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NUCLEAR SPACES OF MAXIMAL DIAMETRAL DIMENSION

by

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The diametral dimension $\Delta(E)$ of a locally convex vector space E is known to be a measure for the nuclearity of E. Therefore it is of interest to characterize the class Ω of those locally convex vector spaces, the diametral dimension of which is maximal. We show that the class Ω has the same stability properties as the class $\mathcal{N}$ of all nuclear spaces, and characterize the members of Ω, that are contained in the smallest stability class, by a property of their bornology. At first let us define what we mean by a stability class:

DEFINITION. (a) A *stability class* is a class of locally convex vector spaces, which is closed under the operations of forming

(S_1) completions
(S_2) subspaces
(S_3) quotients by closed subspaces
(S_4) arbitrary products
(S_5) countable direct sums
(S_6) tensor products
(S_7) isomorphic images.

(b) If E is a locally convex vector space, we denote by $\sigma(E)$, *the stability class of* E, the smallest stability class containing E.

Let us remark, that in (S_6) we choose the projective ($\pi-$) topology on the tensor product; but as we shall be solely concerned with nuclear spaces, we could equally well have chosen the ($\varepsilon-$) topology of bi-equicontinuous convergence. Note further, that because of (S_4) a stability class (if not empty) will always be a proper class.

Examples of stability classes are

the class $\mathcal{S}$ of all Schwartz spaces (cf. e.g. [4]),
the class $\mathcal{N}$ of nuclear spaces, or more generally,
the class $\mathcal{N}_\phi$ of ϕ-nuclear spaces, which are defined as follows:

DEFINITION. Let Φ denote the set of all continuous, subadditive, strictly increasing functions $\phi : [0, \infty) \to [0, \infty)$ vanishing at 0. Let $\phi \in \Phi$; a locally convex vector space E is a member of $\mathcal{N}_\phi$, the class of

ϕ-nuclear spaces, if for every neighbourhood U of 0 in E there is a neighbourhood V of 0 contained in U, such that $\sum \phi(\delta_n(V, U)) < \infty$, where $\delta_n(V, U)$ denotes the n-th Kolmogorov diameter of V with respect to U [1].

By a theorem of Rosenberger [6] for $\phi \in \Phi \mathcal{N}_\phi$ is a stability class, and so is

$$\mathcal{N}_\Phi : = \bigcap_{\Phi} \mathcal{N}_\phi .$$

A further example of a stability class is given by the spaces of maximal diametral dimension:

Definition. If E is a locally convex space, we denote by $\Delta(E)$, *the diametral dimension of* E, the set of all nonnegative sequences δ, such that for each neighbourhood U of 0 in E there is a neighbourhood V of 0 contained in U, such that

$$\lim_{n\to\infty} \delta_n(V, U) \cdot \delta_n^{-1} = 0$$

(cf. [1]).

Call ω the set of all strictly positive non-increasing sequences of real numbers, and Ω the class of all locally convex vector spaces, such that $\omega \subset \Delta(E)$. The following proposition will show, that Ω is a stability class of nuclear spaces.

Proposition. $\mathcal{N}_\Phi = \Omega$.

Proof. (a) $\mathcal{N}_\Phi \subset \Omega$: Let $E \in \mathcal{N}_\Phi$, $\delta \in \omega$, U a neighbourhood of 0 in E. Choose $\phi \in \Phi$, such that for all $n \in N$ $\phi(\delta_n) > 1/(n+1)$: such a function may be obtained by considering the 'upper boundary' of the closed convex hull of $\{(0,0)\} \cup \{(\delta_n, 1/(n+1)); n \in N\}$. As ϕ is subadditive, $\phi(\delta_n/(n+1)) > \phi(\delta_n)/(n+1) > 1/(n+1)^2$. If $\phi \in \Phi$, $\sqrt{\phi}$ will also be in Φ, so there is a neighbourhood W of 0 such that $\sum \sqrt{(\phi(\delta_n(W, U)))} < \infty$, hence

$$\lim_{n\to\infty} (n+1)^2 \phi(\delta_n(W, U)) = 0,$$

and so we may find a neighbourhood V of 0, such that for all $n \in N$ $\phi(\delta_n(V, U)) < (n+1)^{-2} < \phi(\delta_n/(n+1))$, which means

$$\lim_{n\to\infty} \delta_n(V, U)\delta_n^{-1} = 0$$

and consequently $\delta \in \Delta(E)$.

(b) $\Omega \subset \mathcal{N}_\Phi$: Let $E \in \Omega$, $\phi \in \Phi$, U a neighbourhood of 0 in E. Choose a neighbourhood V of 0, such that $\delta_n(V, U) < \phi^{-1}(1/(n+1)^2)$.

Corollary. *Ω is a stability class.*

We shall now be investigating the smallest nontrivial stability class, i.e. $\sigma(\boldsymbol{R})$.

THEOREM. *A locally convex vector space is in* $\sigma(\boldsymbol{R})$, *if and only if* E *is isomorphic to a subspace of a product of* $\oplus_N \boldsymbol{R}$.

PROOF[1]. It suffices to prove the 'only if' part. We introduce an auxiliary class Σ as follows: A locally convex vector space E belongs to Σ, if E possesses a basis $\mathscr{U}(E)$ of neighbourhoods of 0, such that for all $U \in \mathscr{U}(E)$ $E/\ker p_U$ (with the quotient topology) is isomorphic to a subspace of $\oplus_N \boldsymbol{R}$, where p_U denotes the seminorm associated with U. Note that a subspace of $\oplus_N \boldsymbol{R}$ is again an at most countable sum of real lines. Note further, that E is a subspace of a product of $\oplus_N \boldsymbol{R}$ if and only if $E \in \Sigma$. For, suppose E is subspace of

$$X := (\bigoplus_N \boldsymbol{R})^A$$

and choose a neighbourhood U_0 of 0 in X, such that

$$X/\ker p_{U_0} = \bigoplus_N \boldsymbol{R}.$$

Let $U = U_0 \cap E$, then $\ker p_U = \ker p_{U_0} \cap E$, and we have a continuous injection $i : E/\ker p_U \to X/\ker p_{U_0}$. But $X/\ker p_{U_0}$ carries the finest locally convex topology, so $E/\ker p_U$ is itself an at most countable sum of real lines. The theorem will be proven, if we show, that the class of locally convex spaces which are subspaces of a product of $\oplus_N \boldsymbol{R}$ is a stability class. So let us check conditions (S_1) to (S_6):

(S_1): If E is a subspace of $(\oplus_N \boldsymbol{R})^A$, then $\tilde{E}$ is just the closure of E in $(\oplus_N \boldsymbol{R})^A$, since $(\oplus_N \boldsymbol{R})^A$ is complete.

(S_2): If E is a subspace of $(\oplus_N \boldsymbol{R})^A$, and F is a subspace of E, then clearly F is a subspace of $(\oplus_N \boldsymbol{R})^A$.

(S_3): Let F be a closed subspace of $E \in \Sigma, U \in \mathscr{U}(E)$, $\pi_F : E \to E/F$, $\pi_V : E/F \to (E/F)/\ker p_{\pi_F(U)}$, $\pi_U : E \to E/\ker p_U$ canonical projections, $V := \pi_F(U)$. We shall show, that any seminorm $q : (E/F)/\ker p_V \to \boldsymbol{R}$ is continuous. $W := q^{-1}([0, 1))$ is absorbing and absolutely convex, and so is $\pi_U \pi_F^{-1} \pi_V^{-1}(W)$. But since $E/\ker p_U$ is isomorphic to a subspace of $\oplus_N \boldsymbol{R}$, $\pi_U \pi_F^{-1} \pi_V^{-1}(W)$ contains an open neighbourhood $\mathcal{O}$ of 0. Then the open neighbourhood $\pi_V \pi_F \pi_U^{-1}(\mathcal{O})$ will be contained in $W + \pi_V \pi_F(\ker p_U)$; and since $F + \ker p_U \subset \ker p_V$, $\pi_V \pi_F(\ker p_U) = 0$.

[1] We are very grateful to the referee for drawing our attention to a slip in the first version of this proof.

So W is indeed a neighbourhood of 0. As $(E/F)/\ker p_V$ is a nuclear space carrying the finest locally convex topology, it must be isomorphic to a subspace of $\oplus_N \boldsymbol{R}$.

(S_4) and (S_5) are clear.

(S_6): Let

$$E, F \in \Sigma,\ U \in \mathscr{U}(E),\ V \in \mathscr{U}(F),\ \pi_U : E \to E/\ker p_U,$$
$$\pi_V : E \to E/\ker p_V,\ \rho : E \otimes_\pi F \to (E \otimes_\pi F)/\ker (p_U \otimes p_V)$$

canonical projections. Consider a seminorm q on $E \otimes_\pi F/\ker (p_U \otimes p_V)$. Then $W := q^{-1}([0, 1))$ is absorbing and absolutely convex; hence $(\pi_U \otimes \pi_V)\rho^{-1}(W)$ is an absorbing and absolutely convex set in

$$E/\ker p_U \otimes_\pi F/\ker p_V \cong \bigoplus_{N'} \boldsymbol{R} \otimes_\pi \bigoplus_{N''} \boldsymbol{R} = \bigoplus_{N' \times N''} \boldsymbol{R},$$

where N' and N'' are at most countable. So $(\pi_U \otimes \pi_V)\rho^{-1}(W)$ contains an open neighbourhood $\mathcal{O}$ of 0, and so does W, since the open neighbourhood $\rho(\pi_U \otimes \pi_V)^{-1}(\mathcal{O})$ is contained in $W + \rho(\ker \pi_U \otimes \pi_V)$ and $\rho(\ker \pi_U \otimes \pi_V) = 0$. That means, that $E \otimes_\pi F/\ker p_U \otimes p_V$ carries the finest locally convex topology, so it is again a subspace of $\oplus_N \boldsymbol{R}$.

Remark. Diestel, Morris and Saxon [2] define a 'variety' of locally convex spaces as a class, which is closed under the operations (S_2), (S_3), (S_4), and (S_7). They show, that $\sigma(\boldsymbol{R})$ is the second smallest variety.

At this stage the question naturally arises, whether Ω actually equals $\sigma(\boldsymbol{R})$. One feels that this should be true, if the diametral dimension of a space is indeed a measure for its 'nuclearity', for this would mean, that maximal diametral dimension should determine the smallest stability class. On the other hand, the following proposition may perhaps provide a method to refute the equality $\sigma(\boldsymbol{R}) = \Omega$.

Proposition. *Let $E \in \Omega$. Then $E \in \sigma(\boldsymbol{R})$, if and only if E has the following property*

(PB) *there is a basis $\mathscr{U}$ of neighbourhoods of 0 in E, such that for all $U \in \mathscr{U}$ $E/\ker p_U$ is bornological.*

Proof. If $E \in \sigma(\boldsymbol{R}) = \Sigma$, E clearly has property (PB), since an at most countable sum of real lines is bornological. Conversely, let $E \in \Omega$ and $U \in \mathscr{U}$. Now note, that bounded sets in $E/\ker p_U$ are finite-dimensional. For, if B is bounded in $E/\ker p_U$, for each non-increasing sequence δ of positive reals

$$\lim_{n \to \infty} \delta_n^{-1} \delta_n(B, U_0) = 0$$

(where U_0 is the image of U in $E/\ker p_U$), since $E/\ker p_U$ is in Ω, too. This means, that $\delta_n(B, U_0) = 0$ for $n > n_0$, which implies, that B is finite-dimensional, since p_{U_0} is a norm. Now choose an algebraic basis

$$(e_i)_{i \in I} \text{ for } E/\ker p_U.$$

The identity map

$$\mathrm{id} : \bigoplus_I \boldsymbol{R} \cdot e_i \to E/\ker p_U$$

is a continuous bijection; and if $E/\ker p_U$ is bornological, id will be open, too, hence an isomorphism. But then I must be at most countable, since $E/\ker p_U$ is nuclear.

Taking a different approach, one could try to prove the equality $\sigma(\boldsymbol{R}) = \Omega$ by showing, that $\sigma(\boldsymbol{R})$ and Ω are generated by the same ideal of operators. This is, however, not possible, since the equality $\Omega = \mathcal{N}_\Phi$ implies, that neither $\sigma(\boldsymbol{R})$ nor Ω is generated by an ideal:

DEFINITION. Let $\mathcal{I}$ be an ideal of operators. *The class of locally convex vector spaces generated by* $\mathcal{I}$ consists of all locally convex vector spaces E with the following property: For each neighbourhood U of 0 in E there is a neighbourhood V of 0 contained in U, such that the canonical map $E(V, U) : E_V \to E_U$ belongs to $\mathcal{I}$ (E_U denotes the completion of the normed vector space $(E, p_U)/\ker p_U$).

THEOREM. *$\sigma(\boldsymbol{R})$ and Ω are not generated by an ideal.*

PROOF. Suppose, there is an ideal $\mathcal{I}$ generating $\sigma(\boldsymbol{R})$ or Ω. We shall obtain a contradiction by constructing a function $\phi \in \Phi$ and a locally convex vector space E in the class generated by $\mathcal{I}$, such that E is not ϕ-nuclear. As $\sigma(\boldsymbol{R})$ and Ω consist of nuclear spaces, we may assume, that $\mathcal{I}$ is an ideal of compact operators between separable Hilbert spaces. As the ideal of operators with finite-dimensional images does not generate a stability class, $\mathcal{I}$ contains an operator S with infinite-dimensional range. By combining S with a partial isometry, we may obtain a compact self-adjoint operator $T \in \mathcal{I}$, such that the eigenvalues (λ_n) of T form a decreasing sequence of positive reals. By construction, the sequence space

$$\Lambda := \{\xi \mid \forall k \in \boldsymbol{N} \sum |\xi_n| \lambda_n^{-k} < \infty\}$$

will be in the class generated by $\mathcal{I}$. But clearly, Λ is not ϕ-nuclear, if we choose a function $\phi \in \Phi$, such that for all $n \in \boldsymbol{N}$ $\phi(\lambda_n^n) > 1/(n+1)$.

Finally we observe, that $\mathcal{N}$ and $\sigma(\boldsymbol{R})$ share still another 'restricted' stability property, as is shown by the following

PROPOSITION. *A Fréchet space E belongs to $\sigma(\boldsymbol{R})$, if and only if E_b' belongs to $\sigma(\boldsymbol{R})$.*

This proposition may equally well be stated as

PROPOSITION. *A Fréchet space E belongs to Ω, if and only if E'_b belongs to Ω.*

PROOF. (a) Let $E \in \Omega$. E has property (PB), so $E \in \sigma(\boldsymbol{R})$. As we proved already, this implies, that E possesses a basis $\mathscr{U}$ of neighbourhoods of 0, such that for $U \in \mathscr{U}$ $E/\ker p_U$ is a subspace of $\oplus_{\boldsymbol{N}} \boldsymbol{R}$. So $E/\ker p_U$ being a Fréchet space, too, is finite dimensional, which means, that E is a closed subspace of $\Pi_{\boldsymbol{N}} \boldsymbol{R}$. Then E is itself an at most countable product of real lines, so $E' \in \sigma(\boldsymbol{R})$.

(b) If $E' \in \Omega$, E' is nuclear, so E is nuclear and reflexive. Choose a hilbertian neighbourhood U of 0 in E' and a hilbertian bounded set B in E'. Then we have for all $n \in \boldsymbol{N}$ $\delta_n(B, U) = \delta_n(B^0, U^0)$. So, if B is bounded in E and U is a neighbourhood of 0 in E, the image of B in $E/\ker p_U$ is finite-dimensional. As E has property (PB), this implies, as we have already seen, that $E \in \Sigma = \sigma(\boldsymbol{R})$.

We did not include this stability property in our definition of a stability class, since there exist stability classes of nuclear spaces, which do not possess this property, e.g. the class of strongly nuclear spaces (cf. [5]).

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