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**CORRECTION TO ‘ON THE PURITY  
 OF THE BRANCH LOCUS’**

by

Allen B. Altman and Steven L. Kleiman

The proof ([2, p. 464]) fails because the algebra of principal parts  $P^m(A)$  is not a finitely generated  $A$ -module. However, the proof does go through if we replace  $P^m(A)$  by the algebra of topological principal parts  ${}^tP^m(A)$ , defined below. We check this after proving two preliminary results of independent interest.

**PROPOSITION.** *Let  $R$  be a ring,  $q$  an ideal of  $R$ , and  $M$  a finitely generated  $R$ -module. Consider the following separated completions:*

$$\hat{R} = \varprojlim (R/q^n); \quad \hat{M} = \varprojlim (M/q^n M).$$

*Assume  $\hat{R}$  is noetherian and  $q$  is finitely generated. Then there is a canonical isomorphism,*

$$\hat{R} \otimes_R M = \hat{M}.$$

**PROOF.** With no finiteness assumptions on  $\hat{R}$  and  $q$ , the canonical map,

$$\hat{R} \otimes_R M \rightarrow \hat{M},$$

is surjective (the proof is straightforward, see [3, p. 108]). Therefore, since  $q$  is finitely generated, we have an equality,

$$q^n \hat{R} = (q^n)^\wedge,$$

for each positive integer  $n$ . So, since  $q^n \hat{R}$  is obviously equal to  $(q\hat{R})^n$ , we obtain an equality,

$$(\hat{q})^n = (q^n)^\wedge.$$

Consequently, (GD II, 1.10), there is a canonical isomorphism,

$$(1) \quad \hat{R}/(\hat{q})^n = R/q^n.$$

Hence,  $\hat{R}$  is equal to  $\varprojlim (\hat{R}/(\hat{q})^n)$ ; in other words,  $\hat{R}$  is separated and complete with respect to the  $\hat{q}$ -adic topology.

Since  $\hat{R}$  is noetherian and  $\hat{R} \otimes_R M$  is, obviously, finitely generated over  $\hat{R}$ , the  $\hat{q}$ -adic separated completion  $(\hat{R} \otimes_R M)^\wedge$  is equal, by (GD

II, 1.18), to  $\hat{R} \otimes_{\hat{R}} (\hat{R} \otimes_R M)$ , so to  $\hat{R} \otimes_R M$ ; in other words, we have a canonical isomorphism,

$$\hat{R} \otimes_R M = \varinjlim ((\hat{R} \otimes_R M)/(\hat{q})^n(\hat{R} \otimes_R M)).$$

Now, by basic properties of tensor product and by (1), for each  $n$  we have

$$\begin{aligned} (\hat{R} \otimes_R M)/(\hat{q})^n(\hat{R} \otimes_R M) &= (\hat{R}/(\hat{q})^n) \otimes_{\hat{R}} (\hat{R} \otimes_R M) \\ &= (\hat{R}/(\hat{q})^n) \otimes_R M \\ &= (R/q^n) \otimes_R M \\ &= M/q^n M. \end{aligned}$$

Passing to the projective limit over  $n$ , we obtain the proposition.

In the next two results, let  $k$  be a noetherian ring, let  $A$  be a noetherian  $k$ -algebra that is separated and complete with respect to the adic topology of an ideal  $m$  such that  $K = A/m$  is a finitely generated  $k$ -algebra, and let  $B$  be an  $A$ -algebra that is a finitely generated  $A$ -module. The complete tensor products  $A \hat{\otimes}_k A$  and  $B \hat{\otimes}_k B$  are defined as the separated completions of  $A \otimes_k A$  and  $B \otimes_k B$  with respect to the adic topology of the ideals,

$$\begin{aligned} M &= (m \otimes_k A + A \otimes_k m) \\ N &= ((mB) \otimes_k B + B \otimes_k (mB)) = M(B \otimes_k B). \end{aligned}$$

The  $k$ -algebras of  $m$ th order topological principal parts are defined by

$$\begin{aligned} {}^tP^m(A) &= (A \hat{\otimes}_k A)/I^{m+1} \\ {}^tP^m(B) &= (B \hat{\otimes}_k B)/J^{m+1} \end{aligned}$$

where  $I$  (resp.  $J$ ) denotes the kernel of the map  $A \hat{\otimes}_k A \rightarrow A$  (resp.  $B \hat{\otimes}_k B \rightarrow B$ ) that takes  $a \hat{\otimes} b$  to  $ab$ .

**COROLLARY.** *Under the above conditions,  $A \hat{\otimes}_k A$  is noetherian and there is a canonical  $(A \hat{\otimes}_k A)$ -algebra isomorphism,*

$$(A \hat{\otimes}_k A) \otimes_{(A \hat{\otimes}_k A)} (B \otimes_k B) = B \hat{\otimes}_k B.$$

**PROOF.** The ring  $(A \otimes_k A)/M$  is noetherian, for it is equal to  $K \otimes_k K$ , which is, clearly, a finitely generated algebra over the noetherian ring  $k$ . Moreover,  $M$  is a finitely generated ideal of  $(A \otimes_k A)$ , for  $m$  is an ideal in the noetherian ring  $A$ . Hence,  $A \hat{\otimes}_k A$  is noetherian (GD II, 1. 22). Clearly,  $B \otimes_k B$  is a finitely generated  $(A \otimes_k A)$ -module. Therefore, the second assertion follows from the proposition.

**LEMMA.** *Under the above conditions, assume that the structure morphism,*

$$f: \text{Spec}(B) \rightarrow \text{Spec}(A),$$

is étale over a nonempty open subset  $V$  of  $\text{Spec}(A)$ .

(i) For each  $m \geq 0$ , the  $(A \hat{\otimes}_k A)$ -algebra homomorphisms,

$$\begin{aligned} {}_m v : {}^t P^m(A) \otimes_A B &\rightarrow {}^t P^m(B), & v_m : B \otimes_A {}^t P^m(A) &\rightarrow {}^t P^m(B) \\ (a \hat{\otimes} a') \otimes b &\mapsto a \hat{\otimes} (a'b) & b \otimes (a \hat{\otimes} a') &\mapsto (ab) \hat{\otimes} a', \end{aligned}$$

are isomorphisms over  $V$ , where  ${}^t P^m(A)$  and  ${}^t P^m(B)$  are regarded as  $A$ -algebras first from the right, then from the left.

(ii) The canonical map,

$$v : gr_I^*(A \hat{\otimes}_k A) \otimes_A B \rightarrow gr_J^*(B \hat{\otimes}_k B),$$

is an isomorphism over  $V$ , where  $I$  (resp.  $J$ ) denotes the kernel of the map  $A \hat{\otimes}_k A \rightarrow A$  (resp.  $B \hat{\otimes}_k B \rightarrow B$ ) that takes  $a \hat{\otimes} b$  to  $ab$ .

PROOF. (i) Filtered by the powers of  $I$  (resp. of  $I$ , resp. of  $J$ ), the  $(A \hat{\otimes}_k A)$ -algebra  ${}^t P^m(A) \otimes_A B$  (resp.  $B \otimes_A {}^t P^m(A)$ , resp.  ${}^t P^m(B)$ ) is separated and complete, the filtration being finite; so, by (GD II, 1.5, 1.21), it suffices to prove that  $gr^*({}_m v)$  and  $gr^*(v_m)$  are isomorphisms over  $V$ .

Consider the composition

$$[gr_I^*({}^t P^m(A))] \otimes_A B \rightarrow gr_I^*({}^t P^m(A) \otimes_A B) \rightarrow gr_J^*({}^t P^m(B)).$$

The right hand map is obviously equal to both  $gr({}_m v)$  and  $gr(v_m)$ . The left hand map is an isomorphism over  $V$  since  $f$  is flat over  $V$ . Finally, the composition is a truncation of  $v$ , so an isomorphism by (ii). Thus, (i) holds.

(ii) Consider the following diagram:

$$\begin{array}{ccccc} \text{Spec}(B \otimes_A B) & \longrightarrow & \text{Spec}(B \hat{\otimes}_k B) & \longrightarrow & \text{Spec}(B \otimes_k B) \\ \downarrow & & \square & & \downarrow \\ \text{Spec}(A) & \longrightarrow & \text{Spec}(A \hat{\otimes}_k A) & \longrightarrow & \text{Spec}(A \otimes_k A). \end{array}$$

The right hand square is cartesian by the corollary; the left, by the following computation involving the corollary:

$$\begin{aligned} (2) \quad A \otimes_{(A \hat{\otimes}_k A)} (B \hat{\otimes}_k B) &= A \otimes_{(A \hat{\otimes}_k A)} [(A \hat{\otimes}_k A) \otimes_{(A \otimes_k A)} (B \otimes_k B)] \\ &= A \otimes_{(A \otimes_k A)} (B \otimes_k B) = (B \otimes_A B). \end{aligned}$$

The right vertical map is equal to  $f \times f$ , and  $f \times f$  is, by (GD V, 2.7 iv), flat over  $V \times V$ . Hence, by (GD V, 2.7. iii), the middle vertical map is flat over the inverse image of  $V \times V$  in  $\text{Spec}(A \hat{\otimes}_k A)$ . Therefore, by (GD V, 3.2), the canonical map of modules over  $A = (A \hat{\otimes}_k A)/I$ ,

$$v' : [gr_I^*(A \hat{\otimes}_k A)] \otimes_A (B \otimes_A B) \rightarrow gr_I^*(B \hat{\otimes}_k B)$$

is an isomorphism over  $V$ , because  $(B \hat{\otimes}_k B)/I(B \hat{\otimes}_k B)$  is equal to  $B \otimes_A B$  by (2).

Since  $f$  is unramified over  $V$ , the diagonal map,

$$\text{Spec}(B) \rightarrow \text{Spec}(B \otimes_A B),$$

is an open embedding (GD VI, 3.3). However, it is also the closed embedding defined by  $J$ . Now, whenever  $Z, Y$  are two closed subschemes of a scheme  $X$ , and  $Z$  is an open subscheme of  $Y$ , then the canonical map,

$$gr_{I(Y)}^{\bullet}(\mathcal{O}_X)|_Z \rightarrow gr_{I(Z)}^{\bullet}(\mathcal{O}_Z),$$

is an isomorphism, where  $I(Y)$  is the ideal of  $Y$  and  $I(Z)$ , that of  $Z$ ; indeed, the assertion is local on  $X$  and need only be checked on  $Z$ , and the ideals  $I(Y)$  and  $I(Z)$  coincide in a neighborhood of each point of  $Z$ . Therefore, the canonical map,

$$v'' : gr_i^{\bullet}(B \hat{\otimes}_k B) \otimes_{(B \otimes_A B)} B \rightarrow gr_j^{\bullet}(B \hat{\otimes}_k B),$$

is an isomorphism over  $V$ . So, since  $v = v'' \circ (v' \otimes_{(B \otimes_A B)} B)$  holds,  $v$  is also an isomorphism over  $V$ .

**THEOREM.** *Let  $k$  be a noetherian ring,  $A = k[[T_1, \dots, T_n]]$  a formal power series ring. Let  $B$  be a finite  $A$ -algebra that is étale over every prime ideal  $p$  of  $A$  where  $\text{depth}(B_p) \leq 1$  holds. Then, there exists a canonical  $A$ -algebra isomorphism  $u_0 : A \otimes_k B_0 \xrightarrow{\sim} B$  with  $B_0 = B/(T_1 B + \dots + T_n B)$ .*

**PROOF.** Give  $A$  the  $(T_1 A + \dots + T_n A)$ -adic topology. We are going to construct an isomorphism of  $(A \hat{\otimes}_k A)$ -algebras,

$$u : (A \hat{\otimes}_k A) \otimes_A B \rightarrow B \otimes_A (A \hat{\otimes}_k A),$$

where  $A \hat{\otimes}_k A$  is regarded as an  $A$ -algebra via the second factor in  $(A \hat{\otimes}_k A) \otimes_A B$  and via the first in  $B \otimes_A (A \hat{\otimes}_k A)$ . Then,  $u$  yields  $u_0$  as follows. Consider the diagram,

$$\begin{array}{ccc} A & \xleftarrow{w} & A \hat{\otimes}_k A \\ j \uparrow & & \uparrow j_2 \\ k & \xleftarrow{e} & A \end{array}$$

where  $j$  is the structure map, where  $j_2(a) = 1 \hat{\otimes} a$  holds, where  $e(a)$  is the constant term of  $a$ , and where  $w(a_1 \hat{\otimes} a_2) = e(a_2) \cdot a_1$  holds. The diagram is obviously commutative and so we have a canonical isomorphism,

$$A \otimes_{(A \hat{\otimes}_k A)} [(A \hat{\otimes}_k A) \otimes_A B] = A \otimes_k (k \otimes_A B).$$

Since  $k \otimes_A B$  is obviously equal to  $B_0$ , we obtain

$$A \otimes_{(A \hat{\otimes}_k A)} [(A \hat{\otimes}_k A) \otimes_A B] = A \otimes_k B_0.$$

On the other hand, setting  $j_1(a) = a \hat{\otimes} 1$ , we have  $w \circ j_1 = id_A$ , and so we have another canonical isomorphism,

$$A \otimes_{(A \hat{\otimes}_k A)} [B \otimes_A (A \hat{\otimes}_k A)] = B.$$

Therefore,  $A \otimes_{(A \hat{\otimes}_k A)} u$  is equal to the desired isomorphism  $u_0$ .

We now construct  $u$ . Since  $A$  is a formal power series ring,  ${}^tP^m(A)$ , regarded as an  $A$ -algebra either on the left or right, clearly has the form

$${}^tP^m(A) = A[[U_1, \dots, U_n]]/(U_1, \dots, U_n)^{m+1},$$

where  $U_1, \dots, U_n$  are indeterminates (in fact,  $U_i = T_i \hat{\otimes} 1 - 1 \hat{\otimes} T_i$  holds); thus,  ${}^tP^m(A)$  is a free  $A$ -module of finite rank, say  $r$ . Therefore,  $B \otimes_A {}^tP^m(A)$  and  ${}^tP^m(A) \otimes_A B$  are both isomorphic to  $B^{\oplus r}$ . Hence, by the hypothesis on  $B$ , these  $A$ -modules have depth  $\leq 1$  only at points of  $\text{Spec}(A)$  over which  $B$  is étale.

Consider the  $A$ -module,

$$M = \text{Hom}_A ({}^tP^m(A) \otimes_A B, B \otimes_A {}^tP^m(A)),$$

where both arguments are considered as  $A$ -modules on the left (so the second is isomorphic to  $B^{\oplus r}$ , but not necessarily the first). The lemma implies that both arguments are canonically isomorphic to  ${}^tP^m(B)$  as  $(A \hat{\otimes}_k A)$ -algebras over the open subset  $V$  of  $\text{Spec}(A)$  where  $B$  is étale; hence, since by (EGA I, 1.3.12) we have

$$\tilde{M} = \underline{\text{Hom}} (({}^tP^m(A) \otimes_k B)^\sim, (B \otimes_A {}^tP^m(A))^\sim),$$

$\tilde{M}$  has a canonical section over  $V$ . By Lemma 2 ([2], p. 463),  $V$  contains every point  $p$  where  $\text{depth}(M_p) \leq 1$  holds. So, by Lemma 3(ii) ([2], p. 463), this section is defined by an element  $u_m$  of  $M$ ; in fact, by Lemma 3(i) ([2], p. 463),  $u_m$  is an  $(A \hat{\otimes}_k A)$ -algebra homomorphism since it is on  $V$ . Similarly, we obtain an inverse to  $u_m$  (first on  $V$ , then globally).

Clearly  $A \hat{\otimes}_k A$  is  $I$ -adically separated and complete. So, since  $(A \hat{\otimes}_k A) \otimes_A B$  is a finitely generated  $(A \hat{\otimes}_k A)$ -module, it is also  $I$ -adically separated and complete. By right exactness of  $\otimes_A B$ , we have

$${}^tP^m(A) \otimes_A B = ((A \hat{\otimes}_k A) \otimes_A B)/(I^{m+1}((A \hat{\otimes}_k A) \otimes_A B)).$$

Hence, we have

$$(A \hat{\otimes}_k A) \otimes_A B = \varinjlim ({}^tP^m(A) \otimes_A B).$$

Similarly, we have

$$B \otimes_A (A \hat{\otimes}_k A) = \varinjlim (B \otimes_A {}^tP^m(A)).$$

Finally, the various isomorphisms clearly form a compatible system of maps, so they induce the desired  $(A \hat{\otimes}_k A)$ -algebra isomorphism  $u$ .

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