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DERIVATIONS OF VECTOR FIELDS

by

Floris Takens*

1. Statement of the result

Let M be a differentiable, i.e. C^∞ , manifold. We denote the Lie-algebra of C^∞ vectorfields on M by $\chi(M)$. A map $D : \chi(M) \rightarrow \chi(M)$ is called a *derivation* if D is \mathbf{R} -linear and if $D([X, Y]) = [D(X), Y] + [X, D(Y)]$ for all $X, Y \in \chi(M)$. It is clear that every $X \in \chi(M)$ defines a derivation $DX : DX(Y) = [X, Y]$. In this note we want to show that every derivation can be obtained in this way.

THEOREM. *For each derivation $D : \chi(M) \rightarrow \chi(M)$ there is a vectorfield $Z \in \chi(M)$, such that for each $X \in \chi(M)$, $D(X) = [Z, X]$.*

This theorem has a certain relation with recent work of M. Gel'fand, D. B. Fuks, and others [1] on the cohomology of Lie-algebras of smooth vectorfields, because it implies that $H^1(\chi(M); \chi(M)) = 0$; $H^1(\chi(M); \chi(M))$ being the first cohomology group of $\chi(M)$ with coefficient in $\chi(M)$ with the adjoined representation (this was pointed out to me by M. Hazewinkel). There is however one difference in their approach: in defining their cohomology they only use cochains which are continuous mappings (with respect to the C^∞ topology). It is however not difficult to show that the nullity of $H^1(\chi(M); \chi(M))$ follows from our theorem in either case.

The theorem will follow from the following lemmas:

LEMMA 1. *Let $D : \chi(M) \rightarrow \chi(M)$ be a derivation and let $X \in \chi(M)$ be zero on some open subset $U \subset M$. Then $D(X)|U \equiv 0$.*

LEMMA 2. *Let $X \in \chi(\mathbf{R}^n)$ be a vectorfield on \mathbf{R}^n with $j^3(X)(0) = 0$, i.e. the 3-jet of each of the component functions of X is zero in the origin. Then there are vectorfields $Y_1, \dots, Y_q, Z_1, \dots, Z_q$ and there is a neighbourhood U of the origin in \mathbf{R}^n such that:*

$$X|U = \sum_i [Y_i, Z_i]|U \quad \text{and} \\ j^1(Y_i)(0) = 0, \quad j^1(Z_i)(0) = 0 \quad \text{for all } i = 1, \dots, q.$$

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LEMMA 3. Let $D : \chi(M) \rightarrow \chi(M)$ be a derivation and let $X \in \chi(M)$ and $p \in M$ be such that $j^3(X)(p) = 0$. Then $D(X)(p) = 0$. In other words, $DX(p)$ is determined by $j^3(X)(p)$, also if $j^3(X)(p) \neq 0$.

Lemma 3 will be derived from the lemmas 1 and 2. Finally we shall use lemma 3 to derive:

LEMMA 4. Let $U \subset \mathbf{R}^n$ be an open connected and simply connected set and let $D_U : \chi(U) \rightarrow \chi(U)$ be a derivation. Then there is a unique vectorfield $Z \in \chi(U)$ such that $D_U(X) = [Z, X]$ for all $X \in \chi(U)$.

Finally, we shall see that the theorem follows from lemma 1 and lemma 4.

2. The Proofs

PROOF OF LEMMA 1. Suppose $X|_U \equiv 0$ and $D(X)(q) \neq 0$ for some point $q \in U$. We take a vectorfield $Y \in \chi(M)$ such that $\text{supp}(Y) \subset U$ and $[D(X), Y](q) \neq 0$. By definition we have $D[X, Y] = [DX, Y] + [X, DY]$; evaluating this in q we get $0 = [DX, Y](q) \neq 0$, which contracts our assumption. Hence the lemma is proved.

PROOF OF LEMMA 2. It is clearly enough to show the lemma for the case

$$X = X(x_1, \dots, x_n) \frac{\partial}{\partial x_1},$$

with $j^3(X)(0) = 0$. Such vectorfields can be written as a finite sum of vectorfields of the following two types:

type I:

$$\tilde{X} = x_1^{m_1} \cdot \dots \cdot x_n^{m_n} \cdot \alpha(x_2, \dots, x_n) \frac{\partial}{\partial x_1},$$

with $\sum m_i \geq 4$ and α a C^∞ function;

type II:

$$\tilde{X} = x_1^4 \cdot g(x_2, \dots, x_n) \frac{\partial}{\partial x_1}$$

with g a C^∞ function.

To prove the lemma we show that each vectorfield, which is either of type I or of type II, can be written as the Lie-product of two vectorfields with zero 1-jet in the origin. For \tilde{X} of type I as above, we observe that

$$\tilde{X} = \left[\frac{1}{k_1 - h_1} \cdot x_1^{h_2} \cdot \dots \cdot x_n^{h_n} \frac{\partial}{\partial x_1}, x_1^{k_2} \cdot \dots \cdot x_n^{k_n}, \alpha(x_2, \dots, x_n) \frac{\partial}{\partial x_1} \right],$$

$$h_1 + k_1 = m_1 + 1 \text{ and } h_1 \neq k_1$$

$$h_2 + k_2 = m_2$$

$$\vdots$$

$$h_n + k_n = m_n.$$

Using the fact that $\sum m_i \geq 4$, we see that we can choose $h_1, \dots, h_n, k_1, \dots, k_n$ so that $\sum h_i \geq 2$ and $\sum k_i \geq 2$; hence for type one we have the required Lie-product.

Suppose that that

$$\tilde{X} = x_1^4 \cdot g(x_1, \dots, x_n) \frac{\partial}{\partial x_1}$$

is of type II. We want to show that there is a function H , defined on a neighbourhood of the origin in \mathbf{R}^n such that

$$\tilde{X} = \left[x_1^2 H(x_1, \dots, x_n) \frac{\partial}{\partial x_1}, x_1^2 \frac{\partial}{\partial x_1} \right]$$

in a neighbourhood of the origin. The existence of such H follows from

SUB-LEMMA (2.1) *Let Z, X be vectorfields on \mathbf{R}^1 , which depend on real variables μ_1, \dots, μ_r , and which can be written in the form*

$$Z = x^k \cdot f(x, \mu_1, \dots, \mu_r) \frac{\partial}{\partial x}, \quad X = x^l \cdot g(x, \mu_1, \dots, \mu_r) \frac{\partial}{\partial x}$$

where f, g are C^∞ functions on \mathbf{R}^{r+1} (at least on a neighbourhood of the origin), $l \geq 2k$ and $f(0, 0, \dots, 0) \neq 0$.

Then there is a vectorfield Y , also depending on μ_1, \dots, μ_r , of the form

$$Y = x^k \cdot H(x, \mu_1, \dots, \mu_r) \frac{\partial}{\partial x},$$

such that

$$[Y, Z] = X$$

for all (x, μ_1, \dots, μ_r) in a small neighbourhood of the origin in \mathbf{R}^{r+1} .

PROOF of (2.1).

$$[Y, Z] = X \text{ or } \left[x^k \cdot H(x, \mu) \frac{\partial}{\partial x}, x^k \cdot f(x, \mu) \frac{\partial}{\partial x} \right] = x^l \cdot g(x, \mu) \frac{\partial}{\partial x}$$

is equivalent with

$$\begin{aligned} x^k \cdot H(x, \mu) \cdot \left[k \cdot x^{k-1} \cdot f(x, \mu) + x^k \frac{\partial f}{\partial x}(x, \mu) \right] \\ - x^k \cdot f(x, \mu) \left[k \cdot x^{k-1} \cdot H(x, \mu) + x^k \frac{\partial H}{\partial x}(x, \mu) \right] = x^l \cdot g(x, \mu). \end{aligned}$$

The terms with x^{2k-1} cancel and $l \geq 2k$, so we can divide by x^{2k} and obtain:

$$H(x, \mu) \cdot \frac{\partial f}{\partial x}(x, \mu) - f(x, \mu) \frac{\partial H}{\partial x}(x, \mu) = x^{l-2k} \cdot g(x, \mu)$$

Restricting ourselves to a small neighbourhood of the origin in the (x, μ) space, we may divide by f and obtain:

$$\frac{\partial H}{\partial x}(x, \mu) = \frac{\frac{\partial f}{\partial x}(x, \mu)}{f(x, \mu)} \cdot H(x, \mu) - x^{l-2k} \cdot \frac{g(x, \mu)}{f(x, \mu)}$$

This is an ordinary differential equation depending on the parameters $\mu = (\mu_1, \dots, \mu_r)$. Hence, by the existence and smoothness of solutions of differential equations depending on parameters, it follows that there is a function H which has the required properties.

PROOF OF LEMMA 3. For X and p as in the statement of the lemma (i.e. $j^3(X)(p) = 0$) we can find, using local coordinates and lemma 2, a neighbourhood U of p in M and vectorfields on M Y_1, \dots, Y_q and Z_1, \dots, Z_q such that

$$X|U = \sum_i [Y_i, Z_i]|U$$

and

$$j^1(Y_i)(p) = 0, \quad j^1(Z_i)(p) = 0 \text{ for all } i = 1, \dots, q.$$

Let $D : \chi(M) \rightarrow \chi(M)$ be any derivation. It follows from Lemma 1 that

$$D(X)(p) = D\left(\sum_i [Y_i, Z_i]\right)(p).$$

By the definition of derivation, this last expression equals

$$\sum_i [D(Y_i), Z_i](p) + \sum_i [Y_i, D(Z_i)](p)$$

which is zero because the 1-jets of Y_i and Z_i are zero in p . This proves lemma 3.

PROOF OF LEMMA 4. For D_U and $U \subset \mathbb{R}^n$ as in the statement of Lemma 4 and x_1, \dots, x_n coordinates on \mathbb{R}^n , we define the functions $D_{ij} : U \rightarrow \mathbb{R}$, $i, j = 1, \dots, n$ by

$$D_U \left(\frac{\partial}{\partial x_i} \right) = \sum D_{ij} \frac{\partial}{\partial x_j}.$$

We know that

$$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] \equiv 0,$$

for all i, j so

$$\begin{aligned} 0 \equiv D_U \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] &= \left[D_U \left(\frac{\partial}{\partial x_i} \right), \frac{\partial}{\partial x_j} \right] + \left[\frac{\partial}{\partial x_i}, D_U \left(\frac{\partial}{\partial x_j} \right) \right] \\ &= - \sum_h \frac{\partial D_{ih}}{\partial x_j} \frac{\partial}{\partial x_h} + \sum_h \frac{\partial D_{jh}}{\partial x_i} \frac{\partial}{\partial x_h}. \end{aligned}$$

Hence, for all i, j, h , we have

$$\frac{\partial D_{ih}}{\partial x_j} = \frac{\partial D_{jh}}{\partial x_i};$$

as U is 1-connected, there are functions $\bar{D}_h : Y \rightarrow \mathbf{R}$, $h = 1, \dots, n$ such that

$$\frac{\partial \bar{D}_h}{\partial x_i} = -D_{ih}.$$

Now we define $\bar{Z} \in \chi(U)$

$$\bar{Z} = \sum \bar{D}_h \frac{\partial}{\partial x_i}.$$

From the above construction it follows that we have for each

$$i = 1, \dots, n : D_U \left(\frac{\partial}{\partial x_i} \right) = \left[\bar{Z}, \frac{\partial}{\partial x_i} \right].$$

Now we define the derivation

$$D_U^1 : \chi(U) \rightarrow \chi(U) \text{ by } D_U^1(X) = D_U(X) - [\bar{Z}, X];$$

clearly

$$D_U^1 \left(\frac{\partial}{\partial x_i} \right) \equiv 0$$

for all i .

Next we define the functions $D_{ijk} : U \rightarrow \mathbf{R}$, $i, j, k = 1, \dots, n$ by

$$D_U^1 \left(x_i \frac{\partial}{\partial x_j} \right) = \sum D_{ijk} \frac{\partial}{\partial x_k}.$$

First we show that all these functions are constant: as

$$\left[\frac{\partial}{\partial x_i}, x_j \frac{\partial}{\partial x_k} \right] = \delta_{ij} \frac{\partial}{\partial x_k}$$

we have

$$\begin{aligned} 0 \equiv D_U^1 \left[\frac{\partial}{\partial x_i}, x_j \frac{\partial}{\partial x_k} \right] &= \left[D_U^1 \left(\frac{\partial}{\partial x_i} \right), x_j \frac{\partial}{\partial x_k} \right] \\ &\quad + \left[\frac{\partial}{\partial x_i}, D_U^1 \left(x_j \frac{\partial}{\partial x_k} \right) \right] = \sum \frac{\partial D_{jkh}}{\partial x_i} \frac{\partial}{\partial x_h}. \end{aligned}$$

Hence, for all i, j, k, h ,

$$\frac{\partial D_{jkh}}{\partial x_i} \equiv 0,$$

so the functions D_{jkh} must be constant (because U is connected). We de-

note these constants by c_{jkh} . Next we want to show that

- a) $c_{ijk} = 0$ whenever $j \neq k$ and
- b) $c_{ijj} = c_{ikk}$ for all i, j, k .

To prove this we observe that

$$\left[x_i \frac{\partial}{\partial x_j}, x_k \frac{\partial}{\partial x_l} \right] = \delta_{jk} x_i \frac{\partial}{\partial x_l} - \delta_{li} x_k \frac{\partial}{\partial x_j}.$$

Applying D_U^1 to this, we obtain

$$\delta_{jk} \sum_h c_{ilh} \frac{\partial}{\partial x_h} - \delta_{li} \sum_h c_{kjh} \frac{\partial}{\partial x_h} = c_{ijk} \frac{\partial}{\partial x_l} - c_{kli} \frac{\partial}{\partial x_j} \dots^*$$

If we take in $* k \neq l = j = i$ (this assumes that the dimension $n \neq 1$ because for $n = 1$ we cannot take $k \neq l$; if $n = 1$ however a) and b) above are trivially true) we obtain:

$$- \sum_h c_{kjh} \frac{\partial}{\partial x_h} = c_{ilk} \frac{\partial}{\partial x_l} - c_{kli} \frac{\partial}{\partial x_l}$$

from which it follows that $c_{kjh} = 0$ if $l \neq h$ which proves a) above.

Next we take in $* k \neq j$ and $l = i$ and obtain (using the above result):

$$- c_{kjj} \frac{\partial}{\partial x_j} = - c_{kli} \frac{\partial}{\partial x_j} \text{ and hence:}$$

$$c_{kjj} = c_{kli} \text{ if } k \neq j, \text{ which implies b).}$$

From the above calculations it follows that for all i, j ,

$$D_U^1 \left(x_i \frac{\partial}{\partial x_j} \right) = \left[\sum_h c_{hhh} \frac{\partial}{\partial x_h}, x_i \frac{\partial}{\partial x_j} \right].$$

We now define $Z \in \chi(U)$ by

$$Z = \bar{Z} + \sum_h c_{hhh} \frac{\partial}{\partial x_h}$$

and observe that for all i, j ,

$$D_U \left(\frac{\partial}{\partial x_i} \right) = \left[Z, \frac{\partial}{\partial x_i} \right] \text{ and } D_U \left(x_i \frac{\partial}{\partial x_j} \right) = \left[Z, x_i \frac{\partial}{\partial x_j} \right];$$

it is not hard to see that Z is uniquely determined by these properties. In order to complete the proof of this Lemma we have to show that the derivation D_U^2 , defined by $D_U^2(X) = D_U(X) - [Z, X]$ is identically zero:

SUB-LEMMA (4.1). Let D_U and $U \subset \mathbf{R}^n$ be as in Lemma 4. If, for all i, j ,

$$D_U \left(\frac{\partial}{\partial x_i} \right) \equiv 0 \text{ and } D_U \left(x_i \frac{\partial}{\partial x_j} \right) \equiv 0$$

then $D_U(X) \equiv 0$ for all $X \in \chi(U)$.

PROOF of (4.1) We define the functions $D_{ijkl} : U \rightarrow \mathbf{R}$ by

$$D_U \left(x_i x_j \frac{\partial}{\partial x_k} \right) = \sum D_{ijkl} \frac{\partial}{\partial x_l}.$$

To prove that these functions are all constant one can proceed just as in the case with D_{ijk} above, but now we use the fact that

$$D_U \left(\left[\frac{\partial}{\partial x_i}, x_j x_k \frac{\partial}{\partial x_l} \right] \right) \equiv 0;$$

we omit the computation. We denote the corresponding constants again by c_{ijkl} . Next we observe that

$$\left[\sum_h x_h \frac{\partial}{\partial x_h}, x_i x_j \frac{\partial}{\partial x_k} \right] = x_i x_j \frac{\partial}{\partial x_k};$$

applying D_U to this we obtain:

$$\left[\sum_h x_h \frac{\partial}{\partial x_h}, \sum_l c_{ijkl} \frac{\partial}{\partial x_l} \right] = \sum c_{ijkl} \frac{\partial}{\partial x_l}, \text{ or } - \sum_l c_{ijkl} \frac{\partial}{\partial x_l} = \sum c_{ijkl} \frac{\partial}{\partial x_l};$$

hence all the constants c_{ijkl} are zero. In the same way one can show that

$$D_U \left(x_i x_j x_k \frac{\partial}{\partial x_l} \right) \equiv 0$$

for all i, j, k, l . Finally, we apply lemma 3 to obtain the proof: Let $X \in \chi(U)$ and $p \in U$, we want to show $D_U(X)(p) = 0$. There is a vectorfield $\hat{X} \in \chi(U)$ such that the coefficient functions of \hat{X} are polynomials of degree ≤ 3 and such that $j^3(X)(p) = j^3(\hat{X})(p)$. By our previous computations we have $D_U(X) \equiv 0$ and by lemma 3 we have $D(X)(p) = D(\hat{X})(p)$; hence $D_U(X)(p) = 0$, this proves (4.1).

PROOF OF THE THEOREM. For a given derivation $D : \chi(M) \rightarrow \chi(M)$ and an open $U \subset M$, we get an induced derivation $D_U : \chi(U) \rightarrow \chi(U)$. This D_U is constructed as follows:

For $X \in \chi(U)$ an $p \in U$ one defines $D_U(X)(p)$ to be $D(X)(p)$, where $\tilde{X} \in \chi(M)$ is some vectorfield which equals X on some open neighbourhood of p . Clearly $D_U(X)(p)$ is well defined (by Lemma 1) and D_U is a derivation on $\chi(U)$.

Now we take an atlas $\{U_i, \varphi_i(U_i) \rightarrow \mathbf{R}^n\}$ of M such that each U_i is connected and simply connected. Using the coordinates $x_j \varphi_i$ on each U_i we can apply Lemma 4 to each D_{U_i} and obtain on each U_i a vectorfield $Z_i \in \chi(U_i)$ such that $D_{U_i}(X) = [Z_i, X]$ for each $X \in \chi(U_i)$.

As D_{U_i} and D_{U_j} both restricted to $U_i \cap U_j$ are equal, Z_i and Z_j both restricted to $U_i \cap U_j$ also have to be equal. Hence there is a vectorfield $Z \in \chi(M)$ such that for each i , $Z_i = Z|_{U_i}$. It follows easily that, for each $X \in \chi(M)$, $D(X) = [Z, X]$.

3. Remark

One can also take, instead of $\chi(M)$, the set of vectorfields which respect a certain given structure. To be more explicit, let ω be a differential form on M defining a symplectic structure or a volume structure, and let $\chi_\omega(M)$ be the Lie-algebra of those vectorfields X for which $L_X \omega \equiv 0$ (L_X means : Lie derivative with respect to X). Now one can ask again whether every derivation $D : \chi_\omega(M) \rightarrow \chi_\omega(M)$ is induced by a vectorfield $Z \in \chi_\omega(M)$. This is in general not the case. Take for example $M = \mathbb{R}^n$ and $\omega = dx_1 \wedge \cdots \wedge dx_n$ the usual volume form and

$$Z = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}.$$

Then $Z \notin \chi_\omega(\mathbf{R}^n)$ but for each $X \in \chi_\omega(\mathbf{R}^n)$, $[Z, X] \in \chi_\omega(\mathbf{R}^n)$; so $'[Z, -]'$ is a derivation on $\chi_\omega(\mathbf{R}^n)$. This derivation cannot be induced by any $Z' \in \chi_\omega(\mathbf{R}^n)$.

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