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Compositio Mathematica, tome 25, n° 1 (1972), p. 79-86

<http://www.numdam.org/item?id=CM_1972__25_1_79_0>
THE LIE ALGEBRA OF ENDMORPHISMS
OF AN INFINITE-DIMENSIONAL VECTOR SPACE

by

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1. Introduction

The structure of the Lie algebra of all endomorphisms of a finite-dimensional vector space is well known. The purpose of this paper is to investigate the infinite-dimensional case, and in particular to find the lattice of Lie ideals. Rosenberg [6] has carried out the analogous programme for the infinite general linear group.

Notation for Lie algebras will follow that of [9, 10]. Let $\mathbb{F}$ be any field. Let $c$ be any infinite cardinal, with successor $c^+$. Let $V$ be a vector space over $\mathbb{F}$ of dimension $c$, and for any infinite cardinal $d \leq c^+$ define $E(c, d)$ to be the set of all linear transformations $\alpha : V \to V$ such that the image of $\alpha$ has dimension $< d$. Then $E(c, d)$ is an associative $\mathbb{F}$-algebra. Under commutation $[\alpha, \beta] = \alpha \beta - \beta \alpha$ ($\alpha, \beta \in E(c, d)$) it becomes a Lie algebra which we shall denote $L(c, d)$.

Inside $L(c, c^+)$ we let $F = L(c, \mathbb{N}_0)$, $T =$ the set of endomorphisms of trace zero (in the sense of [9] p. 306), $S =$ the set of scalar multiplications $v \to v k$ ($v \in V$, $k \in \mathbb{F}$). We shall prove:

**THEOREM (A).** Let $L = L(c, c^+)$. Then the ideals of $L$ are precisely the following:

a) $L(c, d)$ for $\mathbb{N}_0 \leq d \leq c^+$
b) $L(c, d) + S$ for $\mathbb{N}_0 \leq d \leq c$
c) Any subspace $X$ of $L$ such that $T \leq X \leq F + S$
d) $S$
e) $\{0\}$.

The lattice of ideals has the form as shown on the next page. Further, every subideal of $L$ is an ideal, so that $L$ lies in the class $\mathcal{K}$ of [9].

An immediate corollary of theorem A is that $L(c, c^+)$ satisfies the minimal condition for subideals, Min-si. We shall use this to show that theorem 3.3 of [9] p. 305 is in a sense best possible.

Finally we apply our results to prove that any Lie algebra can be embedded in a simple Lie algebra.
2. The endomorphism algebra

We attack the problem through the associative ideal structure of $E(c, d)$, which is easily determined. By Jacobson [5] p. 108 an associative algebra $A$ is simple if and only if it is simple considered as a ring. This remark combines with a theorem of Herstein [3] (see also Baxter [1]) to yield:

**Lemma (1).** If $A$ is a simple associative $\mathfrak{C}$-algebra and $[A, A] = A$ then any proper Lie ideal of the Lie algebra associated with $A$ is contained in the centre of $A$, unless $A$ is of dimension 4 over its centre which is a field of characteristic 2.

In the sequel all algebras considered will be infinite-dimensional over their centres, so the exceptional case never arises. By a slight extension of Jacobson [5] p. 93 theorem 1 we have:

**Lemma (2).** Let $c, d$ be infinite cardinals with $d \leq c^+$. Then any non-zero associative ideal of $E(c, d)$ is of the form $E(c, e)$ where $\aleph_0 \leq e \leq d$.

**Corollary.** If $c \geq d$ are infinite cardinals then

$$E(c, d^+)/E(c, d)$$

is a simple non-commutative associative algebra.

**Lemma (3).** Let $E = E(c, d)$ where $\aleph_0 < d \leq c^+$. Then

$$[E, E] = E.$$
PROOF. Let \( a \in E \). Decompose \( V \) into a direct sum
\[
V = X \oplus \bigoplus_{i \in \mathbb{Z}} V_i
\]
in such a way that \( \dim V_i = \dim \text{im}(a) \) for all \( i \) and that \( \text{im}(a) \cong W = \bigoplus_{i \in \mathbb{Z}} V_i \). For each \( i \) let \( t_i : V_i \to V_{i+1} \) be an isomorphism. Let the automorphism \( u : W \to W \) be defined by \( u|_{V_i} = t_i \) and let \( t : V \to V \) be defined by \( t|_W = u \) and \( t(X) = \{0\} \). We shall show that there exists \( b \in E \) such that

\[ [b, t] = a. \]

More precisely we show that there is a unique endomorphism \( b \) of \( V \) satisfying (1) such that

\[ b(V_0) = \{0\} \]

and

\[ b(V) \leq W \]

(hence \( b \in E \)).

We set \( a_i = a|_{V_i} \) and \( b_i = b|_{V_i} \). In view of (3) the restrictions of (1) to \( X \), to \( V_{i-1} \) \((i > 0)\) and to \( V_i \) \((i < 0)\) are respectively equivalent to the following equations:

\[
(4) \quad b|_X = -u^{-1}a|_X
\]

\[
(5) \quad b_i = (a_{i-1} + t_i b_{i-1}) t_{i-1}^{-1} \quad (i > 0)
\]

\[
(6) \quad b_i = t_{i+1}^{-1}(a_i + b_{i+1} t_i) \quad (i < 0)
\]

and now the assertion is obvious since (5) and (6) constitute inductive definitions for the \( b_i \).

Note that if \( d = \aleph_0 \) the lemma is false, for then \([E, E]\) is the set of trace zero maps which is smaller than \( E \).

For any associative algebra \( A \) we let \( Z(A) \) denote the centre of \( A \). We then have:

**Lemma (4).** If \( c \geq d \) are infinite cardinals, then

\[
Z(E(c, d^+)|E(c, d))
\]

is trivial except when \( c = d \). It then has dimension 1 and consists of scalar multiplications (modulo \( E(c, d) \)).

This follows from:

**Lemma (5).** If \( c \geq d \) are infinite cardinals and \( z \in L(c, c^+) \) satisfies

\[
[z, L(c, d^+)] \leq L(c, d) + S
\]

then \( z \in L(c, d) + S \).
The proof of this lemma is more intricate than one might wish, and will be postponed until later.

Putting together the results so far obtained we have:

**Lemma (6).** If \( c \geq d \) are infinite cardinals then the Lie algebra \( L(c, d^+) / L(c, d) \)
is simple unless \( c = d \); when its only nontrivial proper ideal is the centre, which has dimension 1 and consists of scalar multiplications (modulo \( L(c, d) \)).

The next result is implicit in [9] (p. 310):

**Lemma (7).** Let \( L \) be a Lie algebra, \( \sigma \) an ordinal, and \( (G^\sigma)_{\sigma \leq \sigma} \) an ascending series of ideals such that for all \( \alpha < \sigma \)

1) \( G_{\alpha+1} / G_{\alpha} \) is simple non-abelian,
2) \( C_{L/G_{\alpha}}(G_{\alpha+1} / G_{\alpha}) = G_{\alpha} / G_{\alpha} \).

Then the only subideals of \( L \) are the \( G_{\alpha} \). Consequently \( L \in \text{Min-si} \cap \mathcal{I} \).

**Proof.** Let \( M \) be a proper subideal of \( L \) and let \( \alpha \) be the least ordinal such that \( G_{\alpha} \leq M \). It is easy to see that \( \alpha \) cannot be a limit ordinal, so \( \alpha = \beta + 1 \) for some \( \beta \). Thus \( (M + G_{\beta}) / G_{\beta} \) is a subideal of \( L / G_{\beta} \) not containing \( G_{\beta+1} / G_{\beta} \). As the latter is a simple non-abelian ideal of \( L / G_{\beta} \) we have

\[
(M + G_{\beta}) / G_{\beta} \cap G_{\beta+1} / G_{\beta} = G_{\beta} / G_{\beta}
\]

so by [9] lemma 4.6 p. 309 \( M \) centralises \( G_{\beta+1} / G_{\beta} \). By part (2) of the hypothesis \( M \leq G_{\beta} \), whence \( M = G_{\beta} \).

Obviously \( L \in \mathcal{I} \), and \( L \in \text{Min-si} \) since the ordinals are well-ordered.

Now we shall show that \( L(c, d) \in \text{Min-si} \cap \mathcal{I} \). The presence of trace zero and scalar maps causes complications, so we study a suitable quotient algebra. Let \( L = L(c, d) \), let \( F, S, T \) be as in theorem A, and put \( I = F + S \). Then \( L^* = L / I \) has an ascending series of ideals

\[
O = L_0^* \leq L_1^* \leq \cdots \leq L_\alpha^* \leq \cdots \leq L_\delta^* = L^*
\]

for a suitable ordinal \( \delta \); the \( L_\alpha^* \) being the ideals \( (L(c, \rho) + S) / I \) arranged in ascending order.

Now \( I \) has a series \( O \leq T \leq F \leq I \) of ideals. But \( T \) is simple ([9] lemma 4.1 p. 306) and \( F / T \) and \( I / F \) are 1-dimensional. Therefore \( I \in (\text{Min-si})(\mathcal{I})(\mathcal{I}) \leq \text{Min-si} \), by [9] lemma 2.2 p. 303. By the same lemma, in order to prove that \( L \in \text{Min-si} \), it suffices to show that \( L^* \in \text{Min-si} \). This will follow from lemma 7 provided we can prove that

\[
C_{L^* / L_\alpha^*}(L_\alpha^* / L_\alpha^*) = I_\alpha^* / L_\alpha^*
\]

which is equivalent to the statement of lemma 5.
We now come to the proof of lemma 5. To simplify the notation we let \( L = L(c, c^+), E = L(c, d), G = L(c, d^+) \). To prove lemma 5 we must show that if \( z \in L \) and \([z, G] \leq E+S\), then \( z \in E+S\).

If \( V \) is a vector space with basis \((v_\lambda)_{\lambda \in \Lambda}\) and \( a \) is an endomorphism of \( V \), we define \( a_{\alpha \beta}(\alpha, \beta \in \Lambda) \) by:

\[
v_\alpha a = \sum a_{\alpha \beta} v_\beta.
\]

**Lemma (8).** If \( V \) is a vector space with basis \((v_\lambda)_{\lambda \in \Lambda}\) where \( \Lambda \) is infinite, and if \( a \) is an endomorphism of \( V \) such that \( \dim (\text{im}(a)) = e \) is infinite, then the set

\[
B = \{ \beta : a_{\alpha \beta} \neq 0 \text{ for some } \alpha \in \Lambda \}
\]

has cardinality \( |B| = e \).

**Proof.** Let \( W = \sum_{\lambda \in B} v_\lambda \). By definition \( \dim (W) = |B| \), and since \( \text{im}(a) \leq W \) we have \( e \leq |B| \). If \((\mu_\nu)_{\mu \in M}\) is a basis for \( \text{im}(a) \), then each \( i_\mu \) is a linear combination of finitely many \( v_\lambda \ (\lambda \in B) \). Therefore \( |B| \leq |Z \times M| = \aleph_0 \cdot e = e \) since \( e \) is infinite.

We now suppose that \( z \) is as above, and that \( V \) is a vector space with basis \((v_\lambda)_{\lambda \in \Lambda}\) where \( |\Lambda| = c \).

**Lemma (9).** There exists \( z' \) such that \( z'_a = 0 \) \((a \in \Lambda)\), \([z', G] \leq E+S\), and \( z - z' \in E+S' \).

**Proof.** Let \( M \) be the set of all pairs \((M, <)\) where \( M \) is a subset of \( \Lambda \) and \( < \) is a well-ordering on \( M \), such that if \( \alpha \in M \) then \( z_\alpha \neq z_{\alpha+1, \alpha+1} \) (where \( \alpha+1 \) is the successor to \( \alpha \) in the ordering \(<\)). Then \( M \) is partially ordered by \( \ll\), where \((M_1, <_1) \ll (M_2, <_2)\) if and only if \( M_1 \) is an initial segment of \( M_2 \). Clearly \( M \) is not empty and satisfies the hypotheses of Zorn’s lemma. Let \((M, <)\) be a maximal element of \( M \). Suppose for a contradiction that \( |M| \leq d \). Take an initial segment \( I \) of \( M \) with \( |I| = d \), and consider

\[
t = [z, \sum_{\alpha \in I} e_{\alpha, \alpha+1}]
\]

where \( e_{\alpha \beta} \ (\alpha, \beta \in \Lambda) \) is the elementary transformation sending \( v_\alpha \) to \( v_\beta \) and all other basis elements to zero. By hypothesis \( t \in E+S \), yet

\[
t = \sum z_{\alpha \beta} e_{\alpha \beta} e_{\beta, \beta+1} - \sum z_{\alpha \beta} e_{\alpha-1, \alpha} e_{\alpha \beta}
\]

\[
= \sum (z_{\alpha, \beta-1} - z_{\alpha+1, \beta}) e_{\alpha \beta}
\]

(where terms involving \( \alpha-1 \) for limit ordinals \( \alpha \) are deemed to be zero).

Now the coefficient of \( e_{\alpha, \alpha+1} \) is \( z_{\alpha} - z_{\alpha+1, \alpha+1} \) which is non-zero for \( d \) values of \( \alpha \). By lemma 8 \( t \notin E+S \) which is a contradiction.

Thus after choosing fewer than \( d \) values of \( \alpha \) all the remaining \( z_{\alpha \alpha} \) are equal. Thus \( \sum z_{\alpha \alpha} e_{\alpha \alpha} \in E+S \). Define \( z' = z - \sum z_{\alpha \alpha} e_{\alpha \alpha} \).
Lemma (10). Suppose that $z' \notin E+S$. Then there exist subsets $A, A'$ of $\Lambda$ and a bijection $\phi : A \rightarrow A'$ such that

1) $A \cap A' = \emptyset$
2) If $\phi(\alpha) = \alpha'$ ($\alpha \in A$) then $z'_{\alpha\alpha'} \neq 0$
3) $|A| = |A'| = d$.

Proof. Let $\mathcal{S}$ be the collection of all triples $(A, A', \phi)$ satisfying (1) and (2). Partially order $\mathcal{S}$ by $\ll$ where $(A, A', \phi) \ll (B, B', \psi)$ if and only if $A \subseteq B$, $A' \subseteq B'$, and $\psi|_{A} = \phi$. By Zorn’s lemma there is a maximal element $(A, A', \phi)$ of $\mathcal{S}$. For brevity let $\phi(\alpha) = \alpha'$ ($\alpha \in A$). We claim that $|A| = d$.

Suppose not. Then $|A| = d' < d$. Let $D = \{\delta : z'_{\phi\delta} \neq 0, \gamma \in A \cup A'\}$.

Since $d$ is infinite we have $|D| < d$. By lemma 8 there must exist $\gamma' \notin (A \cup A' \cup D)$ such that $z'_{\gamma\gamma'} \neq 0$ for some $\gamma \neq \gamma'$ (since $z' \notin E+S$). Then $\gamma \notin (A \cup A')$ since $\gamma' \notin D$. Therefore $\gamma \neq \gamma'$, $\gamma \notin (A \cup A')$, $\gamma' \notin (A \cup A')$.

Define

$$B = A \cup \{\gamma\}$$
$$B' = A' \cup \{\gamma'\}$$
$$\Psi(\beta) = \beta' (\beta \in A)$$
$$\Psi(\gamma) = \gamma'.$$

Then $(B, B', \Psi) \in \mathcal{S}$ and is greater than $(A, A', \phi)$, a contradiction. Hence $|A| \geq d$ as claimed.

We may now derive the final contradiction required to prove lemma 5.

Suppose for a contradiction that $z' \notin E+S$. Then there exists $(A, A', \phi)$ as in lemma 10. Define $\pi : V \rightarrow V$ by

$$v_\alpha \pi = v_{\alpha'} \quad (\alpha \in A)$$
$$v_{\alpha'} \pi = v_\alpha \quad (\alpha' \in A')$$
$$v_\beta \pi = 0 \quad (\beta \in \Lambda \setminus (A \cup A')).$$

By definition $\pi \in G$. By hypothesis $u = [z', \pi] \in E+S$. But for $\alpha \in A$ we have

$$v_\alpha (z' \pi - \pi z') = \sum z'_{\alpha\beta} v_\beta \pi - \sum z'_{\beta\alpha} v_\beta.$$

The coefficient of $v_\alpha$ is

$$z'_{\alpha\alpha'} + z'_{\alpha'\alpha} - z'_{\alpha'\alpha'} = z'_{\alpha\alpha'} \neq 0$$

so that $u_{\alpha\alpha'} \neq 0$ if $\alpha \in A$. Since $|\Lambda| = d$ and $\alpha \neq \alpha'$ we have $u \notin E+S$, a contradiction.
Hence $z' \in E+S$, whence $z \in E+S$, and lemma 5 is proved. By lemma 7 we have:

**Lemma (11).**

1) $L(c, c^+) \in \text{Min-si}$,
2) Every subideal of $L(c, c^+)$ which contains $F+S$ is of the form $L(c, d)+S$.

**Lemma (12).** $L(c, c^+) \in \mathcal{Z}$.

**Proof.** Suppose $L = L(c, c^+)$ has a proper ideal $J$ of finite codimension. Now $L$ has an ascending series, the finite-dimensional factors of which are abelian, the rest simple. Hence $L/J$ is soluble, so that $[L, L] < L$, contrary to lemma 3. Therefore by theorem 3.1 of [9] p. 305 we have $L \in \mathcal{Z}$.

We now proceed to the:

**Proof of Theorem (A).**

All the subalgebras listed are ideals; the only case requiring comment being (c). Since $L = L(c, c^+)$ has no ideals of finite codimension (proof of lemma 12) the factor $(F+S)/T$ is central (see [9] p. 305, proof of theorem 3.1). Therefore any subspace $X$ between $T$ and $F+S$ is an ideal.

Suppose now that $I$ is an ideal of $L$. If $I \geq F+S$ then by lemma 11 $I$ is in the given list. Therefore we may assume $I \not\supset F+S$. If $I \cap T = \{0\}$ then $[I, T] = \{0\}$. But it is easy to see that the only elements of $L$ centralising every elementary transformation $e_{\alpha \beta}$ ($\alpha \neq \beta$) are the elements of $S$. Hence $I \subseteq S$. Since $\dim S = 1$ we have $I = \{0\}$ or $S$. But $T$ is simple ([9] lemma 4.1 p. 306) so if $I \cap T \neq \{0\}$ then $T \supseteq I$. Now $I+F+S \leq L$, and by lemma 11 $I+F+S = L(c, d)+S$ for some $d$. If $d = 0$ then $T \subseteq I \leq F+S$, which is case (c) of the list. There remains the case $d > 0$. Then we have $(I+F+S)/(T+S) = (L(c, d)+S)/(T+S)$ so that $(I+T+S)/(T+S)$ is of codimension $\leq 1$ in $(L(c, d)+S)/(T+S) \cong (L(c, d))/T$ which has no proper ideals of finite codimension by the argument of lemma 12. Therefore $I+T+S = L(c, d)+S$. Now $T \leq I$ so we have $I+S = L(c, d)+S$. If $I \neq L(c, d)+S$ and $I \neq L(c, d)$ then $I \cap L(c, d)$ is of codimension 1 in $L(c, d)$, contradicting lemma 3. Hence $I = L(c, d)$ or $I = L(c, d)+S$.

We have already remarked (in lemma 12) that $L \in \mathcal{Z}$; which completes the proof of the theorem.

### 3. Applications

In [9] it is proved that any Lie algebra satisfying Min-si and having no ideals of finite codimension has an ascending series of ideals whose factors are either infinite-dimensional simple or 1-dimensional central. The re-
The results of theorem A show that the 1-dimensional central factors cannot in general be dispensed with. In [9] this question was left open. The algebras $L(c, d)$ also provide new examples of Lie algebras in $\text{Min-si} \cap \mathcal{X}$.

Following the general lines of Scott [7] p. 316 section 11.5.4 (for groups) we can prove:

**Theorem (B).** Any Lie algebra can be embedded in a simple Lie algebra.

**Proof.** Let $K$ be a Lie algebra over a field $\mathbb{F}$. By Jacobson [4] p. 162 cor. 4 $K$ has a faithful representation by endomorphisms of a vector space $V$ over $\mathbb{F}$. By enlarging $V$ if necessary we may embed $K$ in $L(c^+, c^+)$ for some infinite cardinal $c$. If we split $V$ into $c$ subspaces of dimension $c^+$ and copy the $K$-action on each of these we may assume that $K$ is represented by endomorphisms whose image has dimension $\geq c$. Then the composite embedding

$$K \rightarrow L(c^+, c^+) \rightarrow L(c^+, c^+)/L(c^+, c)$$

maps $K$ into a simple Lie algebra.

One might ask about Lie analogies of other embedding theorems for groups. For example, Dark [2] has proved that every group can be embedded as a subnormal subgroup of a perfect group. Strangely, the analogue of this is false for Lie algebras – an example may be found in [8], p. 98

**References**

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