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The point-outercoarseness of complete $n$-partite graphs


<http://www.numdam.org/item?id=CM_1973__26_2_101_0>
THE POINT-OUTERCOARSENESS OF COMPLETE n-PARTITE GRAPHS

by

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Introduction

A subdivision of a graph $G$ is a graph $G_1$ obtained from $G$ by replacing an edge $x = uv$ of $G$ with a new vertex $w$ together with edges $uw$ and $vw$. Graph $H$ is said to be homeomorphic from graph $G$ if $H$ can be obtained from $G$ by a finite sequence of subdivisions. The subgraph of $G$ induced by a set $W$ of vertices has vertex set $W$ and its edge set is the set of edges of $G$ which are incident with two vertices of $W$. The subgraph of $G$ induced by an edge set $Y$ has $Y$ as its edge set and contains all vertices incident with at least one edge of $Y$. For a real number $r$, $\lfloor r \rfloor$ denotes the greatest integer not exceeding $r$, and $\{r\}$ is the least integer not less than $r$.

Let $p_1 \leq p_2 \leq \cdots \leq p_n$ be positive integers. Then the complete $n$-partite graph $K(p_1, \cdots, p_n)$ has $p = \sum_i p_i$ vertices, its vertex set can be partitioned into subsets $V_i$, $1 \leq i \leq n$, such that $|V_i| = p_i$, and two vertices are adjacent if and only if they are in different $V_i$. The sets $V_1, \cdots, V_n$ are called the partite sets of $K(p_1, \cdots, p_n)$. If each $p_1 = 1$, $K(p_1, \cdots, p_n)$ is denoted by $K_n$ and called the complete graph on $n$ vertices.

An outerplanar graph is a graph which can be embedded in the plane so that every vertex of $G$ lies on the exterior region. In [5] Chartrand and Harary have characterized outerplanar graphs as those graphs which contain no subgraph homeomorphic from $K_4$ or $K(2,3)$.

We define, for each positive integer $n$, the vertex partition number of a graph $G$, denoted by $\pi_n(G)$, as the maximum number of subsets into which the vertex set of $G$ can be partitioned so that each set induces a graph which contains a subgraph homeomorphic from $K_{n+1}$ or the complete 2-partite graph $K([n+2]/2), \{(n+2)/2\})$. This general parameter was first introduced by Chartrand, Geller and Hedetniemi in [4].

For $i = 1, 2, 3, 4$, $\pi_i(G)$ is the maximum number of point induced disjoint subgraphs of $G$ which are totally disconnected, acyclic, outerplanar, and planar, respectively.

The edge partition number $\pi'_n(G)$ is defined analogously to $\pi_n(G)$ with the word 'vertex' replaced by 'edge'. Then $\pi'_1(G)$ is simply the number
of lines of $G$. The only line partition number which has been given considerable study is $\pi_4'(G)$, which is called the coarseness of $G$. This has been investigated by Beineke [1], Beineke and Chartrand [2], Guy [9], and Beineke and Guy [3], with the last paper giving a partial formula for $\pi_4'(K(m, n))$.

The number $\pi_1(G)$ is the well-known line independence number, see Harary [10]. The number $\pi_2(G)$ has been studied by Corrádi and Hajnal [7], Dirac and Erdős [8], and Chartrand, Kronk, and Wall [6]. In this paper we investigate $\pi_3(G)$ which is called the point-outercoarseness of $G$ and is now denoted simply $\pi(G)$.

**Preliminary results**

We make two easy observations and then commence the development of the formula for $\pi(K(p_1, \cdots, p_n))$. Any non-outerplanar graph has at least 4 vertices and 6 edges. This implies

**REMARK 1.** If $G$ is a graph with $p$ points and $q$ edges, then $\pi(G) \leq \lfloor p/4\rfloor$ and $\pi(G) \leq \lfloor q/6\rfloor$.

The maximum number of vertices in any complete subgraph of $G$ is denoted $\omega(G)$ and is called the clique number of $G$.

**REMARK 2.** If $G$ has $p$ vertices and $\omega(G) \leq 3$, then $\pi(G) \leq \lfloor p/5\rfloor$.

**THEOREM 1.** Let $G = K(p_1, \cdots, p_n)$ with $n \geq 2$. If $p_n \geq \left(\frac{3}{2}\right)(p-p_n)$, then $\pi(G) = \lfloor(p-p_n)/2\rfloor$.

**PROOF.** In any decomposition of $G$ into non-outerplanar subgraphs, each subgraph must include at least two vertices from $\bigcup_{i=1}^{n-1}V_i$. There are $p-p_n$ vertices in this set so that $\pi(G) \leq \lfloor(p-p_n)/2\rfloor$.

Any subgraph induced by a set consisting of three vertices from $V_n$ and two vertices from $\bigcup_{i=1}^{n-1}V_i$ is not outerplanar. From the hypothesis that $p-p_n \leq \left(\frac{3}{2}\right)p_n$ it follows that there are $\lfloor(p-p_n)/2\rfloor$ disjoint induced non-outerplanar subgraphs of $G$. Thus $\pi(G) = \lfloor(p-p_n)/2\rfloor$.

**THEOREM 2.** If $G = K(p_1, \cdots, p_n)$ where $n = 2$ or $3$, and $p_n \leq \left(\frac{3}{2}\right)(p-p_n)$, then $\pi(G) = \lfloor p/5\rfloor$.

**PROOF.** Since $\omega(G) \leq 3$, Remark 2 implies that $\pi(G) \leq \lfloor p/5\rfloor$. In order to show that $\lfloor p/5\rfloor$ non-outerplanar, mutually disjoint, induced subgraphs of $G$ exist we consider two cases.

**Case (i).** $n = 2$. Since $p_1 \leq p_2 \leq \left(\frac{3}{2}\right)p_1$, there are $p_2-p_1$ mutually disjoint sets of vertices such that each set contains three vertices from $V_2$ and two vertices from $V_1$. Each of these sets induces a non-outerplanar
copy of $K(2, 3)$. There are $p_1 - 2(p_2 - p_1) = 3p_1 - 2p_2 \geq 0$ other vertices in $V_1$ and $p_2 - 3(p_3 - p_1) = 3p_1 - 2p_2 \geq 0$ other vertices in $V_2$. Call these sets $V'_1$ and $V'_2$ respectively. If $3p_1 - 2p_2 = 0, 1,$ or $2$, then we have partitioned $G$ into $\lceil p/5 \rceil$ non-outerplanar subgraphs. If $3p_1 - 2p_2 \geq 3$, then by alternating the use of three vertices from $V'_2$ and two vertices from $V'_1$ with two from $V'_2$ and three from $V'_1$, we can complete the partition of $\nu(G)$ into $\lceil p/5 \rceil$ sets of cardinality five, each of which induces a non-outerplanar graph. Thus $\pi(G) = \lceil p/5 \rceil$ in this case.

Case (ii). $n = 3$. If $p_1 + p_2 \leq p_3$ we consider graph $H$ which is $G$ minus all edges joining $V_i$ to $V_2$. From case (i), $\pi(G) \geq \pi(H) \geq \lceil p/5 \rceil$. Thus we suppose $p_3 < p_1 + p_2$. For $i = 1, 2, 3$, let $V'_i = V_i$. Form one copy of $K(2, 3)$ with three vertices $v_i$, $v_2$, $v_3$ of $V'_3$ and two vertices $v_4$, $v_5$ of $V'_2$. Let $V'_1$, $V'_2$ be an ordering of $V'_1$ and $V'_2 - \{v_4, v_5\}$ so that $|V'_1| \leq |V'_2|$, and let $V'_3 = V_3 - \{v_1, v_2, v_3\}$. Then repeat this procedure with $V'_1$, $V'_2$, and $V'_3$, and continue this procedure until reaching a non-negative integer $j$ such that $V'_3 \leq V'_2$. (Note that $j$ may be zero.) Let $V'_1, V'_2, V'_3$ be a reordering of $V'_1$, $V'_2, V'_3$ such that

$$|V'_1| \leq |V'_2| \leq |V'_3|$$

and observe that

$$0 \leq |V'_3| - |V'_2| \leq 2.$$

Continue the partition of $G$ into copies of $K(2, 3)$ by using three vertices $w_1$, $w_2$, $w_3$, from $V'_3$ and two vertices $w_4$, $w_5$ from $V'_2$. Let

$$V'_1, V'_2 - \{w_4, w_5\}, V'_3 - \{w_1, w_2, w_3\}$$

be reordered by $V'_1, V'_2, V'_3$ so that

$$|V'_1| \leq |V'_2| \leq |V'_3|.$$

We stop this procedure when $|V'_k| \leq 3$ for some $k \geq j + 1$. If

$$|V'_1| + |V'_2| + |V'_3| \leq 4,$$

then $G$ has been partitioned into $\lceil p/5 \rceil$ non-outerplanar graphs. Otherwise induce one more non-outerplanar graph with the remaining vertices. Thus $\pi(G) \geq \lceil p/5 \rceil$, which completes the proof of the theorem.

Corollary 3. If $G = K(p_1, \cdots, p_n)$ where $n = 2$ or $3$ then $\pi(G) = \min \{\lceil p/5 \rceil, [(p - p_n)/2]\}$.

Theorem 4. Let $G = K(p_1, \cdots, p_n)$ where $p_n \leq \frac{1}{3}(p - p_n)$. Then $\pi(G) \geq \lceil p/5 \rceil$. 

PROOF. We use induction and observe that Theorem 2 verifies the result for \( n = 2 \) or \( 3 \). Assume Theorem 4 holds for \( n \geq 3 \) and let \( G = K(p_1, \ldots, p_{n+1}) \) where \( p_{n+1} \leq (3/2)(p-p_{n+1}) \). The subgraph of \( G \) formed by removing all edges joining \( V_1 \) with \( V_2 \) is a complete \( n \)-partite graph \( H = K(p_1', \ldots, p_n') \) where \( p'_n = \max\{p_{n+1}, p_1 + p_2\} \). Since \( p'_n \leq (3/2)(p_1' + \cdots + p_{n-1}') \), the inductive assumption applies and we have \( \pi(G) \geq \pi(H) \geq [p/5] \).

The following lemma will be helpful.

**Lemma 1.** Let \( c \) be an integer such that \( 1 \leq c \leq n \). If \( p_n - p_{n-c+1} \leq 1 \), then the complete \( n \)-partite graph \( G = K(p_1, \ldots, p_n) \) contains \( [p/c] \) mutually disjoint copies of \( K_c \).

**Proof.** We use induction on \( p \). If the order of \( G \) is less than \( n + c \), then \( p_n - p_{n-c+1} = 1 \). We form one copy of \( K_c \) by selecting one vertex from each \( V_i, i = n-c+1, \ldots, n \). The remaining vertices of \( G \) induce a complete graph on \( p - c \) vertices. Thus \( G \) contains \( [p/c] \) mutually disjoint copies of \( K_c \).

Let the order of \( G \) be \( p \geq n + c \) and suppose the lemma is true for all complete \( n \)-partite graphs with less than \( p \) vertices. Form one copy of \( K_c \) by selecting one vertex from each of \( V_{n-c+1}, \ldots, V_n \). The graph \( H \) induced by the remaining vertices of \( G \) is a complete \( n \)-partite graph with \( p'_n - p'_n - c + 1 \leq 1 \) where \( p'_i \) is the order of the \( i \)-th partite set of \( H \). By the induction hypothesis \( H \) contains \( [(p-c)/c] \) mutually disjoint copies of \( K_c \) and the lemma is proved.

**Theorem 5.** Let \( G = K(p_1, \ldots, p_n) \) where \( n \geq 4 \). If \( p \geq 4 p_n \), then \( \pi(G) = [p/4] \).

**Proof.** We use induction on \( p_n \). If \( p_n = 1 \), \( G \) is the complete graph with \( p = n \) vertices and \( \pi(G) = [p/4] \). Suppose the theorem holds if \( p_n = k \geq 1 \) and let \( p_n = k + 1 \). Remove one vertex from each \( V_n, V_{n-1}, V_{n-2}, \) and \( V_{n-3} \). The resulting graph \( H \) is a complete \( m \)-partite graph with \( n \geq m \geq 4 \) and the largest partite set in \( H \) has \( p_{n-1} = k \) or \( p_n \) vertices. The latter case implies that \( p_n - p_{n-3} = 0 \), and Lemma 1 proves the theorem. In the former case the inductive assumption implies \( \pi(H) = [(p-4)/4] \) and thus \( \pi(G) = [p/4] \).

**The principal result**

Before stating the main theorem, we prove another lemma.

**Lemma 2.** Let \( G = K(p_1, \ldots, p_n) \) with \( n \geq 3 \). If \( r \) is a positive integer such that \( p \geq 3r \), \( p_1 + \cdots + p_{n-1} \geq 2r \), and \( p_1 + \cdots + p_{n-2} \geq r \), then \( G \) contains at least \( r \) mutually disjoint triangles.
Proof. For \( i = 1, \ldots, n \), let \( V_i^0 = V_i \). Form one triangle with vertices
\[
 V_{n-2}, V_{n-1}, V_n \text{ of } V_{n-2}^0, V_{n-1}^0, V_n^0
\]
respectively. Let
\[
 V_i^1 = V_i^0 - \{v_n\} \text{ and } V_i^1, \ldots, V_{n-1}^1
\]
be a recording of
\[
 V_1^0, \ldots, V_{n-3}^0, V_{n-2}^0 - \{v_{n-2}\}, V_{n-1}^0 - \{v_{n-1}\}
\]
such that
\[
 |V_i^1| \leq |V_{i+1}^1| \text{ for } i = 1, 2, \ldots, n-2.
\]
Repeat this procedure until either
\[
 |V_n^k| - |V_{n-2}^k| \leq 1 \text{ and } |V_{n-2}^k| \neq 0
\]
for some \( k \) or \( |V_{n-2}^k| = 0 \) for some \( k \). If the former occurs first, then from Lemma 1, it follows that \( G \) contains at least \( r \) mutually disjoint triangles. Thus suppose \( |V_{n-2}^k| = 0 \) for some \( j \) and consider two cases.

Case (i) \( |V_{n-1}^i| - |V_{n-2}^i| \leq 1 \) for some \( i < k \). Each of the \( k \) triangles which have been formed contain one vertex of \( V_n \) and two vertices from distinct \( V_j, j = 1, \ldots, n-1 \). Since \( |V_{n-1}^i| - |V_{n-2}^i| \leq 1 \), Lemma 1 implies that at most one vertex of \( \bigcup_{1}^{n-1} V_j \) is not included in one of the triangles. Thus \( k = \lfloor (p_1 + \cdots + p_{n-1})/2 \rfloor \geq r \).

Case (ii). \( |V_{n-1}^i| - |V_{n-2}^i| > 1 \) for all \( i > k \). In this case
\[
 V_{n-1}^i \subset V_{n-1} \text{ for } i = 1, \ldots, k-1.
\]
Hence each of the \( k \) triangles contains exactly one vertex from \( \bigcup_{1}^{n-2} V_j \). This implies
\[
 k = |\bigcup_{1}^{n-2} V_j| \geq r
\]
and completes the proof.

Theorem 6. Let \( G = K(p_1, \ldots, p_n) \) with \( n \geq 2 \), then
\[
 \pi(G) = \begin{cases} 
 \lfloor (p-p_n)/2 \rfloor & \text{if } p \leq (\frac{4}{3})p_n \\
 \lfloor p/4 \rfloor & \text{if } p \geq 4p_n \\
 \lfloor (p+r)/5 \rfloor & \text{if } (\frac{3}{5})p_n < p < 4p_n
\end{cases}
\]
where
\[
 r = \min \{(p-p_n-p_{n-1}-p_{n-2}), [(p-p_n-p_{n-1})/2], [(3p-5p_n)/7]\}
\]
Proof. If \( p \leq (\frac{4}{3})p_n \) or \( p \geq 4p_n \), the result follows from Theorems 1 and 5. Thus we consider only \( (\frac{3}{5})p_n < p < 4p_n \) and distinguish three cases depending on \( r \).
CASE (i). $r = p - p_n - p_{n-1} - p_{n-2}$. Since

$$p - p_n - p_{n-1} - p_{n-2} \leq \frac{(p - p_n - p_{n-1})}{2}$$

we have

$$p - p_n - p_{n-1} - p_{n-2} \leq p_{n-2} \leq p_{n-1} \leq p_n.$$ 

That is the cardinality of $\bigcup_{i=1}^{n-3} V_i$ does not exceed the cardinality of $V_{n-2}$. Thus there are $r$ mutually disjoint copies of $K_4$ with one vertex in each of the sets

$$V_n, V_{n-1}, V_{n-2}, \bigcup_{i=1}^{n-3} V_i.$$

Let $G$ minus these $r$ copies of $K_4$ be denoted by $H$. Graph $H$ has $p - 4r$ vertices, and we let $V^1_i = V_i \cap V(H)$ for $i = n - 2, n - 1, n$. Since $r \leq (3p - 5p_n)/7$ we have $\frac{1}{3}(p_n - r) \leq p - p_n - 3r$, where $p_n - r = |V'_n|$ and

$$p - p_n - 3r = |V'_{n-1} \cup V'_{n-2}|.$$ 

Theorem 2 implies that $\pi(H) = [(p - 4r)/5]$. Hence

$$\pi(G) \geq r + [(p - 4r)/5] = [(p + r)/5].$$

Since $G$ does not contain more than $r$ copies of $K_4$, it is clear that $\pi(G) = [(p + r)/5]$. 

CASE (ii). $r = [(p - p_n - p_{n-1})/2] < [(3p - 5p_n)/7]$. In this case we consider the complete $(n - 1)$-partite graph $H = G - V_n$. By hypothesis

(1) 

$$r \leq p - p_n - p_{n-1} - p_{n-2}$$

and

(2) 

$$2r \leq p - p_n - p_{n-1}.$$ 

Inequality (2) together with $[(p - p_n - p_{n-1})/2] < [(3p - 5p_n)/7]$ imply

(3) 

$$r \leq p_{n-1}.$$ 

Adding (2) and (3) we obtain

(4) 

$$3r \leq \sum_{i=1}^{n-1} p_i.$$ 

Since (1), (2), and (4) hold, Lemma 2 implies that $H$ contains at least $r$ mutually disjoint triangles. The set $V_n$ contains $p_n \geq p_{n-1} \geq r$ vertices. Thus, $G$ contains $r$ mutually disjoint copies of $K_4$, each of which has one vertex from $V_n$ and three vertices from $\bigcup_{i=1}^{n-1} V_i$. There are $p - p_n - 3r$ other vertices in $\bigcup_{i=1}^{n-1} V_i$ and $p_n - r$ other vertices in $V_n$. 

The graph $G'$ induced by the remaining vertices of $G$ is a complete $m$-partite graph, $m \leq n$. Since $r < (3p - 5p_n)/7$, we have

(5) 

$$p - p_n - 3r > \frac{1}{3}(p_n - r).$$
That is the number of vertices in $V(G') - V_n$ is more than two-thirds the number of vertices in $V(G') \cap V_n$. From $r = \left(\frac{p - p_n - p_{n-1}}{2}\right)$ it follows that

$$p_n - r + 1 \geq p - p_n - 3r. \quad (6)$$

If a maximum partite set of $G'$ is $V(G') \cap V_n$, then (5) together with Theorem 4 imply that $\pi(G') \geq \left(\frac{p - 4r}{5}\right)$, and thus $\pi(G) \geq r + \pi(G') \geq \left(\frac{p + r}{5}\right)$. From (6) and the fact that $|V(G') - V_n| = p - p_n - 3r$ it follows that if $V(G') \cap V_n$ is not a largest partite set of $G'$, then a largest partite set contains exactly $p_n - r + 1 = p - p_n - 3r$. Thus $G'$ is a bipartite graph with partite sets $V'_1$ and $V'_2$ where $|V'_2| = p - p_n - 3r$ and $|V'_1| = p_n - r$. According to Theorem 4, $\pi(G') \geq \left(\frac{p - 4r}{5}\right)$ and $\pi(G) \geq \left(\frac{p + r}{5}\right)$.

In order to show that equality holds suppose $\pi(G) > \left(\frac{p + r}{5}\right)$. Then there are more than $r$ mutually disjoint copies of $K_4$ in $G$. Each copy of $K_4$ must contain two vertices from $\leq 2V_i$, so that $p - p_n - 3r > 2(r + 1)$. This implies that $\left(\frac{p - p_n - p_{n-1}}{2}\right) > r$ which contradicts the hypothesis for this case. Hence $\pi(G) = \left(\frac{p + r}{5}\right)$.

CASE (iii). $r = \left(\frac{3p - 5p_n}{7}\right)$. In this case we let $H = G - V_n$. From the hypothesis for this case we have

$$r \leq p - p_n - p_{n-1} - p_{n-2} \quad (7)$$

$$2r \leq p - p_n - p_{n-1} \quad (8)$$

Furthermore, $p - p_n - 3r \geq p - p_n - 3\left(\frac{3p - 5p_n}{7}\right) = \left(\frac{8}{7}\right)p_n - \left(\frac{8}{7}\right)p > 0$. Thus $p - p_n > 3r$ which together with (7), (8) and Lemma 2 imply that $H$ contains $r$ mutually disjoint triangles.

Since $4p_n > p$, we have that $3p_n > p - p_n > 3r$. Thus $V_n$ contains more than $r$ vertices. Graph $G$ has at least $r$ mutually disjoint copies of $K_4$ each consisting of one vertex of $V_n$ and three vertices of $\bigcup_{i=1}^{n-1} V_i$. There are $p - 4r$ other vertices in $G$. These vertices induce a complete $m$-partite subgraph $G'$ of $G$ with precisely $p_n - r$ vertices of $V_n$ and $p - p_n - 3r$ vertices of $\bigcup_{i=1}^{n-1} V_i$. Since $r \leq \left(\frac{3p - 5p_n}{7}\right)$ we have

$$\left(\frac{3}{2}\right)(p - p_n - 3r) \geq p_n - r > 0. \quad (9)$$

Let $W$ be a maximum partite set of $G'$. If $W = V_n \cap V(G')$, then (9) together with Theorem 4 imply that $\pi(G') \geq \left(\frac{p - 4r}{5}\right)$, and $\pi(G) \geq \left(\frac{p + r}{5}\right)$.

Suppose $W \neq V_n \cap V(G')$; then let $k = 4p_n - p$. Thus,

$$r = \left[\left(\frac{3p - 5p_n}{7}\right)\right] = \left[p_n - 3k/7\right] = p_n - \{3k/7\}$$
where \( \{x\} \) is the least integer not less than \( x \). We have

\[
(10) \quad p_n - r = \{3k/7\} \quad \text{and} \quad
(11) \quad p - p_n - 3r = 3p_n - k - 3(p_n - \{3k/7\}) = -k + 3\{3k/7\}.
\]

If \( k = 1 \), then the number of vertices in \( G' \) is \( p - 4r = p - 4p_n + 4\{3k/7\} = -1 + 4 = 3 \), and \( \pi(G) \geq r + [(p - 4r)/5] = [(p + r)/5] \). If \( k \geq 2 \), then from (10) and (11) we have \( p_n - r \geq (\frac{3}{7})(p - p_n - 3r) \). This implies that

\[
|V(G') - W| \geq |V_n| - r = p_n - r \geq (\frac{3}{7})(p - p_n - 3r) \geq (\frac{3}{7})|W|.
\]

Hence, according to Theorem 4,

\[
\pi(G') \geq [(p - 4r)/5] \quad \text{and} \quad \pi(G) \geq r + [(p - 4r)/5] = [(p + r)/5].
\]

Suppose \( \pi(G) > [(p + r)/5] \). Any decomposition of \( G \) into more than \([(p + r)/5]\) non-outerplanar graphs will necessarily contain \( r + t \) mutually disjoint copies of \( K_4 \), \( t > 0 \). Let \( V'_1, V'_2, \cdots, V'_m \) be the partite sets of the complete \( m \)-partite graph \( H^1 \) which remains after deleting these \( r + t \) copies of \( K_4 \) from \( G \). The order of \( H^1 \) is \( p - 4r - 4t \) and \( |V'_1| \geq |V'_c| \) where \( V'_c = V_n \cap V(H^1) \). We have \( r + t \leq p/4 < p_n \), and thus

\[
(12) \quad |V'_m| \geq |V'_c| \geq p_n - r - t > 0.
\]

Then

\[
(13) \quad |\bigcup_{i=1}^{m-1} V'_i| \leq |V(H^1)| - (p_n - r - t) = p_n - 3r - 3t.
\]

From the fact that \( r > (3p - 5p_n)/7 - t \) we obtain

\[
(14) \quad p - p_n - 3r - 3t < (\frac{3}{7})(p_n - r - t).
\]

Using (12), (13), and (14) we have

\[
|\bigcup_{i=1}^{m-1} V'_i| \leq p_n - 3r - 3t < (\frac{3}{7})(p_n - r - t) \leq (\frac{3}{7})|V'_m|.
\]

According to Theorem 1,

\[
\pi(H^1) = \lceil |\bigcup_{i=1}^{m-1} V'_i|/2 \rceil = s.
\]

Suppose \( t \geq 2 \). Since \( s \leq [(p - p_n - 3r - 3t)/2] \), the number of mutually disjoint non-outerplanar subgraphs in this decomposition does not exceed \( r + t + [(p - p_n - 3r - 3t)/2] \leq [p - p_n - r - t)/2] \). However, \( r + 2 > (3p - 5p_n)/7 + 1 \), which implies

\[
(15) \quad \left[ \frac{p - p_n - (r + 2)}{2} \right] \leq \left[ \frac{p - p_n - (3p - 5p_n + 7)/7}{2} \right] = \left[ \frac{2p - p_n}{7} - \frac{1}{2} \right].
\]
Also
(16) \[
\left\lceil \frac{p+r}{5} \right\rceil \geq \left\lceil \frac{(10p-5p_n-6)/7}{5} \right\rceil = \left\lceil \frac{2p-p_n-6}{35} \right\rceil.
\]

Since the right side of (15) is not more than the right side of (16), we have
\[
r + t + s \leq \left\lceil \frac{(p-p_n-r-2)/2}{5} \right\rceil \leq \left\lceil \frac{(p+r)/5} \right\rceil.
\]
That is this decomposition yields at most \( [(p+r)/5] \) mutually disjoint non-outerplanar subgraphs of \( G \).

If \( t = 1 \) and \( s = 0 \), then this decomposition yields \( r + 1 \) mutually disjoint non-outerplanar graphs and since \( |V_1| > 0 \) there is at least one vertex which is not included in any of the \( r + 1 \) copies of \( K_4 \). Thus
\[
r + 1 \leq r + \left\lceil \frac{(p-r)/5} \right\rceil = \left\lceil \frac{(p+r)/5} \right\rceil.
\]

Finally, we suppose \( t = 1 \) and \( s > 0 \). Each of these \( s \) graphs has at least five vertices with two vertices in \( \bigcup_1^{m-1} V_i \). Since
\[
| \bigcup_1^{m-1} V_i | < \frac{2}{3}|V_m|,
\]
one of these \( s \) graphs has six or more vertices. That is in the decomposition of \( G \) into \( r + t + s \) non-outerplanar mutually disjoint graphs one graph has more than 5 points. Thus there are \( r + t \) copies of \( K_4 \), one non-outerplanar graph with at least six vertices and at most \( \left\lceil \frac{(p - 4r - 4t - 6)/5} \right\rceil \) other non-outerplanar graphs. Since \( t = 1 \), this decomposition has at most \( r + 2 + \left\lceil \frac{(p - 4r - 10)/5} \right\rceil = \left\lceil \frac{(p+r)/5} \right\rceil \) non-outerplanar graphs.

Thus, in this case, \( \pi(G) = \left\lceil \frac{(p+r)/5} \right\rceil \) and the theorem is proved.

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(Oblatum 3-I-1972)