JAMES C. BECKER
DANIEL HENRY GOTTLIEB

Coverings of fibrations

Compositio Mathematica, tome 26, no 2 (1973), p. 119-128

<http://www.numdam.org/item?id=CM_1973__26_2_119_0>
1. Introduction

Let $E \xrightarrow{p} B$ be a fibration. Then a covering map $\tilde{E} \xrightarrow{\pi} E$ gives rise to a fibration $\tilde{E} \xrightarrow{\pi} B$. If the fibre of this fibration is connected, then the fibre is a covering space of the fibre of $E \to B$. In this case, we shall call $\tilde{E}$ a covering fibration of the fibration $F \to E \xrightarrow{p} B$ extending the covering $\pi : \tilde{F} \to F$.

This note takes up the following questions.

A. Given a fibration $F \to E \to B$ and a covering $\tilde{F} \xrightarrow{\pi} F$, when can we find a covering $\tilde{E}$ extending $\tilde{F}$?

B. Given a space $F$ and a covering space $\tilde{F}$, under what conditions can we always find a covering fibration extending $\tilde{F}$ for any fibration with fibre $F$?

For $\tilde{F}$ a universal covering space, we can answer question A (see Theorem 1) in terms of conditions on the fundamental groups of the fibration $F \to E \to B$. For oriented fibrations and universal coverings $\tilde{F}$, we find the answer to $B$ depends upon $G_1(F)$, (see Theorem 2). Some results on covering fibrations extending non-universal coverings $\tilde{F}$ are found in § 3.

The existence of covering fibrations leads to various applications. The most striking of them is theorem 15 which generalizes a theorem of Borel's [1], lemma 3.2.

By fibration, we shall mean Hurewicz fibration (i.e. $F \to E \xrightarrow{p} B$ has the homotopy covering property). We assume that $E$ and $F$ are path connected, locally path connected, and semi-locally 1-connected. Most of the results proved will obviously be true for other types of fibrations. In § 3, we consider fibre bundles (locally homeomorphic to a product of a neighborhood in the base and the fibre). By an oriented fibration, we shall mean the strongest possible interpretation. That is, $F \to E \to B$ is oriented if $\pi_1(B)$ operates trivially on $\xi(F)$, where $\xi(F)$ is the group of homotopy classes of self homotopy equivalences. By $\xi_0(F)$ we shall mean the group of based homotopy classes of based self homotopy equivalences.

* Partially supported by a National Science Foundation grant.
Also \( \breve{X} \) will always denote a covering space of \( X \). Covering spaces are always assumed to be path connected.

## 2. Covering fibrations extending universal coverings

First we answer question A for universal coverings.

**Theorem 1:** Let \( F \rightarrow E \rightarrow B \) be a fibration with connected fibre \( F \) and let \( \breve{F} \) be the universal cover of \( F \). Then there exists a covering fibration \( \breve{E} \rightarrow E \rightarrow B \) extending \( \breve{F} \) if and only if

\( a) \ i_* : \pi_1(F) \rightarrow \pi_1(E) \) is injective

\( \text{and} \)

\( b) \ p_* : \pi_1(E) \rightarrow \pi_1(B) \) has a right inverse (which is a homomorphism).

**Proof:** Suppose that \( \breve{E} \) is a covering fibration. Then we have

\[
\begin{array}{ccc}
\breve{F} & \xrightarrow{\pi} & F \\
\downarrow i & & \downarrow i \\
\breve{E} & \xrightarrow{\pi} & E \\
\downarrow p & & \downarrow p \\
B & \xrightarrow{1} & B
\end{array}
\]

where \( \pi \) denotes covering maps. From the exact ladder associated with this diagram, we have

\[
\begin{array}{ccc}
\pi_2(B) & \xrightarrow{\sim} & \pi_2(B) \\
\downarrow & \downarrow d & \\
0 & \rightarrow & \pi_1(F).
\end{array}
\]

Thus \( d \) is trivial, hence \( i_* : \pi_1(F) \rightarrow \pi_1(E) \) is injective. On the other hand \((p\pi)_* : \pi_1(\breve{E}) \rightarrow \pi_1(B)\). So \((p\pi)^* \) has an inverse \( j : \pi_1(B) \rightarrow \pi_1(\breve{E}) \). Then \( \pi_* j : \pi_1(B) \rightarrow \pi_1(E) \) is the required inverse to \( p_* : \pi_1(E) \rightarrow \pi_1(B) \).

Conversely, suppose a) and b) hold, Let \( \breve{E} \) be the covering space of \( E \) corresponding to a subgroup \( H \) of \( \pi_1(E) \) such that \( p_* : H \rightarrow \pi_1(B) \) is an isomorphism. Then \( p' = p\pi : \breve{E} \rightarrow B \) is a fibration with fibre \( F' \) and we
have the commutative diagram

\[
\begin{array}{ccc}
F' & \xrightarrow{\overline{\pi}} & F \\
\downarrow{v'} & & \downarrow{i} \\
\overline{E} & \xrightarrow{\pi} & E \\
\downarrow{\overline{p}'} & & \downarrow{p} \\
B & \xrightarrow{1} & B
\end{array}
\]

where \(\overline{\pi}\) is the restriction of \(\pi\).

Now we will show that \(F' = \overline{F}\). Notice that

\[
p'_* : \pi_1(\overline{E}) \rightarrow \pi_1(B)
\]

is the composition of isomorphisms

\[
\pi_1(\overline{E}) \xrightarrow{\cong} H \xrightarrow{p_*} \pi_1(B)
\]

and is therefore an isomorphism. Hence \(F'\) is path connected. Since \(\overline{\pi} : F' \rightarrow F\) is a fibration with discrete fiber, it is a covering space [9, Theorem 10, p. 78].

We will show now that \(\pi_1(F') = 0\). If \(v \in \pi_1(F')\) then \(i'_*(v) = 0\) by exactness. Hence \(0 = \pi_1*i'_*(v) = i_*\overline{\pi}_*(v)\). But \(\overline{\pi}_*\) is injective since \(\overline{\pi}\) is a covering and \(i_*\) is injective by hypothesis a). Therefore \(v = 0\) as was to be shown.

Next we turn to question B in the case of universal coverings and oriented fibrations. We need a few technical remarks.

First, let us recall the definition of the first evaluation subgroup \(G_1(F) \subset \pi_1(F)\). A homotopy \(h : F \rightarrow F\) is called a cyclic homotopy if \(h_0 = h_1 = 1_F\). The loop \(\tau\) given by \(\tau(t) = h_t(*)\) (where \(* \in F\) is the base point) is called the trace of the homotopy. Then \(G_1(F)\) consists of the subgroups of \(\pi_1(F,*)\) whose elements are represented by loops which are traces of cyclic homotopies. For a list of properties of \(G_1(F)\), see [5].

We also need the fact (see theorem 16.9, [3]) that for any fibre \(F\), there exists a universal fibration \(F \rightarrow E_\infty \rightarrow B_\infty\). Now the transgression \(d_\infty : \pi_2(B_\infty) \rightarrow \pi_1(F)\) is related to \(G_1(F)\) by the fact that \(d_\infty(\pi_2(B_\infty)) = G_1(F)\). Now, as always, we assume that \(F\) is connected. Then let \(\overline{B}_\infty\) be the universal covering space of \(B_\infty\). Let \(F \rightarrow D \rightarrow \overline{B}_\infty\) denote the fibration classified by the covering map \(\pi : \overline{B}_\infty \rightarrow B_\infty\). It is easy to see that \(d(\pi_2(\overline{B}_\infty)) = G_1(F)\), where \(d\) is the transgression \(d : \pi_2(\overline{B}_\infty) \rightarrow \pi_1(F)\) It also is true that every oriented fibration may be regarded as a pullback of \(F \rightarrow D \rightarrow \overline{B}_\infty\). This follows since a classifying map \(k : B \rightarrow B_\infty\) of an oriented fibration \(F \rightarrow E \rightarrow B\) maps \(\pi_1(B)\) to the identity of \(\pi_1(B_\infty)\).
Hence, there is a lifting of \( k \) to \( \tilde{k} : B \to \widetilde{B}_\infty \) such that \( k = \pi \tilde{k} \).

Now we can state and prove

**Theorem 2:** There exists a covering fibration of any oriented fibration with connected fibre \( F \) extending the universal covering space \( \tilde{F} \) if and only if \( G_1(F) \) is trivial.

**Proof:** Let \( F \xrightarrow{i} D \xrightarrow{p} \widetilde{B}_\infty \) be the fibration mentioned above. We shall show that there exists a covering fibration \( \tilde{D} \to \widetilde{B}_\infty \) which extends \( \tilde{F} \) if and only if \( G_1(F) = 0 \). The pull back of the classifying map for \( F \to E \to B \), denoted by \( k : B \to B_\infty \), of \( \tilde{F} \to D \to B \) will be the required covering fibration. That is, \( \tilde{D} = k^*(D) \).

Now \( \pi_1(\widetilde{B}_\infty) \) is trivial. Thus (b) of theorem 1 is satisfied. On the other hand, \( G_1(F) = 0 \) implies that \( d : \pi_2(\widetilde{B}_\infty) \to \pi_1(F) \) is trivial. Hence \( i_* : \pi_1(F) \to \pi_1(D) \) is injective. So condition (a) is satisfied and so we may apply theorem 1.

If we assume that \( G_1(F) \neq 0 \), then by theorem 1, we cannot find a covering fibration of \( F \to D \to \widetilde{B}_\infty \) extending \( \tilde{F} \).

**Corollary 3:** Let \( F \) be a compact polyhedron and \( \chi(F) \neq 0 \). Then there exists a covering fibration of any oriented fibration extending \( F \), the universal covering \( F \).

**Proof:** We know that \( \chi(F) \neq 0 \) implies that \( G_1(F) = 0 \).

**Corollary 4:** Let \( F \) be a compact polyhedron with \( \chi(F) \neq 0 \). Then there exists a cross-section over the two-skeleton of the base space of any oriented fibration with fibre \( F \).

**Proof:** By the corollary above, we have

\[
\begin{array}{cccc}
\tilde{F} & \xrightarrow{\pi} & F & \\
\downarrow & & \downarrow & \\
\tilde{E} & \xrightarrow{\pi} & E & \\
\downarrow & & \downarrow & \\
B & \xrightarrow{p} & B &
\end{array}
\]

where \( B \) is a 2-dimensional \( CW \) complex.

The groups in which the obstructions to a cross-section of \( \tilde{E} \to \tilde{B}_\infty \) lie must vanish. Thus there is a cross-section \( c : B \to \tilde{E} \). Now \( \pi c \) is the required cross-section.

Now we shall study those fibrations with fibres \( RP^n \), real projective space. We begin by giving alternative proofs to some results of Olum.
Lemma 5 (Olum [8]):
\[ \xi_0(RP^n) \cong \mathbb{Z}_2. \]

Proof: Let \( \hat{\xi}_0(S^n) \) denote the group of homotopy classes of base point preserving equivariant homotopy equivalences of \( S^n \). We have an isomorphism
\[ \xi_0(RP^n) = \hat{\xi}_0(S^n) \]
defined by sending \( f \) to the unique base point preserving map \( \hat{f} \) that covers \( f \). By the method of proof of [2, TH. 2.5], the operation of suspension defines an isomorphism
\[ \hat{\xi}_0(S^n) = \hat{\xi}_0(S^{n+1}), \quad n \geq 1. \]
Therefore, we have
\[ \xi_0(RP^n) = \hat{\xi}_0(RP^{n+1}), \quad n \geq 1. \]
Finally,
\[ \xi_0(RP^1) = \xi_0(S^1) \cong \mathbb{Z}_2. \]

Corollary 6.
\[ \xi(RP^{2n+1}) \cong \mathbb{Z}_2; \xi(RP^{2n}) \cong 0. \]

Proof: Let \( L \) denote the space of self homotopy equivalences of \( RP^n \). Let \( L_0 \) be the subspace of maps of \( L \) which preserve the base point. Now \( G_1(RP^{2n}) = 0 \) and
\[ G_1(RP^{2n+1}) = \pi_1(RP^{2n+1}); \]
see [4], corollary I.6 and theorem II.5. From the exact sequence arising from the fibration \( L_0 \to L \to RP^n \), and noting that the image of \( \omega_* \) is \( G_1(RP^n) \), we see that \( d : \pi_1(RP^n) \to \pi_0(L_0) \) is zero if \( n \) is odd and injective if \( n \) is even. Thus \( i_* : \pi_0(L_0) \to \pi_0(L) \) is injective if \( n \) is odd and has kernel \( \mathbb{Z}_2 \) if \( n \) is even. But \( \pi_0(L_0) = \hat{\xi}_0(RP^n) \cong \mathbb{Z}_2 \) by lemma 5. Also \( i_* : \pi_0(L_0) \to \pi_0(L) \) must be onto since \( RP^n \) is connected. Thus \( \xi(RP^n) = \pi_0(L) \) is \( \mathbb{Z}_2 \) when \( n \) is odd and 0 when \( n \) is even.

Corollary 7. Every fibration with fibre \( RP^{2n} \) is orientable.

Proof: Since \( \xi(RP^{2n}) \) is trivial, the fundamental group of the base must act trivially on \( \xi(RP^{2n}) \).

Theorem 8. Every fibration with fibre \( RP^{2n} \) is covered by an \( S^{2n} \) fibration with an involution.

Proof: The conclusion means that we can always find a covering fibration which extends the universal cover \( S^{2n} \) of \( RP^{2n} \). But this follows immediately from theorem 2 and corollary 7. The extended total space must be a 2-fold covering of the original total space, and the deck transformation is the involution.
REMARK: Theorem 8 is not true for \( RP^{2n+1} \) even when the fibration is orientable. Consider the “universal oriented fibration”,

\[
RP^{2n+1} \rightarrow D \rightarrow B^2.
\]

Now

\[
i_*: \pi_1(RP^{2n+1}) \rightarrow \pi_1(D)
\]

is trivial since

\[
G_1(R^{2n+1}) = \mathbb{Z}_2.
\]

Thus in theorem 1 condition a) fails.

The next corollary is an application of theorem 8.

**Corollary 9.** Let \( RP^{2n} \rightarrow E \rightarrow B \) be a fibration. Then

\[
H^*(E; \mathbb{Z}_2) = H^*(RP^{2n}; \mathbb{Z}_2) \otimes H^*(B; \mathbb{Z}_2)
\]
as \( \mathbb{Z}_2 \)-vector spaces.

**Proof:** Let \( S^{2n} \rightarrow \tilde{E} \rightarrow B \) be an \( S^{2n} \)-fibration with an involution which covers \( RP^{2n} \rightarrow E \rightarrow B \). Let \( \lambda: \tilde{E} \rightarrow S^\infty \) be an equivariant map and let \( \lambda: E \rightarrow RP^\infty \) denote the quotient map. Then the composition \( RP^{2n} \overset{i}{\rightarrow} E \overset{\lambda}{\rightarrow} RP^\infty \) sends the generator \( c \in H^1(RP^\infty; \mathbb{Z}_2) \) to the generator \( \tilde{c} \in H^1(RP^{2n}; \mathbb{Z}_2) \). A cohomology extension of the fiber

\[
\theta: H^*(RP^{2n}; \mathbb{Z}_2) \rightarrow H^*(E; \mathbb{Z}_2)
\]
is defined by \( \theta(\tilde{c}^t) = \lambda^*(c^t), \ t \geq 0 \). The corollary now follows from the Leray-Hirsch theorem [9, p. 257].

### 3. Arbitrary coverings

We shall restrict our attention to coverings of fibre bundles. Our technique can probably be extended to Hurewicz fibrations, or Dold fibrations, but the needed propositions have not been written down yet. First we answer question B for fibre bundles. Then follow applications.

Let \( G \) be a group of homeomorphisms of \( F \) onto itself. Let \( G^* \) be the self homeomorphisms of a covering of \( F \), denoted \( \tilde{F} \), which are liftings of homeomorphisms in \( G \). Let \( \Phi: G^* \rightarrow G \) be the map which takes a map \( f \in G^* \) to the induced map \( \Phi(f) \in G \). Then \( \Phi \) is continuous and also is a homomorphism.

**Theorem 10:** If there exists a cross-section \( c: G \rightarrow G^* \) to \( \Phi \) which is also a homomorphism, then there is a covering bundle of any \( G \)-bundle with fibre \( F \) which extends \( \tilde{F} \).

**Proof:** Let \( F \rightarrow E \rightarrow B \) be a \( G \)-bundle and let \( G \rightarrow E^* \rightarrow B \) be the associated principal \( G \)-bundle. Then consider the \( G \)-bundle \( \tilde{F} \rightarrow E \times_{\tilde{F}} \)
B where $G$ acts on $F$ by means of the cross-section $c$. Then $\tilde{E} = E \times_g F$ is the required covering. The projection $\pi: \tilde{E} \to E$ is given by
\[
\pi(\langle e, \tilde{x} \rangle) = \langle e, \pi(\tilde{x}) \rangle \in E^* \times_g F = E.
\]

The next theorem gives conditions for Theorem 10 to hold. We shall let $G_e^*$ be the identity component of $G^*$ and $G_e$ the identity component of $G$.

**THEOREM 11:** A covering bundle extending $\tilde{F}$ always exists if a) $G_1(F) \subseteq \pi_1(\tilde{F})$, and b) there is a homomorphism
\[
\tilde{c}: G_e^* \to G^*/G_e^*
\]
which is a cross-section to the homomorphism
\[
\Phi: G^*/G_e^* \to G/G_e
\]
induced by $\Phi$.

**PROOF:** Note that $\Phi: G^* \to G$ is a fibration with a discrete fibre. The fibre over $1_F$ consists of the liftings of $1_F$. Assume that $\tilde{f} \in G_e^*$ is a lifting of $1_F$. Then the path from $1_F$ to $\tilde{f}$ induces a cyclic homotopy $h_t: F \to F$. The trace of $h_t$ must represent an element in $\pi_1(\tilde{F})$ by a). Thus the trace lifts to a closed path in $\tilde{F}$. Thus $\tilde{f}$ has a fixed point and hence $\tilde{f} = 1_{\tilde{F}}$.

This fact allows us to conclude that $G_e$ and $G_e^*$ are isomorphic. Thus we may construct the cross-section required by theorem 10 over $G_e$, and condition b) allows us to extend the cross-section over the other components.

**REMARK:** If $G$ is connected and $G_1(F) = 0$, we may extend any covering $\tilde{F}$ to a covering $G$-bundle of any arbitrary $G$-bundle.

**COROLLARY 12:** If $F$ is a compact polyhedron and $\chi(F) \neq 0$, then $\tilde{F}$ can be extended to a covering bundle for any fibre bundle with connected structural group.

Let $M$ be a closed topological manifold which is unorientable and let $\tilde{M}$ denote its oriented double covering. Let $0(M)$ be the subgroup of $\pi_1(M)$ of elements represented by orientation preserving loops. Then $0(M)$ is the subgroup of $\pi_1(M)$ corresponding to the oriented double covering $\tilde{M}$.

From this point on, we shall consider fibre bundles with fibre $M$ and study covering bundles which extend $\tilde{M}$. The end result will yield the main application of these techniques, Theorem 15.

**THEOREM 13:** For any fibre bundle with fibre $M$, a closed unorientable topological manifold, the oriented double covering $\tilde{M}$ extends to a covering
bundle. In addition, the structural group of this covering bundle preserves the orientation of $\tilde{M}$.

**PROOF:** Every homeomorphism $h : M \to M$ has two liftings $\tilde{M} \to \tilde{M}$, one of which preserves and the other reverses the orientation of $\tilde{M}$. Consider the correspondence $i$ which sends every $h \in G$ to its orientation preserving lifting in $G^*$. It is easy to see that $i : G \to G^*$ is continuous, a homomorphism of groups and a cross-section of $\Phi : G^* \to G$. Thus theorem 10 is satisfied. The group of the covering fibre bundle is $i(G)$, so the second statement of the theorem is clearly true.

If we compare theorem 13 with condition a) of theorem 11, we are lead to conjecture that $G_1(M) \subset O(M)$. This is in fact true, as the following theorem shows.

**THEOREM 14:**

$$G_1(M) \subset O(M).$$

**PROOF:** Let $\alpha \in G_1(M)$. Then there is a cyclic homotopy $h_t : M \to M$ whose trace represents $\alpha$. Now $h_t$ lifts to a homotopy $\tilde{h}_t : \tilde{M} \to \tilde{M}$ and $\tilde{h}_1$ is a lifting of the identity. Since $\tilde{h}_1$ is homotopic to $1_{\tilde{M}}$, $\tilde{h}_1$ preserves the orientation on $\tilde{M}$. There are only two liftings of the identity and one of them reverses orientation. So $\tilde{h}_1 = 1_{\tilde{M}}$. Thus the trace of $\tilde{h}_t$ is a loop which covers the trace of $h_t$. Hence $\alpha$ must be in $O(M)$.

**THEOREM 15:** Let $M \to E \to B$ be a fibre bundle with fibre a closed topological $n$-manifold and with structural group $G$. If $\chi(M) \not\equiv O \pmod{p}$, where $p$ is a prime, then

$$\pi^* : H^*(B; \mathbb{Z}_p) \to H^*(E; \mathbb{Z}_p)$$

is injective.

**PROOF:** First consider the case where $\pi_1(B)$ operates trivially on $H^*(M; \mathbb{Z}_p)$. Then the theorem is true if $M$ is orientable or if $p = 2$ [6, Theorem 12]. Suppose now that $M$ is unorientable and $p \neq 2$. Let $\tilde{M}$ denote the orientable double covering of $M$. By theorem 13, we have

$$\begin{array}{c}
\tilde{M} \\
\downarrow \\
\tilde{E} \\
\downarrow \\
B
\end{array} \quad \begin{array}{c}
M \\
\downarrow \\
E \\
\downarrow \\
B
\end{array}$$

and $\pi_1(B)$ operates trivially on $H^*(\tilde{M}; \mathbb{Z}_p)$. Now $\chi(\tilde{M}) = 2\chi(M) \not\equiv 0 \pmod{p}$. Since $\tilde{M}$ is orientable it follows that $\tilde{\pi}^*$ is injective. Then by commutativity $\pi^*$ is injective.
Now consider the case where $\pi_1(B)$ operates non-trivially on $H^n(M; \mathbb{Z}_p)$ $p$ odd. Let $K \subset \pi_1(B)$ denote the normal subgroup of index 2 consisting of elements which operate trivially on $H^n(M; \mathbb{Z})$. Let $\tilde{B} \xrightarrow{q} B$ denote the 2-fold covering which corresponds to $K$. We have

\[
\begin{array}{ccc}
M & \rightarrow & M \\
\downarrow & & \downarrow \\
\tilde{E} & \xrightarrow{\tilde{q}} & E \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
\tilde{B} & \xrightarrow{q} & B
\end{array}
\]

where $\tilde{E} \xrightarrow{\tilde{q}} \tilde{B}$ is the bundle induced by $q$. Since $\pi_1(B)$ operates trivially on $H^n(M; \mathbb{Z}_p)$ we have from the preceding paragraph that $\tilde{\pi}^*$ is injective. To show that $\pi^*$ is injective it is now sufficient to show that

$$q^* : H^*(\tilde{B} ; \mathbb{Z}_p) \rightarrow H^*(B ; \mathbb{Z}_p)$$

is injective. There is the transfer map [10, Chapter 5]

$$\tau : H^*(\tilde{B} ; \mathbb{Z}_p) \rightarrow H^*(B ; \mathbb{Z}_p),$$

and $\tau q^*$, being multiplication by 2, is an isomorphism. Therefore $q^*$ is injective. This completes the proof of the theorem.

This theorem was first noted by A. Borel in [1] with the extra hypotheses that the bundle is differentiable and oriented in some sense and $M$ is oriented. In [6], the second author removed the differentiability hypothesis and the above theorem removes the orientability hypotheses.

BIBLIOGRAPHY

Armand Borel

Pierre E. Conner and E. E. Floyd

Albrecht Dold

Daniel Henry Gottlieb

Daniel Henry Gottlieb

Daniel Henry Gottlieb
Daniel Henry Gottlieb
[7] Witnesses, transgressions, Hurewicz homomorphisms, and the evaluation map, mimeographed, Purdue Univ.

Paul Olum

Edwin H. Spanier

Norman E. Steenrod, D. B. A. Epstein

(Oblatum 8-V-72)