

COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 26, n° 2 (1973), p. 119-128

http://www.numdam.org/item?id=CM_1973__26_2_119_0

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COVERINGS OF FIBRATIONS

by

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1. Introduction

Let $E \xrightarrow{p} B$ be a fibration. Then a covering map $\tilde{E} \xrightarrow{\pi} E$ gives rise to a fibration $\tilde{E} \xrightarrow{p\pi} B$. If the fibre of this fibration is connected, then the fibre is a covering space of the fibre of $E \rightarrow B$. In this case, we shall call \tilde{E} a *covering fibration of the fibration $F \rightarrow E \xrightarrow{p} B$ extending the covering $\pi : \tilde{F} \rightarrow F$* .

This note takes up the following questions.

A. Given a fibration $F \rightarrow E \rightarrow B$ and a covering $\tilde{F} \xrightarrow{\pi} F$, when can we find a covering \tilde{E} extending \tilde{F} ?

B. Given a space F and a covering space \tilde{F} , under what conditions can we always find a covering fibration extending \tilde{F} for any fibration with fibre F ?

For \tilde{F} a universal covering space, we can answer question A (see Theorem 1) in terms of conditions on the fundamental groups of the fibration $F \rightarrow E \rightarrow B$. For oriented fibrations and universal coverings \tilde{F} , we find the answer to B depends upon $G_1(F)$, (see Theorem 2). Some results on covering fibrations extending non-universal coverings \tilde{F} are found in § 3.

The existence of covering fibrations leads to various applications. The most striking of them is theorem 15 which generalizes a theorem of Borel's [1], lemma 3.2.

By fibration, we shall mean Hurewicz fibration (i.e. $F \rightarrow E \xrightarrow{p} B$ has the homotopy covering property). We assume that E and F are path connected, locally path connected, and semi-locally 1-connected. Most of the results proved will obviously be true for other types of fibrations. In § 3, we consider fibre bundles (locally homeomorphic to a product of a neighborhood in the base and the fibre). By an oriented fibration, we shall mean the strongest possible interpretation. That is, $F \rightarrow E \rightarrow B$ is *oriented* if $\pi_1(B)$ operates trivially on $\zeta(F)$, where $\zeta(F)$ is the group of homotopy classes of self homotopy equivalences. By $\xi_0(F)$ we shall mean the group of based homotopy classes of based self homotopy equivalences.

* Partially supported by a National Science Foundation grant.

Also \tilde{X} will always denote a covering space of X . Covering spaces are always assumed to be path connected.

2. Covering fibrations extending universal coverings

First we answer question A for universal coverings.

THEOREM 1: *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration with connected fibre F and let \tilde{F} be the universal cover of F . Then there exists a covering fibration \tilde{E} of $E \xrightarrow{p} B$ extending \tilde{F} if and only if*

a) $i_* : \pi_1(F) \rightarrow \pi_1(E)$ is injective

and

b) $p_* : \pi_1(E) \rightarrow \pi_1(B)$ has a right inverse (which is a homomorphism).

PROOF: Suppose that \tilde{E} is a covering fibration. Then we have

$$\begin{array}{ccc}
 \tilde{F} & \xrightarrow{\pi} & F \\
 i \downarrow & & \downarrow i \\
 \tilde{E} & \xrightarrow{\pi} & E \\
 p \downarrow & & \downarrow p \\
 B & \xrightarrow{1} & B
 \end{array}$$

where π denotes covering maps. From the exact ladder associated with this diagram, we have

$$\begin{array}{ccc}
 \pi_2(B) & \xrightarrow{\cong} & \pi_2(B) \\
 \downarrow & & \downarrow d \\
 0 & \longrightarrow & \pi_1(F).
 \end{array}$$

Thus d is trivial, hence $i_* : \pi_1(F) \rightarrow \pi_1(E)$ is injective. On the other hand $(p\pi)_* : \pi_1(\tilde{E}) \xrightarrow{\cong} \pi_1(B)$. So $(p\pi)^*$ has an inverse $j : \pi_1(B) \rightarrow \pi_1(\tilde{E})$. Then $\pi_* j : \pi_1(B) \rightarrow \pi_1(E)$ is the required inverse to $p_* : \pi_1(E) \rightarrow \pi_1(B)$.

Conversely, suppose a) and b) hold, Let \tilde{E} be the covering space of E corresponding to a subgroup H of $\pi_1(E)$ such that $p_* : H \rightarrow \pi_1(B)$ is an isomorphism. Then $p' = p\pi : \tilde{E} \rightarrow B$ is a fibration with fibre F' and we

have the commutative diagram

$$\begin{array}{ccc}
 F' & \xrightarrow{\bar{\pi}} & F \\
 i' \downarrow & & \downarrow i \\
 \tilde{E} & \xrightarrow{\pi} & E \\
 p' \downarrow & & \downarrow p \\
 B & \xrightarrow{1} & B
 \end{array}$$

where $\bar{\pi}$ is the restriction of π .

Now we will show that $F' = \tilde{F}$. Notice that

$$p'_* : \pi_1(\tilde{E}) \rightarrow \pi_1(B)$$

is the composition of isomorphisms

$$\pi_1(\tilde{E}) \xrightarrow{\pi_*} H \xrightarrow{p_*} \pi_1(B)$$

and is therefore an isomorphism. Hence F' is path connected. Since $\bar{\pi} : F' \rightarrow F$ is a fibration with discrete fiber, it is a covering space [9, Theorem 10, p. 78].

We will show now that $\pi_1(F') = 0$. If $v \in \pi_1(F')$ then $i'_*(v) = 0$ by exactness. Hence $0 = \pi_* i'_*(v) = i_* \bar{\pi}_*(v)$. But $\bar{\pi}_*$ is injective since $\bar{\pi}$ is a covering and i_* is injective by hypothesis a). Therefore $v = 0$ as was to be shown.

Next we turn to question B in the case of universal coverings and oriented fibrations. We need a few technical remarks.

First, let us recall the definition of the first evaluation subgroup $G_1(F) \subset \pi_1(F)$. A homotopy $h_t : F \rightarrow F$ is called a *cyclic homotopy* if $h_0 = h_1 = 1_F$. The loop τ given by $\tau(t) = h_t(*)$ (where $*$ $\in F$ is the base point) is called the *trace* of the homotopy. Then $G_1(F)$ consists of the subgroups of $\pi_1(F, *)$ whose elements are represented by loops which are traces of cyclic homotopies. For a list of properties of $G_1(F)$, see [5].

We also need the fact (see theorem 16.9, [3]) that for any fibre F , there exists a universal fibration $F \rightarrow E_\infty \rightarrow B_\infty$. Now the transgression $d_\infty : \pi_2(B_\infty) \rightarrow \pi_1(F)$ is related to $G_1(F)$ by the fact that $d_\infty(\pi_2(B_\infty)) = G_1(F)$. Now, as always, we assume that F is connected. Then let \tilde{B}_∞ be the universal covering space of B_∞ . Let $F \rightarrow D \rightarrow \tilde{B}_\infty$ denote the fibration classified by the covering map $\pi : \tilde{B}_\infty \rightarrow B_\infty$. It is easy to see that $d(\pi_2(\tilde{B}_\infty)) = G_1(F)$, where d is the transgression $d : \pi_2(\tilde{B}_\infty) \rightarrow \pi_1(F)$. It also is true that every oriented fibration may be regarded as a pullback of $F \rightarrow D \rightarrow \tilde{B}_\infty$. This follows since a classifying map $k : B \rightarrow B_\infty$ of an oriented fibration $F \rightarrow E \rightarrow B$ maps $\pi_1(B)$ to the identity of $\pi_1(B_\infty)$.

Hence, there is a lifting of k to $\tilde{k} : B \rightarrow \tilde{B}_\infty$ such that $k = \pi\tilde{k}$.

Now we can state and prove

THEOREM 2: *There exists a covering fibration of any oriented fibration with connected fibre F extending the universal covering space \tilde{F} if and only if $G_1(F)$ is trivial.*

PROOF: Let $F \xrightarrow{i} D \xrightarrow{p} \tilde{B}_\infty$ be the fibration mentioned above. We shall show that there exists a covering fibration $\tilde{D} \rightarrow \tilde{B}_\infty$ which extends \tilde{F} if and only if $G_1(F) = 0$. The pull back of the classifying map for $F \rightarrow E \rightarrow B$, denoted by $k : B \rightarrow B_\infty$, of $\tilde{F} \rightarrow D \rightarrow B$ will be the required covering fibration. That is, $\tilde{D} = k^*(D)$.

Now $\pi_1(\tilde{B}_\infty)$ is trivial. Thus (b) of theorem 1 is satisfied. On the other hand, $G_1(F) = 0$ implies that $d : \pi_2(\tilde{B}_\infty) \rightarrow \pi_1(F)$ is trivial. Hence $i_* : \pi_1(F) \rightarrow \pi_1(D)$ is injective. So condition (a) is satisfied and so we may apply theorem 1.

If we assume that $G_1(F) \neq 0$, then by theorem 1, we cannot find a covering fibration of $F \rightarrow D \rightarrow \tilde{B}_\infty$ extending \tilde{F} .

COROLLARY 3: *Let F be a compact polyhedron and $\chi(F) \neq 0$. Then there exists a covering fibration of any oriented fibration extending F , the universal covering F .*

PROOF: We know that $\chi(F) \neq 0$ implies that $G_1(F) = 0$.

COROLLARY 4: *Let F be a compact polyhedron with $\chi(F) \neq 0$. Then there exists a cross-section over the two-skeleton of the base space of any oriented fibration with fibre F .*

PROOF: By the corollary above, we have

$$\begin{array}{ccc}
 \tilde{F} & \xrightarrow{\pi} & F \\
 \downarrow & & \downarrow \\
 \tilde{E} & \xrightarrow{\pi} & E \\
 \downarrow p & & \downarrow \\
 B & \longrightarrow & B
 \end{array}$$

where B is a 2-dimensional CW complex.

The groups in which the obstructions to a cross-section of $\tilde{E} \xrightarrow{p} B$ lie must vanish. Thus there is a cross-section $c : B \rightarrow \tilde{E}$. Now πc is the required cross-section.

Now we shall study those fibrations with fibres RP^n , real projective space. We begin by giving alternative proofs to some results of Olum.

LEMMA 5 (Olum [8]) :

$$\xi_0(RP^n) \cong Z_2.$$

PROOF: Let $\hat{\xi}_0(S^n)$ denote the group of homotopy classes of base point preserving equivariant homotopy equivalences of S^n . We have an isomorphism

$$\xi_0(RP^n) = \hat{\xi}_0(S^n)$$

defined by sending f to the unique base point preserving map \tilde{f} that covers f . By the method of proof of [2, TH. 2.5], the operation of suspension defines an isomorphism

$$\hat{\xi}_0(S^n) = \hat{\xi}_0(S^{n+1}), \quad n \geq 1.$$

Therefore, we have

$$\xi_0(RP^n) = \xi_0(RP^{n+1}), \quad n \geq 1.$$

Finally,

$$\xi_0(RP^1) = \xi_0(S^1) \cong Z_2.$$

COROLLARY 6.

$$\xi(RP^{2n+1}) \cong Z_2; \xi(RP^{2n}) \cong 0.$$

PROOF: Let L denote the space of self homotopy equivalences of RP^n . Let L_0 be the subspace of maps of L which preserve the base point. Now $G_1(RP^{2n}) = 0$ and

$$G_1(RP^{2n+1}) = \pi_1(RP^{2n+1});$$

see [4], corollary I.6 and theorem II.5. From the exact sequence arising from the fibration $L_0 \rightarrow L \xrightarrow{\omega} RP^n$, and noting that the image of ω_* is $G_1(RP^n)$, we see that $d: \pi_1(RP^n) \rightarrow \pi_0(L_0)$ is zero if n is odd and injective if n is even. Thus $i_*: \pi_0(L_0) \rightarrow \pi_0(L)$ is injective if n is odd and has kernel Z_2 if n is even. But $\pi_0(L_0) = \xi_0(RP^n) \cong Z_2$ by lemma 5. Also $i_*: \pi_0(L_0) \rightarrow \pi_0(L)$ must be onto since RP^n is connected. Thus $\xi(RP^n) = \pi_0(L)$ is Z_2 when n is odd and 0 when n is even.

COROLLARY 7. *Every fibration with fibre RP^{2n} is orientable.*

PROOF: Since $\xi(RP^{2n})$ is trivial, the fundamental group of the base must act trivially on $\xi(RP^{2n})$.

THEOREM 8. *Every fibration with fibre RP^{2n} is covered by an S^{2n} fibration with an involution.*

PROOF: The conclusion means that we can always find a covering fibration which extends the universal cover S^{2n} of RP^{2n} . But this follows immediately from theorem 2 and corollary 7. The extended total space must be a 2-fold covering of the original total space, and the deck transformation is the involution.

REMARK: Theorem 8 is not true for RP^{2n+1} even when the fibration is orientable. Consider the “universal oriented fibration”,

$$RP^{2n+1} \xrightarrow{i} D \rightarrow \tilde{B}_\infty.$$

Now

$$i_* : \pi_1(RP^{2n+1}) \rightarrow \pi_1(D)$$

is trivial since

$$G_1(R^{2n+1}) = Z_2.$$

Thus in theorem 1 condition a) fails.

The next corollary is an application of theorem 8.

COROLLARY 9. *Let $RP^{2n} \rightarrow E \rightarrow B$ be a fibration. Then*

$$H^*(E; Z_2) = H^*(RP^{2n}; Z_2) \otimes H^*(B; Z_2)$$

as Z_2 -vector spaces.

PROOF: Let $S^{2n} \rightarrow \tilde{E} \rightarrow B$ be an S^{2n} -fibration with an involution which covers $RP^{2n} \rightarrow E \rightarrow B$. Let $\lambda : \tilde{E} \rightarrow S_\infty$ be an equivariant map and let $\lambda : E \rightarrow RP^\infty$ denote the quotient map. Then the composition $RP^{2n} \xrightarrow{i} E \xrightarrow{\lambda} RP^\infty$ sends the generator $c \in H^1(RP^\infty; Z_2)$ to the generator $\hat{c} \in H^1(RP^{2n}; Z_2)$. A cohomology extension of the fiber

$$\theta : H^*(RP^{2n}; Z_2) \rightarrow H^*(E; Z_2)$$

is defined by $\theta(\hat{c}^t) = \lambda^*(c^t)$, $t \geq 0$. The corollary now follows from the Leray-Hirsch theorem [9, p. 257].

3. Arbitrary coverings

We shall restrict our attention to coverings of fibre *bundles*. Our technique can probably be extended to Hurewicz fibrations, or Dold fibrations, but the needed propositions have not been written down yet. First we answer question B for fibre bundles. Then follow applications.

Let G be a group of homeomorphisms of F onto itself. Let G^* be the self homeomorphisms of a covering of F , denoted \tilde{F} , which are liftings of homeomorphisms in G . Let $\Phi : G^* \rightarrow G$ be the map which takes a map $f \in G^*$ to the induced map $\Phi(f) \in G$. Then Φ is continuous and also is a homomorphism.

THEOREM 10: *If there exists a cross-section $c : G \rightarrow G^*$ to Φ which is also a homomorphism, then there is a covering bundle of any G -bundle with fibre F which extends \tilde{F} .*

PROOF: Let $F \rightarrow E \rightarrow B$ be a G -bundle and let $G \rightarrow E^* \rightarrow B$ be the associated principal G -bundle. Then consider the G -bundle $\tilde{F} \rightarrow E \times_G \tilde{F} \rightarrow$

B where G acts on \tilde{F} by means of the cross-section c . Then $\tilde{E} = E \times_G \tilde{F}$ is the required covering. The projection $\pi : \tilde{E} \rightarrow E$ is given by

$$\pi(\langle e, \tilde{x} \rangle) = \langle e, \pi(\tilde{x}) \rangle \in E^* \times_G F = E.$$

The next theorem gives conditions for Theorem 10 to hold. We shall let G_e^* be the identity component of G^* and G_e the identity component of G .

THEOREM 11: *A covering bundle extending \tilde{F} always exists if a) $G_1(F) \subset \pi_1(\tilde{F})$, and b) there is a homomorphism*

$$\bar{c} : G/G_e \rightarrow G^*/G_e^*$$

which is a cross-section to the homomorphism

$$\Phi : G^*/G_e^* \rightarrow G/G_e$$

induced by Φ .

PROOF: Note that $\Phi : G^* \rightarrow G$ is a fibration with a discrete fibre. The fibre over 1_F consists of the liftings of 1_F . Assume that $\tilde{f} \in G_e^*$ is a lifting of 1_F . Then the path from \tilde{f} to $1_{\tilde{F}}$ induces a cyclic homotopy $h_t : F \rightarrow F$. The trace of h_t must represent an element in $\pi_1(\tilde{F})$ by a). Thus the trace lifts to a closed path in \tilde{F} . Thus \tilde{f} has a fixed point and hence $\tilde{f} = 1_{\tilde{F}}$.

This fact allows us to conclude that G_e and G_e^* are isomorphic. Thus we may construct the cross-section required by theorem 10 over G_e , and condition b) allows us to extend the cross-section over the other components.

REMARK: If G is connected and $G_1(F) = 0$, we may extend any covering \tilde{F} to a covering G -bundle of any arbitrary G -bundle.

COROLLARY 12: *If F is a compact polyhedron and $\chi(F) \neq 0$, then \tilde{F} can be extended to a covering bundle for any fibre bundle with connected structural group.*

Let M be a closed topological manifold which is unorientable and let \tilde{M} denote its oriented double covering. Let $0(M)$ be the subgroup of $\pi_1(M)$ of elements represented by orientation preserving loops. Then $0(\tilde{M})$ is the subgroup of $\pi_1(\tilde{M})$ corresponding to the oriented double covering \tilde{M} .

From this point on, we shall consider fibre bundles with fibre M and study covering bundles which extend \tilde{M} . The end result will yield the main application of these techniques, Theorem 15.

THEOREM 13: *For any fibre bundle with fibre M , a closed unorientable topological manifold, the oriented double covering \tilde{M} extends to a covering*

bundle. In addition, the structural group of this covering bundle preserves the orientation of \tilde{M} .

PROOF: Every homeomorphism $h : M \rightarrow M$ has two liftings $\tilde{M} \rightarrow \tilde{M}$, one of which preserves and the other reverses the orientation of \tilde{M} . Consider the correspondence i which sends every $h \in G$ to its orientation preserving lifting in G^* . It is easy to see that $i : G \rightarrow G^*$ is continuous, a homomorphism of groups and a cross-section of $\Phi : G^* \rightarrow G$. Thus theorem 10 is satisfied. The group of the covering fibre bundle is $i(G)$, so the second statement of the theorem is clearly true.

If we compare theorem 13 with condition a) of theorem 11, we are lead to conjecture that $G_1(M) \subset O(M)$. This is in fact true, as the following theorem shows.

THEOREM 14:

$$G_1(M) \subset O(M).$$

PROOF: Let $\alpha \in G_1(M)$. Then there is a cyclic homotopy $h_t : M \rightarrow M$ whose trace represents α . Now h_t lifts to a homotopy $\tilde{h}_t : \tilde{M} \rightarrow \tilde{M}$ and \tilde{h}_1 is a lifting of the identity. Since \tilde{h}_1 is homotopic to $1_{\tilde{M}}$, \tilde{h}_1 preserves the orientation on \tilde{M} . There are only two liftings of the identity and one of them reverses orientation. So $\tilde{h}_1 = 1_{\tilde{M}}$. Thus the trace of \tilde{h}_t is a loop which covers the trace of h_t . Hence α must be in $O(M)$.

THEOREM 15: *Let $M \rightarrow E \xrightarrow{\pi} B$ be a fibre bundle with fibre a closed topological n -manifold and with structural group G . If $\chi(M) \neq 0 \pmod{p}$, where p is a prime, then*

$$\pi^* : H^*(B; Z_p) \rightarrow H^*(E; Z_p)$$

is injective.

PROOF: First consider the case where $\pi_1(B)$ operates trivially on $H^n(M; Z_p)$. Then the theorem is true if M is orientable or if $p = 2$ [6, Theorem 12]. Suppose now that M is unorientable and $p \neq 2$. Let \tilde{M} denote the orientable double covering of M . By theorem 13, we have

$$\begin{array}{ccc} \tilde{M} & & M \\ \downarrow & & \downarrow \\ \tilde{E} & \longrightarrow & E \\ \tilde{\pi} \downarrow & & \downarrow \pi \\ B & \xrightarrow{1} & B \end{array}$$

and $\pi_1(B)$ operates trivially on $H^n(\tilde{M}; Z_p)$. Now $\chi(\tilde{M}) = 2\chi(M) \neq 0 \pmod{p}$. Since \tilde{M} is orientable it follows that $\tilde{\pi}^*$ is injective. Then by commutativity π^* is injective.

Now consider the case where $\pi_1(B)$ operates non-trivially on $H^n(M; Z_p)$ p odd. Let $K \subset \pi_1(B)$ denote the normal subgroup of index 2 consisting of elements which operate trivially on $H^n(M; Z)$. Let $\tilde{B} \xrightarrow{q} B$ denote the 2-fold covering which corresponds to K . We have

$$\begin{array}{ccc}
 M & & M \\
 \downarrow & & \downarrow \\
 \tilde{E} & \xrightarrow{\tilde{q}} & E \\
 \tilde{\pi} \downarrow & & \downarrow \pi \\
 \tilde{B} & \xrightarrow{q} & B
 \end{array}$$

where $\tilde{E} \xrightarrow{\tilde{q}} \tilde{B}$ is the bundle induced by q . Since $\pi_1(B)$ operates trivially on $H^n(M; Z_p)$ we have from the preceding paragraph that $\tilde{\pi}^*$ is injective. To show that π^* is injective it is now sufficient to show that

$$q^* : H^*(B; Z_p) \rightarrow H^*(\tilde{B}; Z_p)$$

is injective. There is the transfer map [10, Chapter 5]

$$\tau : H^*(\tilde{B}; Z_p) \rightarrow H^*(B; Z_p),$$

and τq^* , being multiplication by 2, is an isomorphism. Therefore q^* is injective. This completes the proof of the theorem.

This theorem was first noted by A. Borel in [1] with the extra hypotheses that the bundle is differentiable and oriented in some sense and M is oriented. In [6], the second author removed the differentiability hypothesis and the above theorem removes the orientability hypotheses.

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(Oblatum 8-V-72)