathfrak{N}$-subideals of L^*. Thus if $M = \langle H^*, H^{*^e}\rangle \leqq L^*$, then $M \leqq \beta^*(L^*)$ and so $M \in \mathrm{L}\mathfrak{N}$.

If $x \in H$, then xd is either zero or the first coefficient of $x^e - x \in M$. Thus we can find, by our previous remarks, a $\mathrm{L}\mathfrak{N}$-subideal N of L^* with $x^e - x \in N$. Thus $xd \in N^{\downarrow}$ and $N^{\downarrow}$ si L and $N^{\downarrow} \in LN$ so $N^{\downarrow} \leqq \beta^*(L)$.

4. Applications

A Lie algebra L satisfies the *Engel condition* if and only if for any $x, y \in L$ there is an integer $m = m(x, y)$ such that $[x, {}_m y] = 0$. We denote by $\mathfrak{E}$, the class of all Lie algebras satisfying the Engel condition.

THEOREM (E): (Over fields of characteristic zero) *If $\mathfrak{X}$ is a subjunctive class and $\mathfrak{X} \leqq \mathfrak{E}$, then $\mathfrak{X}$ is locally coalescent.*

The proof of theorem E is similar to that of theorem A but we need a few more facts.

A derivation d of a Lie algebra L is called a *nil derivation* if for every $x \in L$, there is a positive integer $m = m(x)$ such that $xd^m = 0$. Thus every element of an Engel algebra induces a nil derivation. If d is a nil derivation of a Lie algebra L over a field of characteristic zero, then the map

$$\exp(d) = \sum_{n=0}^{\infty} d^n/n!$$

is a well defined Lie automorphism of L. By lemma 3 of Hartley [4] we have

LEMMA 7: *If A is a finite dimensional subspace and d a nil derivation of a Lie algebra L, then*

$$A^d = \sum_{i=1}^{t} A^{\alpha_i},\ A^d = \sum_{i=0}^{\infty} Ad^i$$

for some $\alpha_1, \cdots, \alpha_t \in$ group $\langle \exp(d)\rangle$. ($A^\alpha = \{a^\alpha; a \in A\}$) As an obvious corollary we have

COROLLARY (7.1): *Suppose that K is an $\mathfrak{E}$-subideal of a Lie algebra L, B is a finite dimensional subspace of K and A is a finite dimensional subspace of L. Then for each $r \geqq 0$, there exists a finite number of automorphisms $\alpha_1, \cdots, \alpha_s \in \langle \exp(\mathrm{ad}(b)) | b \in B\rangle$ of L such that*

$$\sum_{i=0}^{r} [A, {}_i B] = \sum_{j=1}^{s} A^{\alpha_j}.$$

(We note that $s = s(r)$).

From [2] we have

THEOREM 8: (*The Derived Join Theorem*). *Let L be a Lie algebra, $H \lhd^m L$, $K \lhd^n L$ and $J = \langle H, K\rangle$. Then there exists $q = q(m, n)$ such that*

$$J^{(q)} \lhd^q L \quad \textit{and} \quad J^{(q)} \leqq H+K.$$

PROOF of Theorem E: Let L be any Lie algebra (over a field of characteristic zero), $H \lhd^m L$, $K \lhd^n L$, $J = \langle H, K\rangle$ and A, B finite dimensional subspaces of H, K respectively and suppose $H, K \in \mathfrak{X}$. To prove theorem E it is enough to show that there is an $\mathfrak{X}$-subideal X of L such that $A^B + B \subseteq X \leqq J$.

As before for $m = 1$, the result is true. Let $m > 1$, and assume the usual inductive hypothesis. Let q be defined as in theorem 8, and put $M = J^{(q)}$. It is well known (see for instance [3]) that

$$\mathfrak{E} \cap \mathrm{E}\mathfrak{A} \leqq \mathrm{L}\mathfrak{N}.$$

Thus $(H+M)/M \in \mathfrak{E} \cap \mathrm{E}\mathfrak{A} \leqq \mathrm{L}\mathfrak{N}$ and similarly $(K+M)/M \in \mathrm{L}\mathfrak{N}$. Hence $J/M = \beta^*(J/M) \in \mathrm{L}\mathfrak{N}$ and so we can find r_1 such that $[A, {}_{r_1}B] \subseteq M$. Let $r = r_1 - 1$. Then

$$[A, {}_{r+1}B] \subseteq M \cap A^B.$$

Let $N = \langle H, M\rangle$. Then $N = H+M \leqq H+K$, by theorem 8, so $N = H + N \cap K$. But $N \cap K$ si K (since $N = H+M$ si L) and so $N \cap K \in \mathfrak{X}$. Thus N is the join of two permutable $\mathfrak{X}$-subideals, H and $N \cap K$, and so be lemma 1, $N \in \mathfrak{X}$. But $M \lhd N$, and so $M \in \mathfrak{X}$.

From (1) and the result above,

$$A^B = \sum_{i=1}^{r} [A, {}_iB] + A^B \cap M$$

and so if $Y = \langle A^B\rangle$, then

$$Y = A_1 + Y \cap M,$$

where $A_1 = \langle \Sigma_{i=1}^r [A, {}_iB]\rangle$. By corollary 7.1, we can find $\alpha_1, \cdots, \alpha_s \in \langle \exp(\mathrm{ad}(b)) | b \in B\rangle$ for which

$$A_1 = \langle A^{\alpha_1}, \cdots, A^{\alpha_s}\rangle.$$

By the inductive hypothesis on $m-1$, in the same way as in the proof of theorem A, we can find an $\mathfrak{X}$-subideal X_0 of L such that

$$A_1 \leqq X_0 \leqq \langle H^{\alpha_1}, \cdots, H^{\alpha_s}\rangle \leqq J.$$

Now $Y = A_1 + Y \cap M$, implies A_1 and $Y \cap M$ are permutable and so applying lemma 1 to X_0 and M, we can find an $\mathfrak{X}$-subideal X_1 of L such that

$$Y = \langle A^B \rangle \leqq X_1 \leqq X_0 + M \leqq J.$$

Finally $\langle A^B \rangle$ and B are permutable and so by lemma 1 applied to X_1 and K, there is an $\mathfrak{X}$-subideal X of L such that

$$\langle A^B \rangle + B \subseteq X \leqq X_1 + K \subseteq J.$$

This completes our induction on m and the proof of theorem E.

REMARKS: Strictly speaking theorem E is independent of theorem A. To see this we note that as $(K+M)/M \in \mathrm{L}\mathfrak{N}$ and $[A, {}_nB]$ and B are both finite dimensional subspaces of K, then there exists a k such that $[A, {}_{n+k}B] \subseteq M$. We could then use $n+k$ in place of r_1.

We can also obtain a corollary similar to corollary C.

REFERENCES

[1] R. K. AMAYO: The Derived Join Theorem and coalescence in Lie algebras. *Compositio Math.* (to appear).
[2] R. K. AMAYO: Soluble subideals of Lie algebras. *Compositio Math.* (to appear).
[3] K. W. GRUENBERG: Two theorems on Engel groups. *Proc. Camb. Phil. Soc.*, *49* (1953) 377–380.
[4] B. HARTLEY: Locally nilpotent ideals of a Lie algebra. *Ibid.*, *63* (1967) 257–272.
[5] J. E. ROSEBLADE and S. E. STONEHEWER.: Subjunctive and locally coalescent classes of groups. *J. Algebra*, *8* (1968) 423–435.
[6] I. N. STEWART: Structure theorems for a class of locally finite Lie algebras. *Proc. London Math. Soc.*, *(3) 24* (1972) 79–100.

(Oblatum: 13–X–1972)

Mathematics Institute
University of Warwick
Coventry, England