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by

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1. Introduction

The purpose of this note is to prove that every connected pseudoconcave complex Lie group is a complex torus. We outline the proof here, leaving the details for later sections.

Let G be a connected pseudoconcave complex Lie group of dimension n. The adjoint representation, Ad, maps G holomorphically into $\boldsymbol{C}^{n^2}$. If f is a function holomorphic on $\boldsymbol{C}^{n^2}$ then $f \circ Ad$ is a holomorphic function on G. Every pseudoconcave complex manifold has only constant holomorphic functions. Thus, since the holomorphic functions on $\boldsymbol{C}^{n^2}$ separate points, Ad maps G onto the identity matrix. This implies that G is abelian. Hence there is a discrete subgroup of $\boldsymbol{C}^n$, Γ such that G is biholomorphically isomorphic to $\boldsymbol{C}^n/\Gamma$. Since G has only constant holomorphic functions, Γ has rank n over $\boldsymbol{C}$. Thus we may assume that Γ is generated as a $\boldsymbol{Z}$-module by the $\boldsymbol{R}$-linearly independent vectors $v_1 \cdots, v_m, e_1, \cdots, e_n$, where e_i is the i-th unit vector in $\boldsymbol{C}^n$ and $m \leqq n$.

If we take vectors v_i' near enough to v_i then the vectors $v_1', \cdots, v_m'$, $e_1, \cdots, e_n$ will still be $\boldsymbol{R}$-linearly independent. Let Γ' be the group generated by these vectors and $G' = \boldsymbol{C}^n/\Gamma'$. Lemma 7, which uses some ideas of Morimoto [2], states that if $m < n$ then we can find vectors v_i' arbitrarily close to v_i so that there is a non-constant function which is holomorphic on G'. But Lemma 8 states that if the vectors v_i' are near enough to v_i then G' is pseudoconcave. The only alternative is that $m = n$ and therefore G is a complex torus.

2. Preliminary definitions and lemmata

For our purposes complex manifolds are assumed to be connected. Let X be an n-dimensional complex manifold and Y an open subset of X. Y is said to have smooth boundary if for every $p \in \partial Y$ there is an open neighborhood $U = U(p)$ and a real-valued C^∞-function φ with nowhere vanishing gradient defined on U such that $\{u \in U : \varphi(u) < 0\} = Y \cap U$. The analytic tangent plane to ∂Y at p is the unique $(n-1)$-dimensional complex vector space contained in the real tangent space at p. The Levi form of φ at p, $L_p(\varphi)$, is the hermitian form determined by the matrix $(\partial^2\varphi(p))/(\partial z_\alpha \partial \bar{z}_\beta)$. The signature of $L_p(\varphi)$ restricted to the analytic tangent plane at p is a biholomorphic invariant and is independent of the defining function φ. We say ∂Y is pseudoconcave at p if $L_p(\varphi)$ has at least one negative eigenvalue when restricted to the analytic tangent plane at p.

Let $\bar{D} = \{z \in \boldsymbol{C} : |z| \leqq 1\}$. Δ closed disc in X with center x is the image of $\bar{D}$ under a biholomorphic mapping into X with x corresponding to the origin. If ∂Y is pseudoconcave at p then there is a closed disc $\Delta \subset \bar{Y}$ with center p such that $\Delta \cap \partial Y = \{p\}$.

DEFINITION. Let X be a complex manifold of complex dimension $\geqq 2$. We say X is pseudoconcave if there is an open, non-empty, relatively compact subset $Y \Subset X$ such that ∂Y is smooth and everywhere pseudoconcave.

LEMMA 1. *Every function f holomorphic on a pseudoconcave manifold X is constant.*

PROOF. Let Y display the pseudoconcavity of X. Then $|f|$ takes its maximum on $\bar{Y}$ at $p \in \partial Y$. There is a closed disc $\Delta \subset \bar{Y}$ with center at p such that $\Delta \cap \partial Y = \{p\}$. $f|\Delta$ is holomorphic on Δ and assumes its maximum modulus at the center of Δ. Thus $f|\Delta$ is constant. Therefore $|f|$ takes its maximum on $\bar{Y}$ at points of Y. Thus the restriction of f to the connected component of Y containing $\Delta \sim \{p\}$ is constant. Thus f is constant.

Let G be an n-dimensional complex Lie group. For every $x \in G$ define the holomorphic automorphism A_x on G by $A_x(g) = xgx^{-1}$. The Lie algebra of G, $L(G)$, is defined to be the tangent space of G at the identity. Thus, for every $x \in G$, $dA_x : L(G) \to L(G)$ is an automorphism of $L(G)$. We have, therefore, the holomorphic mapping $Ad : G \to \mathrm{Aut}(L(G))$ defined by $Ad(x) = dA_x$. Since $\mathrm{Aut}(L(G)) \subset \boldsymbol{C}^{n^2}$, $Ad : G \to \boldsymbol{C}^{n^2}$.

LEMMA 2. *Let G be an n-dimensional pseudoconcave complex Lie group. There is a discrete subgroup of $\boldsymbol{C}^n$ of rank n over $\boldsymbol{C}$, Γ, such that G is biholomorphically isomorphic to $\boldsymbol{C}^n/\Gamma$.*

PROOF. Every function f holomorphic on $\boldsymbol{C}^{n^2}$ gives us a function $f \circ Ad$ holomorphic on G. $f \circ Ad$ is constant by Lemma 1. The holomorphic functions on $\boldsymbol{C}^{n^2}$ separate points. Thus $Ad[G] = \{I\}$, where I is the $n \times n$ identity matrix. Therefore G is abelian.

By standard results in Lie theory [1], G is bihomorphically isomorphic to $\boldsymbol{C}^n/\Gamma$, where Γ is a discrete subgroup of $\boldsymbol{C}^n$ of rank l over $\boldsymbol{C}$. After making a $\boldsymbol{C}$-linear change of coordinates in $\boldsymbol{C}^n$ we may assume that Γ

is generated as a $\boldsymbol{Z}$-module by the $\boldsymbol{R}$-linearly independent vectors $v_1, \cdots, v_m, e_1, \cdots, e_l$, where e_i is the i-th unit vector in $\boldsymbol{C}^n$. Thus $G \cong \boldsymbol{C}^l/\Gamma \times \boldsymbol{C}^{n-l}$. By Lemma 1, $l = n$ and thus Γ has rank n over $\boldsymbol{C}$.

If Γ has rank n over $\boldsymbol{C}$ and is generated over $\boldsymbol{Z}$ by the $\boldsymbol{R}$-linearly independent vectors $v_1 \cdots, v_m, e_1, \cdots e_n$ where e_i is the i-th unit vector in $\boldsymbol{C}^n$, then we follow the notation of [2] by writing $V = (v_1, \cdots, v_m)$, $\Gamma = \Gamma(V)$, and $M^*(n, m; \boldsymbol{C})$ as the set of such matrices V.

A proof of the following classical lemma can be found in [2].

LEMMA 3. *Suppose $V, V' \in M^*(n, m; \boldsymbol{C})$. Then $\boldsymbol{C}^n/\Gamma(V)$ and $\boldsymbol{C}^n/\Gamma(V')$ are biholomorphically isomorphic if and only if there is a matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL(n+m, \boldsymbol{Z})$, with A an $n \times n$ matrix, such that $(A+V'B)V = (C+V'D)$. Further, if only V'* (resp. V) *is known to be in $M^*(n, m; \boldsymbol{C})$ and such a matrix exists then V(resp. V') $\in M^*(n, m; \boldsymbol{C})$.*

3. The main result

$V \in M^*(n, m; \boldsymbol{C})$ is said to satisfy the Morimoto condition if there exist non-zero vectors $a \in \boldsymbol{Z}^n$ and $c \in \boldsymbol{Z}^m$ such that $aV = c$. The following lemma is due to Morimoto [2], but the proof given here is somewhat simpler.

LEMMA 4. *Let $G = \boldsymbol{C}^n/\Gamma(V)$, where $V \in M^*(n, m; \boldsymbol{C})$ satisfies the Morimoto condition. Then there exists $V' \in M^*(n, m; \boldsymbol{C})$ such that $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$ and G is biholomorphically isomorphic to $\boldsymbol{C}^n/\Gamma(V')$.*

PROOF. Assume that we have constructed a matrix

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \in GL(n+m, \boldsymbol{Z})$$

such that $BV-D$ is invertible and the last row of $C-AV$ is identically zero. Defining $V' = (C-AV)(BV-D)^{-1}$, we have $(A+V'B)V = C+V'D$. Thus ,by Lemma 3, $V' \in M(n, m; \boldsymbol{C})$ and $\boldsymbol{C}^n/\Gamma(V') \cong \boldsymbol{C}^n/\Gamma(V)$. Since the last row of $C-AV$ is identically zero, $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$. It remains to construct such a matrix.

V satisfies the Morimoto condition. Thus there are nonzero vectors $a = (a_1, \cdots, a_n) \in \boldsymbol{Z}^n$ and $c = (a_{n+1}, \cdots a_{n+m}) \in \boldsymbol{Z}^m$ such that $aV = c$. We may assume by renumbering the vectors v_i that $a_1 \neq 0$ and that $gcd(a_1, \cdots, a_{n+m}) = 1$. Set $gcd(a_1, \cdots, a_k) = p_k$ for $k = 2, \cdots, n+m$ and $p_1 = a_1$. We can find relatively prime integers α_k and γ_k such that $p_{k-1}\alpha_k + a_k\gamma_k = p_k$, $k = 2, \cdots, n+m$. For $j = 1, \cdots, i-1$ and $i = 2, \cdots, n$, the numbers $\beta_{ij} \equiv -a_j\gamma_i/p_{i-1}$ are integers, since p_{i-1} divides $a_1, a_2, \cdots, a_{i-1}$. Define $D_k \equiv \det M_k$, where

$$M_k = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_k \\ \beta_{21} & \alpha_2 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \beta_{k1} & \cdots & \beta_{k,k-1} & & \alpha_k \end{pmatrix}.$$

Let $r_i = -\gamma_i/p_{i-1}$. Subtract the first row of M_k multiplied by r_i from its i-th row, $i = 2, \cdots, k$. The resulting matrix is triangular with determinant

$$D_k = a_1 \prod_{1=2}^{k} (\alpha_i - r_i a_i) = p_1 \prod_{i=2}^{k} ((\alpha_i p_{i-1} + a_i \gamma_i)/p_{i-1}) = \prod_{i=1}^{k} p_i / \prod_{i=1}^{k-1} p_i = p_k,$$

$k = 2, \cdots, m+n$. Let $M = \begin{pmatrix} A & C \\ B & D \end{pmatrix}$, with A an $n \times n$-matrix, be M_{n+m} with first and n-th rows interchanged. Since $p_{n+m} = 1$, $M \in GL(n+m, \boldsymbol{Z})$.

Now

$$B = \begin{pmatrix} r_{n+1}\, a \\ \vdots \\ r_{n+m}\, a \end{pmatrix} \quad \text{and} \quad BV = \begin{pmatrix} r_{n+1}\, c \\ \vdots \\ r_{n+m}\, c \end{pmatrix}.$$

Let $c(k) = (a_{n+1}, \cdots, a_{n+k}, 0, \cdots, 0)$. Therefore

$$D = \begin{pmatrix} \alpha_{n+1} {}^t e_1 \\ \alpha_{n+2} {}^t e_2 + r_{n+2}\, c(1) \\ \vdots \\ \alpha_{n+m} {}^t e_m + r_{n+m}\, c(m-1) \end{pmatrix}.$$

where e_i is the i-th unit vector in $\boldsymbol{C}^m$.

Thus

$$BV - D = \begin{pmatrix} r_{n+1}\, a_{n+1} - \alpha_{n+1} & & * \\ & \ddots & \\ 0 & & r_{n+m}\, a_{n+m} - \alpha_{n+m} \end{pmatrix}.$$

Using the same argument as in the case of M_k, we see that $BV - D$ is invertible. Since $aV = c$, the last row of $C - AV$ is identically zero.

COROLLARY 5 *If $V \in M^*(n, m; \boldsymbol{C})$ and satisfies the Morimoto condition then there is a non-constant function holomorphic on $G = \boldsymbol{C}^n/\Gamma(V)$.*

PROOF. By Lemma 4 there exists $V' = \begin{pmatrix} V'' \\ 0 \end{pmatrix}$ such that G is biholomorphically isomorphic to $\boldsymbol{C}^n/\Gamma(V')$. But $\boldsymbol{C}^n/\Gamma(V') \cong \boldsymbol{C}^{n-1}/\Gamma(V'') \times \boldsymbol{C}^*$.

Since there are non-constant functions holomorphic on C^*, there are non-constant functions holomorphic on G.

COROLLARY 6. *Let G be a pseudoconcave, complex Lie group. Then G is biholomorphically isomorphic to $C^n/\Gamma(V)$ where $V \in M^*(n, m; C)$ does not satisfy the Morimoto condition.*

PROOF. By Lemma 2 there exists $V \in M^*(n, m; C)$ such that G is biholomorphically isomorphic to $C^n/\Gamma(V)$. Suppose V satisfies the Morimoto condition. Then, by Corollary 5, there is a non-constant function holomorphic on G. But, by Lemma 1, every function holomorphic on a pseudoconcave manifold is constant. Thus V does not satisfy the Morimoto condition.

For $n \times m$-matrices V and V' we define $d(V, V')$ as the euclidean distance between V and V' induced from C^{mn}. We topologize $M^*(n, m; C)$ via the metric d. V is said to have rational coordinates if $V = (v_{\alpha\beta}) = (x_{\alpha\beta} + iy_{\alpha\beta})$ with $x_{\alpha\beta}, y_{\alpha\beta} \in Q$. Define $M_Q^*(n, m; C)$ as the set of matrices in $M^*(n, m; C)$ with rational coordinates. $M_Q^*(n, m; C)$ is dense in $M^*(n, m; C)$.

LEMMA 7. *Suppose $m < n$. Then the matrices in $M_Q^*(n, m; C)$ satisfy the Morimoto condition. These are therefore dense in $M^*(n, m; C)$.*

PROOF. Suppose $V \in M_Q^*(n, m; C)$. Then $V = X + iY$, where $X = (x_{\alpha\beta})$, $Y = (y_{\alpha\beta})$ and $x_{\alpha\beta}, y_{\alpha\beta} \in Q$. Since $m < n$ there exists a non-zero row vector $a \in Q^n$ such that $aY = 0$. Define $c \equiv aX \in Q^m$. Thus $aV = c$. By clearing denominators we may take $a \in Z^n$ and $c \in Z^m$. Thus every matrix $V \in M_Q^*(n, m; C)$ satisfies the Morimoto condition.

The remaining step for the proof of the main theorem is to show that pseudoconcavity is invariant under small changes of the group.

LEMMA 8. *Suppose $V^0 \in M^*(n, m; C)$ and $G^0 = C^n/\Gamma(V^0)$ is a pseudoconcave Lie group. There exists $\varepsilon > 0$ such that if $V \in M^*(n, m; C)$ and $d(V, V^0) < \varepsilon$ then $G = C^n/\Gamma(V)$ is a pseudoconcave Lie group.*

PROOF. For every $V \in M^*(n, m; C)$ choose $v_{m+1}, \cdots, v_n$ such that $\{v_1, \cdots, v_m, v_{m+1}, \cdots, v_n, e_1, \cdots, e_n\}$ is a basis for R^{2n}. Define the real linear maps $S_V : R^{2n} \to R^{2n}$ by $v_i \to v_i^0$ and $e_i \to e_i$ for $i = 1, \cdots, n$. Thus $S_V : C^n \to C^n$ is a diffeomorphism such that $S_V(V, E_n) = (V^0, E_n)$. Hence S_V induces a diffeomorphism $T_V : G \to G^0$ such that

$$\begin{array}{ccc} C^n & \xrightarrow{S_V} & C^n \\ {\scriptstyle \pi_V}\downarrow & & \downarrow{\scriptstyle \pi_{V^0}} \\ G & \xrightarrow[T_V]{} & G^0 \end{array}$$

is commutative, where π_V(resp. π_{V^0}) is the quotient map which divides C^n by the group $\Gamma(V)$ (resp. $\Gamma(V^0)$).

If $d(V, V^0) < \varepsilon$ and ε is sufficiently small, then we may assume that $v_{m+1} = v^0_{m+1}, \cdots, v_n = v^0_n$. If $|| \; ||$ is any norm on C^n, then there exists a constant $c > 0$ such that

$$||Sz - z|| \leqq c\varepsilon ||z||$$

for any $z \in C^n$. This means that $||S - I|| \leqq c\varepsilon$ where I is the identity map of C^n onto itself.

Let Ω display the pseudoconcavity of G^0. It is enough to show that there exists $\varepsilon > 0$ such that $T_V^{-1}[\Omega]$ is a pseudoconcave, open subset of G when $d(V^0, V) < \varepsilon$. Since $\partial T_V^{-1}[\Omega]$ is compact, it is enough to prove that $\partial T_V^{-1}[\Omega]$ is pseudoconcave at each $p \in \partial T_V^{-1}[\Omega]$ when $d(V^0, V) < \varepsilon(p)$. Now π_V and π_{V^0} are locally biholomorphic. Hence we only need to prove that $\partial(S_V^{-1} \circ \pi_V^{-1}[\Omega])$ is pseudoconcave at some point $q \in \pi_V^{-1}(p)$. We may assume that $q = 0$, because translation acts biholomorphically on G (resp. G^0). Thus we have reduced the proof of this lemma to the following:

LEMMA 9. *Let $S_V : C^n \to C^n$ be the real linear transformation defined above. Let U be a neighborhood of 0 and φ a real valued C^∞-function defined on U with nowhere vanishing gradient and $\varphi(0) = 0$. Assume that 0 is a pseudoconcave boundary point of $U^- = \{u \in U : \varphi(u) < 0\}$. Then there exists $\varepsilon > 0$ such that $d(V^0, V) < \varepsilon$ implies that 0 is a pseudoconcave boundary point of $S_V^{-1}[U^-]$.*

PROOF. Set $\zeta = Sz$. By a suitable choice of the basis for coordinates in C^n, we may assume that

$$\varphi(\zeta_1, \cdots \zeta_n) = 2Re(\zeta_n + \sum \alpha_{ij}\zeta_i\zeta_j) + \sum \beta_{ij}\zeta_i\bar{\zeta}_j + 0(||\zeta||^3).$$

Since the origin is a pseudoconcave boundary point of U^-, we may as well assume that $\beta_{11} < 0$.

Let $\zeta = Sz$ be given by the equations

$$\zeta_i = \sum_{j=1}^{n} a_{ij} z_j + \sum_{j=1}^{n} b_{ij} \bar{z}_j. \qquad 1 \leqq i \leqq n$$

Because of the assumption $d(V, V^0) < \varepsilon$, we must have $a_{ij} - \delta_{ij} = 0(\varepsilon)$ and $b_{ij} = 0(\varepsilon)$, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$, as we have observed before.

Substituting the ζ_i by the above expressions, we obtain for the function $\varphi \circ S_V$ that the analytic tangent plane to $\varphi \circ S_V(z) = 0$ at the origin is given by the equations

$$\sum_{j=1}^{n} (a_{nj} + \bar{b}_{nj}) z_j = 0.$$

Let L be the complex line defined by the equations of the analytic tangent plane and $\{z_2 = \cdots = z_{n-1} = 0\}$. On L we have $(a_{n1}+\bar{b}_{n1})z_1 + (a_{nn}+b_{nn})z_n = 0$. Thus $z_n = 0(\varepsilon)z_1$ on L. Moreover the Levi form of $\varphi \circ S_V$ at the origin restricted to L reduces to

$$(\beta_{11}+0(\varepsilon))|z_1|^2.$$

Recall that $\beta_{11} < 0$. Hence if ε is sufficiently small, the Levi form of $\varphi \circ S_V$ at the origin restricted to L has one negative eigenvalue.

THEOREM. *Every pseudoconcave, complex Lie group G^0 is a complex torus.*

PROOF. We have already shown that there exists $V^0 \in M^*(m,,n; \boldsymbol{C})$ such that G^0 is biholomorphically isomorphic to $\boldsymbol{C}^n/\Gamma(V^0)$. By Lemma 8, there exists $\varepsilon > 0$ such that if $d(V^0, V) < \varepsilon$ then $G = \boldsymbol{C}^n/\Gamma(V)$ is pseudoconcave. If $m > n$ then, by Lemma 7, there exists $V \in M^*(n, m; \boldsymbol{C})$ satisfying $d(V^0, V) < \varepsilon$ and the Morimoto condition. But, by Corollary 6, this is absurd. Thus $m = n$ and G^0 is a complex torus.

BIBLIOGRAPHY

ADAMS, FRANK J.
Lectures on Lie Groups, W. A. Benjamin, 1969.

MORIMOTO, AHIHIKO
On the classification of noncompact complex abelian Lie groups, Trans. Amer. Math., Soc., 123 (1966), pp 200–228.

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