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by

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1. Introduction

Let $s$ denote the countable infinite product of open intervals and let $I^\infty$ denote the Hilbert cube, i.e. the countable infinite product of closed intervals. A Fréchet manifold (or $F$-manifold) is a separable metric space having an open cover by sets each homeomorphic to an open subset of $s$. A Hilbert cube manifold (or $Q$-manifold) is a separable metric space having an open cover by sets each homeomorphic to an open subset of $I^\infty$.

In [2] it is shown that real Hilbert space $l_2$ is homeomorphic to $s$ and indeed it is known that all separable infinite-dimensional Fréchet spaces are homeomorphic (see [2] for references). Thus $F$-manifolds can be viewed as separable metric manifolds modeled on any separable infinite-dimensional Fréchet space. Using linear space apparatus and a number of earlier results, Henderson [9] has obtained embedding, characterization, and representation theorems concerning $F$ manifolds (see [10] for generalizations to manifolds modeled on more general infinite-dimensional linear spaces).

In [6] a number of results similar in nature to those of [9] were obtained concerning certain incomplete, sigma-compact countably infinite-dimensional manifolds. Some results were also established in [6] concerning the relationship of such incomplete manifolds to $Q$-manifolds. Since the nature of these results is such that a good bit of information about $Q$-manifolds can be obtained from the 'related' incomplete manifolds, we thus have a device for attacking $Q$-manifold problems.

It is the purpose of this paper to use 'related' incomplete manifolds to establish for $Q$-manifolds some more results similar to those of [9]. We list the main results of this paper in section 2.

Unfortunately we leave important questions concerning $Q$-manifolds unanswered. We call particular attention to the paper Hilbert cube manifolds [Bull. Amer. Math. Soc. 76 (1970), 1326–1330], in which the author gives an extensive list of open questions concerning $Q$-manifolds.

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2. Statements of results

A (topological) polyhedron is a space homeomorphic (\(\cong\)) to \(|K|\), where \(K\) is a complex (i.e. a countable locally-finite simplicial complex). Unless otherwise specified all polyhedra will be topological polyhedra. West [15] has shown that \(P \times s\) is an \(F\)-manifold and \(P \times I^\infty\) is a \(Q\)-manifold, for any polyhedron \(P\).

A closed set \(F\) in a space \(X\) is said to be a \(Z\)-set in \(X\) provided that for each non-null homotopically trivial (i.e. all homotopy groups are trivial) open subset \(U\) of \(X\), \(U \setminus F\) is non-null and homotopically trivial. We use the representation \(s = \prod_{i=1}^{\infty} I_i^0\) and \(I^\infty = \prod_{i=1}^{\infty} I_i\), where for each \(i > 0\) \(I_i^0\) is the open interval \((-1, 1)\) and \(I_i\) is the closed interval \([-1, 1]\).

In Theorem 1 we show how to ‘fatten-up’ a polyhedron which is a \(Z\)-set in a \(Q\)-manifold to a ‘nice’ neighborhood of the polyhedron. This will be useful in the sequel.

**Theorem 1.** Let \(X\) be a \(Q\)-manifold and let \(P\) be a polyhedron which is also a \(Z\)-set in \(X\). If \(q \in I^\infty \setminus \{(0, 0, \cdots)\}\), then there is an open embedding \(h : P \times (I^\infty \setminus \{q\}) \to X\) such that \(h(x, (0, 0, \cdots)) = x\), for all \(x \in P\).

In [9] the following results are established.

1. Every \(F\)-manifold can be embedded as an open subset of \(l_2\).
2. If \(X\) and \(Y\) are \(F\)-manifolds having the same homotopy type (i.e. \(X \sim Y\)), then \(X \cong Y\).
3. If \(X\) is any \(F\)-manifold, then there is a polyhedron \(P\) for which \(X \cong P \times l_2\).

If \(J\) is a simple closed curve, then \(J \times I^\infty\) is a \(Q\)-manifold which cannot be embedded as an open subset of \(I^\infty\). Also, \(I^\infty\) and \(I^\infty \setminus \{\text{point}\}\) are \(Q\)-manifolds of the same homotopy type which are not homeomorphic. Thus the obvious straightforward analogues of (1) and (2) for \(Q\)-manifolds are not valid. Most of the results that follow are concerned with obtaining partial analogues of (1), (2), and (3) for \(Q\)-manifolds.

**Theorem 2.** Let \(X\) be a \(Q\)-manifold and let \(P\) be any polyhedron such that \(X \sim P\). Then there is a \(Z\)-set \(F \subset X\) such that \(X \setminus F \cong P \times (I^\infty \setminus \{\text{point}\})\).

Each \(Q\)-manifold is an ANR and it follows from [11] that each separable metric ANR has the homotopy type of some polyhedron. Thus each \(Q\)-manifold has the homotopy type of some polyhedron.

**Theorem 3.** Let \(X\) be any \(Q\)-manifold and let \(P\) be any polyhedron such that \(X \sim P\). Then \(X \times [0, 1) \cong P \times (I^\infty \setminus \{\text{point}\})\).

**Corollary 1.** If \(X\) is any \(Q\)-manifold, then there is a polyhedron \(P\) such that \(X \times [0, 1) \cong P \times I^\infty\).
COROLLARY 2. If $X$ and $Y$ are $Q$-manifolds such that $X \sim Y$, then $X \times [0, 1) \cong Y \times [0,1)$.

COROLLARY 3. If $P$ and $R$ are polyhedra such that $P \sim R$, then $P \times (I^\omega \setminus \{\text{point}\}) \cong R \times (I^\omega \setminus \{\text{point}\})$.

In a sense Corollary 3 is analogous to a result of West [15]. It is shown there that if a polyhedron $P$ is a formal deformation of a polyhedron $R$ (in the sense of Whitehead [16]), then $P \times I^\omega \cong R \times I^\omega$.

THEOREM 4. If $X$ is a $Q$-manifold, then $X \times [0, 1)$ can be embedded as an open subset of $I^\omega$.

COROLLARY 4. If $X$ is a $Q$-manifold, then $X = U \cup V$, where $U$ and $V$ are open subsets of $X$ which are homeomorphic to open subsets of $I^\omega$.

If $X$ is any $Q$-manifold, then it is shown in [5] that $X \cong X \times I^\omega$ (and therefore $X \cong X \times [0, 1]$). Thus the above results offer some information about the internal structure of $Q$-manifolds.

In [10] it is shown that if $X$ and $Y$ are $F$-manifolds and $f : X \to Y$ is a homotopy equivalence, then $f$ is homotopic to a homeomorphism of $X$ onto $Y$. We obtain a corresponding property for $Q$-manifolds which strengthens Corollary 2.

THEOREM 5. Let $X, Y$ be $Q$-manifolds and let $f : X \to Y$ be a homotopy equivalence. Then there is a homeomorphism of $X \times [0, 1)$ onto $Y \times [0, 1)$ which is homotopic to $f \times \text{id} : X \times [0, 1) \to Y \times [0, 1)$.

The following results are some partial answers to questions concerning compact $Q$-manifolds.

THEOREM 6. Let $X$ be a compact $Q$-manifold and assume that $X \sim P$, where $P$ is a compact polyhedron. Then there is a copy $P'$ of $P$ in $X$ such that $P'$ is a $Z$-set in $X$ and $X \setminus P' \cong P \times (I^\omega \setminus \{\text{point}\})$.

COROLLARY 5. If $X$ is a compact homotopically trivial $Q$-manifold, then $X \cong I^\omega$.

THEOREM 7. Let $X$ be a compact $Q$-manifold and assume that $X \sim P$, where $P$ is a compact polyhedron. Then there is an embedding $h : X \to I^\omega$ such that $\text{Bd}(h(X)) \cong P \times I^\omega$ and $\text{Cl}(I^\omega \setminus h(X)) \cong I^\omega$.

In regard to Theorem 7 we remark that in [8] a similar, and somewhat stronger, result is established for $F$-manifolds.

We show that if $X$ is an open subset of $I^\omega$, then the factor $[0,1)$ of Corollary 1 can be omitted.

THEOREM 8. If $X$ is an open subset of $I^\omega$, then there is a polyhedron $P$ such that $X \cong P \times I^\omega$. 
We remark that the proof of this result is quite different from the proof of the corresponding property for open subsets of $I_2$ (see [8]).

We also establish a Schoenflies-type result for $Q$-manifolds.

**Theorem 9.** Let $X$ and $Y$ be $Q$-manifolds and let $f, g : X \to Y$ be closed embeddings which are homotopy equivalences and such that $f(X), g(X)$ are bicollared in $Y$ ("bicollared" is defined in Section 3). Then the homeomorphism $g \circ f^{-1} \times \text{id} : f(X) \times [0,1) \to g(X) \times [0,1)$ can be extended to a homeomorphism of $Y \times [0,1)$ onto itself.

We remark that in the case $X = Y = I^\infty$, the factor $[0,1)$ can be omitted in the statement of Theorem 9. The proof of this follows routinely from [17].

The proof of Theorem 9 applies to give us a corresponding result for $F$-manifolds.

**Theorem 10.** Let $X$ and $Y$ be $F$-manifolds and let $f, g : X \to Y$ be closed embeddings which are homotopy equivalences and such that $f(X), g(X)$ are bicollared in $Y$. Then the homeomorphism $g \circ f^{-1} : f(X) \to g(X)$ can be extended to a homeomorphism of $Y$ onto itself.

In case $X = Y = I_2$, Theorem 10 follows routinely from the Schoenflies result of [13].

### 3. Preliminaries

In this section we describe some of the apparatus that will be used in the succeeding sections.

For spaces $X$ and $Y$, a continuous function $f : X \to Y$ is said to be *proper* provided that the inverse image of each compact subset of $Y$ is compact. Then a *proper homotopy* is a homotopy $F : X \times I \to Y$ which is a proper map (we let $I = [0,1)$).

For each integer $n > 0$ let $W_n^+ = \{(x_i)_{i \in I} | x_n = 1\}$ and $W_n^- = \{(x_i)_{i \in I} | x_n = -1\}$. We call $W_n^+$ and $W_n^-$ *endslices* of $I^\infty$. For each integer $n > 0$ we let $\pi_n : I^\infty \to \prod_{i=1}^n I_i$ be the natural projection and put $B(I^\infty) = I^\infty \setminus s$.

A subset of $I^\infty$ of the form $\prod_{i=1}^\infty J_i$ is called a *basic closed set* in $I^\infty$ provided that $J_i$ is a closed subinterval of $I_i$ for each $i > 0$, and $J_i = I_i$ for all but finitely many $i$. Note that any basic closed subset of $I^\infty$ may be viewed as a Hilbert cube, with its topological boundary being a finite union of endslices.

Let $X$ and $Y$ be spaces and $\mathcal{U}$ be an open cover of $Y$. Then functions $f, g : X \to Y$ are said to be *$\mathcal{U}$-close* provided that for each $x \in X, f(x)$ and $g(x)$ lie in some element of $\mathcal{U}$. A function $f : Y \to Y$ is said to be *limited by*
provided that $f$ and $\text{id}_Y$ (the identity function on $Y$) are $\mathcal{U}$-close. A function $f : X \times I \to Y$ is said to be limited by $\mathcal{U}$ provided that for each $x \in X$, $f(\{x\} \times I)$ lies in a member of $\mathcal{U}$.

Following Anderson [1] we say that a subset $M$ of a metric space $X$ has the compact absorption property in $X$ (or $M$ is a cap-set for $X$) if

1. $M = \bigcup_{n=1}^{\infty} M_n$, where each $M_n$ is a compact $Z$-set in $X$ such that $M_n \subseteq M_{n+1}$, and
2. for each $\varepsilon > 0$, each integer $m > 0$, and each compact subset $F$ of $X$, there is an integer $n > 0$ and an embedding $h : F \to M_n$ such that $h|F \cap M_m = \text{id}$ and $d(h, \text{id}) < \varepsilon$.

For each integer $n > 0$ let $\Sigma_n = \Pi_{i=1}^{\infty} [-n/(n+1), n/(n+1)]$ and $\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n$. In [1] it is shown that $\Sigma$ and $B(I^\infty)$ are cap-sets for $I^\infty$.

We will need the following properties of cap-sets in $Q$-manifolds. All of these can be found in [6]. We let $X$ represent a $Q$-manifold.

**Lemma 3.1.** Cap-sets exist in $Q$-manifolds, and any cap-set for $X$ is of the form $P \times \Sigma$, for any polyhedron $P$ satisfying $P \sim X$.

**Lemma 3.2.** If $M$ is a cap-set for $X$ and $F \subset X$ is a $Z$-set, then $M \cup F$ and $M \setminus F$ are cap-sets for $X$.

**Lemma 3.3.** If $M$ and $N$ are cap-sets for $X$ and $\mathcal{U}$ is an open cover of $X$, then there is a homeomorphism of $X$ onto itself which takes $M$ onto $N$ and which is limited by $\mathcal{U}$.

**Lemma 3.4.** If $M$ is a cap-set for $X$ and $F \subset X$ is a closed set satisfying $F \cap M = \emptyset$, then $F$ is a $Z$-set in $X$.

**Lemma 3.5.** If $P$ is a polyhedron, then $P \times \Sigma_n$ is a $Z$-set in $P \times \Sigma$. If $M$ is a cap-set for $X$ and $F \subset M$ is a $Z$-set in $M$, then $\text{Cl}_X(F)$ (the closure of $F$ in $X$) is a $Z$-set in $X$.

**Lemma 3.6.** If $M$ is a cap-set for $X$, then $X \setminus M$ is an $F$-manifold satisfying $X \setminus M \sim X$. In fact, $M \cong X \times B(I^\infty)$, which is a cap-set for $X \times I^\infty$. If $F \subset X \setminus M$ is a $Z$-set in $X \setminus M$, then $\text{Cl}_X(F)$ is a $Z$-set in $X$.

Let $X$ be a space and let $\mathcal{U}$ be any open cover of $X$. Then define $\text{St}^0(\mathcal{U}) = \mathcal{U}$ and for each $n > 0$ define $\text{St}^n(\mathcal{U})$ to consist of all sets of the form $A \cup (\bigcup \{U \in \mathcal{U} \mid U \cap A \neq \emptyset\})$, where $A \in \text{St}^{n-1}(\mathcal{U})$.

The following result on extensions of homeomorphisms in $Q$-manifolds is established in [3].

**Lemma 3.7.** Let $X$ be a $Q$-manifold, $\mathcal{U}$ be an open cover of $X$, $F_1$ and $F_2$ be $Z$-sets in $X$, and let $h : F_1 \to F_2$ be a homeomorphism. If there is a proper homotopy $H : F_1 \times I \to X$ such that $H_0 = \text{id}$, $H_1 = h$, and $H$...
is limited by \( u \), then \( h \) can be extended to a homeomorphism of \( X \) onto itself which is limited by \( \text{St}^4(u) \).

The following characterization of \( Z \)-sets in \( Q \)-manifolds is established in [6].

**Lemma 3.8.** Let \( X \) be a \( Q \)-manifold and let \( F \subset X \) be a closed set. Then \( F \) is a \( Z \)-set in \( X \) if and only if there is a homeomorphism of \( X \) onto \( X \times I^\infty \) taking \( F \) into \( X \times \{(0, 0, \cdots )\} \).

It is shown in [3] that for any \( Z \)-set \( F \) in a \( Q \)-manifold \( X \), there is a homeomorphism of \( X \) onto \( X \times I^\infty \) such that \( x \) is taken to \((x, (0, 0, \cdots))\), for all \( x \in F \). It is shown in [7] that a corresponding property for \( F \)-manifolds is also true.

We say that a subset \( A \) of a space \( X \) is *bicollared* provided that there exists an open embedding \( h : A \times (-1, 1) \to X \) satisfying \( h(x, 0) = x \), for all \( x \in A \).

Let \( X \) be a metric space and \( A \) be a closed subset of \( X \). An open cover \( u \) of \( X \setminus A \) is said to be *normal with respect to \( A \)* provided that for each \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that if \( U \in u \) and \( d(A, U) < \delta \), then \( \text{diam}(U) < \varepsilon \). Under these circumstances it is easy to see that any homeomorphism \( h : X \setminus A \to X \setminus A \) which is limited by \( u \) has an extension to a homeomorphism \( h^\prime : X \to X \) which satisfies \( h^\prime |A = id \).

## 4. Proof of Theorem 1

For any complex \( K \), we use \( K^{(n)} \) to denote the \( n \)th barycentric subdivision of \( K \) and \( K_n \) to denote the \( n \)-skeleton of \( K \). For any subset \( C \) of \( |K| \) and integers \( m, n > 0 \), we let \( \text{St}(C, K_n^{(m)}) \) denote the subset of \( |K| \) consisting of the union of the closed simplexes of \( K_n^{(m)} \) which intersect \( C \), where \( K_n^{(m)} \) will always mean the \( m \)th barycentric subdivision of \( K_n \).

We now present a sequence of lemmas that will lead up to a proof of Theorem 1. The proof we give uses an induction on the \( n \)-skeletons of a triangulation of the polyhedron \( P \). The fourth lemma we establish is the actual inductive step, and the first three are technical results that we need there.

**Lemma 4.1.** Let \( K \) be a complex, \( n > 0 \) be an integer, \( C \) be a compact subset of \( |K| \) such that \( \text{St}(C, K_{n+1}) \subset |K_n| \), and let \( L = \text{St}(|K_n|, K_n^{(2)}) \). Then there is a homeomorphism \( h : L \times I^\infty \to |K_n| \times I^\infty \) such that \( h|C \times I^\infty = id \), \( h(L \times W_1^+) = |K_n| \times W_1^+ \), and \( h(x, (0, 0, \cdots)) = (x, (0, 0, \cdots)) \), for all \( x \in |K_n| \).

**Proof.** Let \( Q = \prod_{i=2}^\infty I_i \). It follows from Theorem 4.2 of [15] that there is a homeomorphism \( h' : L \times Q \to |K_n| \times Q \). Since the collapse (see
for definitions) from $L$ to $|K^n| \setminus C$, an open set missing $C$, the proof given there immediately implies that we may additionally require that $h'|C = id$. Although the condition $h'(x, (0, 0, \cdots)) = (x, (0, 0, \cdots))$, for all $x \in |K^n|$, is not mentioned in [15], it can easily be obtained from the apparatus given there. All one has to do is follow the steps in the proof of Theorem 4.2 of [15], correcting at each stage of the collapse to achieve our required condition.

Now define $h : L \times I^\infty \to |K^n| \times I^\infty$ so that $h(x, (x_1, x_2, \cdots)) = (y, (x_1, y_2, y_3, \cdots))$, for all $x \in L$ and $(x_1, x_2, \cdots) \in I^\infty$, where $h'(x, (x_2, x_3, \cdots)) = (y, (y_2, y_3, \cdots))$. Then $h$ obviously fulfills our requirements.

Let $B^n_r$ be the $n$-dimensional ball of radius $r$ ($0 < r \leq 1$) and $S^{n-1}_r$ the boundary of $B^n_r$. For convenience we will assume that

$$B^n_r = \{(x_i) \in I^\infty | \sum_{i=1}^n x_i^2 \leq r^2 \text{ and } x_i = 0 \text{ for } i > n\},$$

$$S^{n-1}_r = \{(x_i) \in I^\infty | \sum_{i=1}^n x_i^2 = r^2 \text{ and } x_i = 0 \text{ for } i > n\}.$$

**Lemma 4.2.** Let $X$ be a $Q$-manifold, $F \subset X$ be a closed set, and let $f : B^n_1 \to X$ be an embedding such that $f(B^n_1)$ is a $Z$-set and $f(B^n_1) \cap F \subset f(S^{n-1}_1)$. For any $r \in (0,1)$ there is an embedding $h : B^n_r \times I^\infty \to X$ satisfying the following properties.

1. $h(x, (0, 0, \cdots)) = f(x)$, for all $x \in B^n_r$,
2. $\text{Bd}(h(B^n_r \times I^\infty)) = h(B^n_r \times W^+_r) \cup h(S^{n-1}_r \times I^\infty)$,
3. $\text{Bd}(h(B^n_r \times I^\infty))$ is bicollared,
4. $h(B^n_r \times I^\infty) \cap (F \cup f(B^n_1)) = f(B^n_1)$.

**Proof.** It is clear that there is an embedding $g_1 : I^\infty \to X$ and a finite union $W$ of endslices of $I^\infty$ such that $f((0, 0, \cdots)) \in g_1(I^\infty \setminus W)$ and $\text{Bd}(g_1(I^\infty)) = g_1(W)$. Choose $\epsilon > 0$ so that $f(B^n_\epsilon) \subset g_1(I^\infty \setminus W)$ and use Lemma 3.7 to get a homeomorphism $g_2 : X \to X$ satisfying $g_2 \circ f(B^n_\epsilon) = f(B^n_\epsilon)$. Then $(g_2 \circ g_1)^{-1} \circ f(B^n_\epsilon)$ is a $Z$-set in $I^\infty$ missing $W$.

Applying Lemma 3.7 to $I^\infty$ there is a homomorphism $g_3 : I^\infty \to I^\infty$ satisfying $g_3(W) = W$ and $g_3 \circ (g_2 \circ g_1)^{-1} \circ f(x) = x$, for all $x \in B^n_n$, where $r < r_1 < 1$. Choose $m > n$ and $\delta \in (0,1)$ such that $K \cap W = \emptyset$ and $K \cap g_1 \circ (g_2 \circ g_1)^{-1} (f(B^n_\epsilon) \cup F) = B^n_r$, where

$$K = \pi_n(B^n_r) \times \prod_{i=n+1}^m [-\delta, \delta] \times \prod_{i=m+1}^\infty I_1.$$

Then put

$$Q = \pi_n(B^n_r) \times \prod_{i=n+1}^m [-\delta, \delta] \times [-\frac{1}{2}, 1] \times \prod_{i=m+2}^\infty I_1.$$
It is obvious that there is a homeomorphism $g_4 : B^n_r \times I^\infty \to Q$ satisfying
\[
g_4(B^n_r \times W^+_1) = \pi_n(B^n_r) \times \prod_{i=n+1}^{m} [-\delta, \delta] \times \{ -\frac{1}{2} \} \times \prod_{i=m+2}^{\infty} I_i,
\]
and $g_4((x, (0, 0, \cdots)) = x$, for all $x \in B^n_r$. Then $h = g_2 \circ g_1 \circ g_3^{-1} \circ g_4$ is our required embedding.

**Lemma 4.3.** Let $K$ be a complex, $n > 0$ be an integer, $C$ be a compact subset of $|K|$ satisfying $\text{St}(C, K_{n+1}) \subseteq |K_n|$, and let $L = \text{St}(|K_n|, K_{n+1}^{(2)})$. Let $X$ be a $Q$-manifold and let $h : L \times I^\infty \to X$ be a closed embedding such that $\text{Bd}(h(L \times I^\infty)) = h(L \times W^+_1)$ and it is bicollared. Let $F \subseteq X$ be a $Z$-set such that

\[ F \cap [h(L \times \{(0, 0, \cdots)\}) \cup h(C \times I^\infty) \cup h(\text{Bd}(L) \times (I^\infty \setminus W^+_1))] = \emptyset, \]

where $\text{Bd}(L)$ is the topological boundary of $L$ in $|K_{n+1}|$. Then there exists a homeomorphism $f : X \to X$ such that

\[ f[h(L \times \{(0, 0, \cdots)\}) \cup h(\text{Bd}(L) \times I^\infty) \cup h(C \times I^\infty)] = \text{id} \]

and $f(F) \cap h(L \times I^\infty) \subseteq h(\text{Bd}(L) \times W^+_1)$.

**Proof.** Let $A = h(L \times [-1, 0] \times \{(0, 0, \cdots)\}) \cup h(L \times W^-_1)$ which is a $Z$-set in $X$, and let $B = h(C \times I^\infty) \cup h(L \times \{(0, 0, \cdots)\}) \cup h(\text{Bd}(L) \times I^\infty)$, which is closed in $X$. Let $X' = X \setminus B$, $A' = A \cap X'$, and $F' = F \cap X'$. Since $A'$ and $F'$ are intersections of $Z$-sets in $X$ with the open subset $X'$ of $X$, it follows that $A'$ and $F'$ are $Z$-sets in $X'$. Now choose an open cover $U$ of $X'$ which is normal with respect to $B$.

Using Lemma 3.8 there is a homeomorphism $f_1 : X' \to X' \times I^\infty$ such that $f_1(A' \cup F') \subseteq X' \times \{(0, 0, \cdots)\}$. We can obviously obtain a homeomorphism $f_2 : X' \times I^\infty \to X' \times I^\infty$ such that $f_2 \circ f_1(F') \cap f_1(A') = \emptyset$ and $f_2$ is limited by $f_1(U)$. Then $f_1^{-1} \circ f_2 \circ f_1 : X' \to X'$ is a homeomorphism limited by $U$ and satisfying $f_1^{-1} \circ f_2 \circ f_1(F') \cap A' = \emptyset$. From Section 3 it follows that $f_1^{-1} \circ f_2 \circ f_1$ extends to a homeomorphism $g : X \to X$ such that $g[B] = \text{id}$ and $g(F) \cap A \cup B \subseteq h(\text{Bd}(L) \times W^+_1)$.

We can use a motion in $L \times I^\infty$ in only the $I_i$-direction and transfer it back to $X$ by means of $h$ to obtain a homeomorphism $g_1 : X \to X$ such that $g_1[B] = \text{id}$ and $g_1 \circ g(F) \cap h(L \times [-1, \frac{1}{2}] \times \Pi_{i=2}^{\infty} I_i) = \emptyset$. The problem is now to move $g_1 \circ g(F) \setminus (h(\text{Bd}(L) \times W^+_1))$ the rest of the way out of $h(L \times I^\infty)$, with no motion taking place on $B$. Because $\text{Bd}(h(L \times I^\infty))$ is bicollared, we can easily find a homeomorphism $g_2 : X \to X$ satisfying $g_2[B] = \text{id}$ and $g_2 \circ g_1 \circ g(F) \cap h(L \times I^\infty) \subseteq h(\text{Bd}(L) \times W^+_1)$. Then put $f = g_2 \circ g_1 \circ g$ to satisfy our requirements.
We now combine these results to obtain the inductive step in the proof of Theorem 1.

**Lemma 4.4** Let $K$ be a complex, let $n > 0$ be an integer, and let $C$ be a compact subset of $|K|$ such that $\text{St}(C, K_{n+1}) \subset |K_n|$. Let $X$ be a $Q$-manifold and let $\varphi : |K| \to X$ be an embedding such that $\varphi(|K|)$ is a Z-set. Let $h_n : |K_n| \times I^\infty \to X$ be a closed embedding such that $\text{Bd}(h_n(|K_n| \times I^\infty)) = h_n(|K_n| \times W_1^+)$ and it is bicollared, $h_n(|K_n| \times I^\infty) \cap \varphi(|K|) \subset \varphi(\text{St}(|K_n|, K^{(3)}))$, and $h_n(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K_n|$. Then there exists a closed embedding $h_{n+1} : |K_{n+1}| \times I^\infty \to X$ such that $\text{Bd}(h_{n+1}(|K_{n+1}| \times I^\infty)) = h_{n+1}(|K_{n+1}| \times W_1^+)$ and it is bicollared, $h_{n+1}(|K_{n+1}| \times I^\infty) \cap \varphi(|K|) \subset \varphi(\text{St}(|K_{n+1}|, K^{(3)}))$, $h_{n+1}C \times I^\infty = h_n|C \times I^\infty$, and $h_{n+1}(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in |K_{n+1}|$.

**Proof.** Let $L = \text{St}(|K_n|, K^{(2)}_{n+1})$ and let $\text{Bd}(L)$ represent the boundary of $L$ in $|K_{n+1}|$. Let $\{\sigma_i\}_{i=1}^\infty$ be the collection of $(n+1)$-simplexes of $K$ and note that $\sigma'_i = \text{Cl}(\sigma_i \setminus L)$ is an $(n+1)$-cell contained in the combinatorial interior of $\sigma_i$. For each $i$ let $\text{Bd}(\sigma'_i)$ denote the combinatorial boundary of $\sigma'_i$. (We are assuming that if $i \neq j$, then $\sigma_i \neq \sigma_j$. If the collection of $(n+1)$-simplexes of $K$ is finite, then the argument is similar). It follows from the given conditions that $\varphi(\bigcup_{i=1}^\infty \sigma'_i) \cap h_n(|K_n| \times I^\infty) = \emptyset$.

Using Lemma 4.2 there is a closed embedding $f : \bigcup_{i=1}^\infty \sigma'_i \times I^\infty \to X$ such that the following properties are satisfied.

1. $f(\bigcup_{i=1}^\infty \sigma'_i \times I^\infty) \cap h_n(|K_n| \times I^\infty) = \emptyset$,
2. $f(x, (0, 0, \cdots)) = \varphi(x)$, for all $x \in \bigcup_{i=1}^\infty \sigma'_i$,
3. $f(\bigcup_{i=1}^\infty \sigma'_i \times I^\infty) \cap \varphi(|K|) = \varphi(\bigcup_{i=1}^\infty \sigma'_i)$, and
4. $\text{Bd} \left( f(\bigcup_{i=1}^\infty \sigma'_i \times I^\infty) \right) = f(\bigcup_{i=1}^\infty \sigma'_i \times W_1^+) \cup f(\bigcup_{i=1}^\infty \text{Bd}(\sigma'_i) \times I^\infty)$

and it is bicollared.

For each $i$ let $\text{Int}(\sigma'_i) = \sigma'_i \setminus \text{Bd}(\sigma'_i)$ and put

$X' = X \setminus f(\bigcup_{i=1}^\infty \text{Int}(\sigma'_i) \times (I^\infty \setminus W_1^+))$,

which is a $Q$-manifold containing

$f(\bigcup_{i=1}^\infty \sigma'_i \times W_1^+) \cup f(\bigcup_{i=1}^\infty \text{Bd}(\sigma'_i) \times I^\infty)$

as a Z-set. (This last assertion easily follows since $\text{Bd}(f(\bigcup_{i=1}^\infty \sigma'_i \times I^\infty))$ is bicollared). Using Lemma 4.1 there is a homeomorphism $\theta : L \times I^\infty \to |K_n| \times I^\infty$ such that $\theta(x, (0, 0, \cdots)) = (x, (0, 0, \cdots))$, for all $x \in |K_n|$, and
\[ \theta | C \times I^\infty = \text{id}, \quad \text{and} \quad \theta (L \times W_1^+) = |K_n| \times W_1^+. \]

Then \( h_n = h_n \circ \theta : L \times I^\infty \to X' \) is a closed embedding such that \( h_n(x, (0, 0, \cdots)) = \phi(x) \), for all \( x \in |K_n|, \quad \text{Bd}(h_n(L \times I^\infty)) = h_n(L \times W_1^+) \) and it is bicollared, and \( h_n|C \times I^\infty = h_n|C \times I^\infty. \)

Let us consider the two sets \( h_n(L \times \{(0, 0, \cdots)\}) \cup h_n(\text{Bd}(L) \times I^\infty) \) and \( f(( \bigcup_{i=1}^{n} \text{Bd}(|\sigma_i|) \times I^\infty) \cup \phi(L) \), which are Z-sets in \( X' \). Define a homeomorphism \( \alpha \) of the former onto the latter such that \( \alpha \circ h_n(x, (0, 0, \cdots)) = \phi(x) \), for all \( x \in L, \quad \text{and} \quad \alpha \circ h_n(x, t) = f(x, t) \), for all \( x \in \text{Bd}(L) \) and \( t \in I^\infty. \)

Using the fact that \( \phi(x) = h_n(x, (0, 0, \cdots)) \), for all \( x \in |K_n| \), and the fact that \( f(x, (0, 0, \cdots)) = \phi(x) \), for all \( x \in \bigcup_{i=1}^{n} \sigma_i \), it is clear that \( \alpha \) is properly homotopic to the identity in \( X' \). In fact, there is an open cover \( u \) of \( X' \setminus h_n(C \times I^\infty) \) which is normal with respect to \( h_n(C \times I^\infty) \) and for which there is a proper homotopy

\[ H : \left( h_n(L \times \{(0, 0, \cdots)\}) \cup h_n(\text{Bd}(L) \times I^\infty) \right) \setminus h_n(C \times I^\infty) \times I \to X' \setminus h_n(C \times I^\infty) \]

satisfying \( H_0 = \text{id}, \)

\[ H_1 = \alpha h_n(L \times \{(0, 0, \cdots)\}) \cup h_n(\text{Bd}(L) \times I^\infty) \setminus h_n(C \times I^\infty), \]

and \( H \) is limited by \( u \). Using Lemma 3.7 we can extend \( \alpha \) to a homeomorphism \( \tilde{\alpha} : X' \to X' \) satisfying \( \tilde{\alpha}|h_n(C \times I^\infty) = \text{id}. \)

Then \( \tilde{\alpha} \circ h_n : L \times I^\infty \to X' \)

is a closed embedding which satisfies \( \text{Bd}(\tilde{\alpha} \circ h_n(L \times I^\infty)) = \tilde{\alpha} \circ h_n(L \times W_1^+) \)

and it is bicollared,

\[ \tilde{\alpha} \circ h_n|C \times I^\infty = h_n|C \times I^\infty, \quad \tilde{\alpha} \circ h_n|\text{Bd}(L) \times I^\infty = f|\text{Bd}(L) \times I^\infty, \]

and \( \tilde{\alpha} \circ h_n(x, (0, 0, \cdots)) = \phi(x) \), for all \( x \in L. \)

Now let \( F = f(( \bigcup_{i=1}^{n} \sigma_i) \times W_1^+) \), which is a Z-set in \( X' \) satisfying \( F \cap (\tilde{\alpha} \circ h_n(L \times \{(0, 0, \cdots)\}) \cup (\tilde{\alpha} \circ h_n(\text{Bd}(L \times (I^\infty \setminus W_1^+)))] = \emptyset. \)

Using Lemma 4.3 there is a homeomorphism \( \beta : X' \to X' \) satisfying

\[ \beta(F) \cap (\tilde{\alpha} \circ h_n(L \times I^\infty) \subset (\tilde{\alpha} \circ h_n(\text{Bd}(L) \times W_1^+) \]

and

\[ \beta(\tilde{\alpha} \circ h_n(L \times \{(0, 0, \cdots)\}) \cup (\tilde{\alpha} \circ h_n(\text{Bd}(L) \times I^\infty) \cup (\tilde{\alpha} \circ h_n(C \times I^\infty) = \text{id}. \]

Thus \( f : ( \bigcup_{i=1}^{n} \sigma_i \times I^\infty \to X \) and \( \beta^{-1} \circ \tilde{\alpha} \circ h_n : L \times I^\infty \to X \) are closed embeddings which are compatible, i.e. we can patch them together to obtain a closed embedding \( h'_{n+1} : |K_{n+1}| \times I^\infty \to X \) which satisfies \( \text{Bd}(h'_{n+1}(|K_{n+1}| \times I^\infty)) = h'_{n+1}(|K_{n+1}| \times W_1^+), \quad h'_{n+1}|C \times I^\infty = h_n|C \times I^\infty, \)

and \( h'_{n+1}(x, (0, 0, \cdots)) = \phi(x) \), for all \( x \in |K_{n+1}| \).

Of course we have made no provision to require that
be bicollared, but this presents no problem since
\[ \text{Bd}(h_{n+1}'(\{|K_{n+1}| \times [I^\infty]) \}
\] is bicollared. It is also true that we might not have
\[ h_{n+1}'(\{|K_{n+1}| \times I^\infty) \cap \varphi(\{|K|) \subset \varphi(\text{St}(\{|K_{n+1}|, K^{(3)})) \]
but this can be clearly achieved by ‘squeezing’
\[ h_{n+1}'(\{|K_{n+1}| \times I^\infty) \text{ close to } \varphi(\{|K_{n+1}|) \]
Thus we can modify \( h_{n+1}' \) to obtain our required \( h_{n+1} \).

**Proof of Theorem 1.**

Write \( X = \bigcup_{n=1}^{\infty} X_n \), where each \( X_n \) is a compact set contained in the interior of \( X_{n+1} \). Let \( K \) be a complex and let \( \varphi : |K| \to P \) be a homeomorphism. Let \( H_1 \) be a finite subcomplex of \( K \) such that \( P \cap X_1 \subset \varphi(|H_1|) \) and choose \( n_1 \) large enough so that
\[ \text{St}(|H_1|, K_{n_1+1}) \subset |K_{n_1}|. \]

One can clearly construct a closed embedding \( h_0 : |K_0| \times I^\infty \to X \) which satisfies \( h_0(x, (0, 0, \cdots)) = \varphi(x) \), for all \( x \in |K_0| \), and
\[ \text{Bd}(h_0(|K_0| \times I^\infty)) = h_0(|K_0| \times W_1^+) \]
and it is bicollared. Then using Lemma 4.4 and an obvious inductive argument we can obtain a closed embedding \( h_{n_1} : |K_{n_1}| \times I^\infty \to X \) satisfying \( h_{n_1}(x, (0, 0, \cdots)) = \varphi(x) \), for all \( x \in |K_{n_1}| \), and
\[ \text{Bd}(h_{n_1}(|K_{n_1}| \times I^\infty)) = h_{n_1}(|K_{n_1} \times W_1^+) \]
and it is bicollared.

Now let \( H_2 \) be a finite subcomplex of \( K \) so that \( |H_1| \subset \text{Int}(|H_2|) \) and \( \varphi(|K|) \cap X_2 \subset \varphi(|H_2|) \). Choose \( n_2 > n_1 \) such that
\[ \text{St}(|H_2|, K_{n_2+1}) \subset |K_{n_2}|. \]

Using Lemma 4.4 and an inductive argument we can find a closed embedding \( h_{n_2} : |K_{n_2}| \times I^\infty \to X \) such that \( h_{n_2}(x, (0, 0, \cdots)) = \varphi(x) \), for all \( x \in |K_{n_2}| \), \( \text{Bd}(h_{n_2}(|K_{n_2}| \times I^\infty)) = h_{n_2}(|K_{n_2} \times W_1^+) \) and it is bicollared, and \( h_{n_2}(|H_1| \times I^\infty) = h_{n_2}(|H_1| \times W_1^+ \times I^\infty) \).

In general let \( \{H_i\}_{i=1}^{\infty} \) be a collection of finite subcomplexes of \( K \) so that for each \( i, |H_i| \subset \text{Int}(|H_{i+1}|) \) and \( \varphi(|K|) \cap X_i \subset \varphi(|H_i|) \). Choose integers \( \{n_i\}_{i=1}^{\infty} \) such that for each \( i, n_i < n_{i+1} \) and
\[ \text{St}(|H_i|, K_{n_i+1}) \subset |K_{n_i}|. \]
Using the above techniques we find that for each \(i > 0\) there is a closed embedding \(h_{ni} : |K_{ni}| \times I^\infty \to X\) such that \(h_{ni}(x, (0, 0, \cdots)) = \varphi(x)\), for all \(x \in |K_{ni}|\), \(\text{Bd}(h_{ni}(|K_{ni}| \times I^\infty)) = h_{ni}(|K_{ni}| \times W_1^+\) and it is bicollared, and \(h_{ni+1}(|H| \times I^\infty = h_{ni}|H_i| \times I^\infty\). For each \(x \in |H_i| \times (I^\infty \setminus W_1^+)\) define \(h'(x) = h_{ni}(x)\). It is clear that \(h' : |K| \times (I^\infty \setminus W_1^+) \to X\) is an open embedding satisfying \(h'(x, (0, 0, \cdots)) = \varphi(x)\), for all \(x \in |K|\). Since \(I^\infty \setminus W_1^+ \cong I^\infty \setminus \{\text{point}\}\) we can clearly modify \(h'\) to obtain our required open embedding \(h\).

5. Proof of Theorem 2

We will first establish two technical results concerning cap-sets in \(Q\)-manifolds. These are used only in the proof of Theorem 2.

**Lemma 5.1.** Let \(X\) be a \(Q\)-manifold, \(P\) be a polyhedron, \(\varphi : P \times \Sigma \to X\) be an embedding such that \(\varphi(P \times \Sigma)\) is a cap-set for \(X\) and \(\varphi(P \times \Sigma_1)\) is closed in \(X\), and let \(F\) be a compact Z-set in \(X\). Then there is a homeomorphism \(h : X \to X\) such that \(h(F) \subset \varphi(P \times \Sigma_2)\) and \(h|\varphi(P \times \Sigma_1) = \text{id}\).

**Proof.** By Lemma 3.5, it follows that \(\varphi(P \times \Sigma_1)\) is a Z-set in \(X\). Let \(X' = X \setminus \varphi(P \times \Sigma_1)\), \(F' = F \cap X'\), and \(M = \varphi(P \times \Sigma) \setminus \varphi(P \times \Sigma_1)\). Then \(X'\) is a \(Q\)-manifold, \(F'\) is a Z-set in \(X'\), and \(M\) is a cap-set for \(X'\). Choose an open cover \(\mathcal{U}\) of \(X'\) which is normal with respect to \(\varphi(P \times \Sigma_1)\).

Lemma 3.2 implies that \(M \cup F'\) is a cap-set for \(X'\). Using Lemma 3.3 there is a homeomorphism \(f : X' \to X'\) such that \(f(M \cup F') = M\) and \(f\) is limited by \(\mathcal{U}\). Then \(f\) clearly extends to a homeomorphism \(\tilde{f} : X \to X\) satisfying \(\tilde{f}|\varphi(P \times \Sigma_1) = \text{id}\) and \(\tilde{f}(F) \subset \varphi(P \times \Sigma)\).

Put \(F^* = \pi_2 \circ \varphi^{-1} \circ \tilde{f}(F)\), which is a compact set in \(\Sigma\). Clearly there is a proper isotopy \(g_t : F^* \cup \Sigma_1 \to \Sigma\) such that \(g_0 = \text{id}\), \(g_1(F^*) \subset \Sigma_2\) and \(g_t|\Sigma_1 = \text{id}\) for all \(t\). Now define an isotopy

\[h_t : \tilde{f}(F) \cup \varphi(P \times \Sigma_1) \to \varphi(P \times \Sigma)\] by \(h_t \circ \varphi(x, y) = \varphi(x, g_t(y))\),

for all \((x, y) \in P \times \Sigma\) that satisfy \(\varphi(x, y) \in \tilde{f}(F) \cup \varphi(P \times \Sigma_1)\). Note that \(h_t(\tilde{f}(F) \cup \varphi(P \times \Sigma_1))\) is a Z-set in \(X\) and \(h_t\) is a proper isotopy. Using Lemma 3.7 we can extend \(h_1\) to a homeomorphism \(g : X \to X\). Then \(h = g \circ \tilde{f}\) fulfills our requirements.

**Lemma 5.2.** Let \(X\) be a \(Q\)-manifold, \(P\) be a polyhedron, and let \(\varphi : P \times \Sigma \to X\) be an embedding such that \(\varphi(P \times \Sigma)\) is a cap-set for \(X\) and \(\varphi(P \times \Sigma_2)\) is closed in \(X\). Let \(h : P \times I^\infty \to X\) be a closed embedding so that \(h(x, (0, 0, \cdots)) = \varphi(x, (0, 0, \cdots))\), for all \(x \in P\), and \(\text{Bd}(h(P \times I^\infty)) = h(P \times W_1^+)\). If \(F \subset X\) is a compact Z-set, then there is a homeomorphism \(f : X \to X\) such that \(f(F) \subset h(P \times I^\infty)\) and \(f|h(P \times W_1^-) = \text{id}\).
PROOF. Let \( \varphi : \varphi(P \times \Sigma_2) \to h(P \times \Sigma_2) \) be the homeomorphism defined by \( \vartheta \circ \varphi(x, y) = h(x, y) \), for all \((x, y) \in P \times \Sigma_2\). It is clear that \( \vartheta \) is properly homeotopic to the identity. Let \( \varphi_1 \) be an extension of \( \vartheta \) to a homeomorphism of \( X \) onto itself. Then \( \varphi_1 \circ \varphi : P \times \Sigma \to X \) is an embedding such that \( \varphi_1 \circ \varphi(P \times \Sigma) \) is a cap-set for \( X \), \( \varphi_1 \circ \varphi(P \times \Sigma_1) = h(P \times \Sigma_1) \), \( \varphi_1 \circ \varphi(P \times \Sigma_2) = h(P \times \Sigma_2) \), and \( \varphi_1 \circ \varphi(x, (0, 0, \cdots)) = h(x, (0, 0, \cdots)) \), for all \( x \in P \).

It is clear that there exists a homeomorphism \( \alpha : h(P \times \Sigma_1) \to h(P \times W_1^+) \) such that \( \alpha \circ h(x, (0, 0, \cdots)) = h(x, (-1, 0, 0, \cdots)) \) for all \( x \in P \), and for which \( \alpha \) is properly homotopic to the identity, with the homotopy taking place inside \( h(P \times I^\infty) \). By choosing covers appropriately and using Lemma 3.7 we can extend \( \alpha \) to a homeomorphism \( \varphi_2 : X \to X \) which satisfies \( \varphi_2|X \setminus h(P \times I^\infty) = \text{id} \). It is clear now that \( \bar{\varphi} = \varphi_2 \circ \varphi_1 \circ \varphi : P \times \Sigma \to X \) is an embedding such that \( \bar{\varphi}(P \times \Sigma) \) is a cap-set for \( X \) and \( \bar{\varphi}(P \times \Sigma_2) \) is a \( Z \)-set in \( X \).

Using Lemma 5.1 there is a homeomorphism \( f : X \to X \) such that \( f(F) \subset \bar{\varphi}(P \times \Sigma_2) \) and \( f|\bar{\varphi}(P \times \Sigma_1) = \text{id} \). This implies that \( f|h(P \times W_1^+) = \text{id} \). Note that \( \varphi_1 \circ \varphi(P \times \Sigma_2) = h(P \times \Sigma_2) \) and \( \varphi_2 \circ \varphi_1 \circ \varphi(P \times \Sigma_2) = \varphi_2 \circ h(P \times \Sigma_2) \subset h(P \times I^\infty) \), which implies that \( f(F) \subset h(P \times I^\infty) \).

PROOF OF THEOREM 2.

Roughly the idea of the proof is to find a copy of \( P \) in \( X \) which is a \( Z \)-set, use Theorem 1 to build a 'nice' open set around this polyhedron, and use Lemma 5.2 to 'blow up' this open set to engulf a cap-set. The part of \( X \) that this open set misses is the \( Z \)-set \( F \) which we are looking for.

Using Lemma 3.1 let \( \varphi : P \times \Sigma \to X \) be an embedding such that \( \varphi(P \times \Sigma) \) is a cap-set for \( X \). A routine argument proves that if \( A \) is any locally compact subset of \( X \), then \( Cl(A) \setminus A \) is a closed subset of \( X \). Thus, \( F_1 = Cl(\varphi(P \times \Sigma_2)) \setminus \varphi(P \times \Sigma_2) \) is a closed subset of \( X \) missing \( \varphi(P \times \Sigma) \). It follows from Lemma 3.4 that \( F_1 \) is a \( Z \)-set in \( X \). Put \( X' = X \setminus F_1 \) and note that \( \varphi(P \times \Sigma) \) is a cap-set for \( X' \). But we now have \( \varphi(P \times \Sigma_2) \) a \( Z \)-set in \( X' \), because it is closed.

Using Theorem 1 there is a closed embedding \( h : P \times I^\infty \to X' \) such that \( h(x, (0, 0, \cdots)) = \varphi(x, (0, 0, \cdots)) \), for all \( x \in P \), and

\[
\text{Bd}(h(P \times I^\infty)) = h(P \times W_1^+).
\]

Write \( \varphi(P \times \Sigma) = \bigcup_{n=1}^{\infty} M_n \), a tower of compact \( Z \)-sets. Using Lemma 5.2 there is a homeomorphism \( f_1 : X' \to X' \) such that

\[
f_1(M_1) \subset h(P \times [-1, \frac{1}{n}] \times \Pi_{i=2}^{\infty} I_i).
\]

Then put \( g_1 = f_1^{-1} \) to complete the first step of our construction.
Now let $X'' = X' \setminus g_1 \circ h \left( P \times [-1, \frac{1}{2}) \times \prod_{i=2}^{\infty} I_i \right)$, which is obviously a $Q$-manifold containing $g_1 \circ h(P \times \{\frac{1}{2}\} \times \prod_{i=2}^{\infty} I_i)$ as a $Z$-set. Put $M_2' = M_2 \cap X''$, which is clearly a compact $Z$-set in $X''$. One can obviously construct a homeomorphism $\alpha : X' \to X''$ such that

$$\alpha \circ g_1 \circ h(x, (0, 0, \cdots)) = g_1 \circ h(x, (\frac{3}{2}, 0, 0, \cdots)),$$

for all $x \in P$. Then $\phi' = \alpha \circ g_1 \circ \phi : P \times \Sigma \to X''$ is an embedding such that $\phi'(P \times \Sigma)$ is a cap-set for $X''$ and

$$\phi'(x, (0, 0, \cdots)) = g_1 \circ h(x, (\frac{3}{2}, 0, 0, \cdots)),$$

for all $x \in P$. Also $g_1 \circ h : P \times [\frac{3}{2}, 1] \times \prod_{i=2}^{\infty} I_i \to X''$ is a closed embedding satisfying $\text{Bd}_X \cdot (g_1 \circ h(P \times \{\frac{1}{2}\} \times \prod_{i=2}^{\infty} I_i)) = g_1 \circ h(P \times W_1^+)$. 

Once more applying Lemma 5.2 there is a homeomorphism $f_2 : X'' \to X''$ such that $f_2|g_1 \circ h(P \times \{\frac{1}{2}\} \times \prod_{i=2}^{\infty} I_i) = \text{id}$ and

$$f_2(M_2') \subseteq g_1 \circ h \left( P \times [\frac{1}{2}, \frac{3}{2}] \times \prod_{i=2}^{\infty} I_i \right).$$

Then let $\tilde{f}_2$ be the extension of $f_2$ to all of $X'$ such that

$$\tilde{f}_2|g_1 \circ h \left( P \times [-1, \frac{1}{2}] \times \prod_{i=2}^{\infty} I_i \right) = \text{id}.$$

Now put $g_2 = \tilde{f}_2^{-1}$, which is a homeomorphism of $X'$ onto itself satisfying $g_2|g_1 \circ h(P \times [-1, \frac{1}{2}] \times \prod_{i=2}^{\infty} I_i) = \text{id}$ and

$$M_2' \subseteq g_2 \circ g_1 \circ h(P \times [-1, \frac{3}{4}] \times \prod_{i=2}^{\infty} I_i).$$

It is then clear that we can obtain a sequence $\{g_{i}\}_{i=1}^{\infty}$ of homeomorphisms of $X'$ onto itself such that $M_n \subseteq g_n \circ g_{n-1} \circ \cdots \circ g_1 \circ h \left( P \times \left[ -1, 1 - \frac{1}{2^n} \right] \times \prod_{i=2}^{\infty} I_i \right)$ and

$$g_n|g_{n-1} \circ \cdots \circ g_1 \circ h \left( P \times \left[ -1, 1 - \frac{1}{2^{n-1}} \right] \times \prod_{i=2}^{\infty} I_i \right) = \text{id},$$

for all $n > 1$. Then let $g(x) = \lim_{n \to \infty} g_n \circ \cdots \circ g_1(x)$ for all $x \in h(P \times (I^{\infty}_+ \setminus W_1^+))$. It is clear that $g : h(P \times (I^{\infty}_+ \setminus W_1^+)) \to X'$ is an open embedding such that $g \circ h(P \times (I^{\infty}_+ \setminus W_1^+))$ contains $\phi(P \times \Sigma)$. Thus

$$F_2 = X' \setminus g \circ h(P \times (I^{\infty}_+ \setminus W_1^+))$$

is a $Z$-set in $X'$ and therefore $F = F_1 \cup F_2$ is a $Z$-set in $X$ such that $X \setminus F \simeq P \times (I^{\infty}_+ \setminus W_1^+)$. 

6. Proofs of Theorems 3, 4, 5 and their Corollaries

The following result will be used in the proof of Theorem 3.

**Lemma 6.1.** Let $X$ be a $Q$-manifold and let $F \subseteq X$ be a $Z$-set. Then $(X \setminus F) \times [0, 1) \cong X \times [0, 1)$, where the homeomorphism can be chosen to be homotopic to the inclusion of $(X \setminus F) \times [0, 1)$ in $X \times [0, 1)$.

**Proof.** If $X_1$ is any $Q$-manifold and $C \subseteq X_1$ is any $Z$-set, then $C \times [0, 1]$ is a $Z$-set in $X_1 \times [0, 1]$. In order to see this let us take a homeomorphism $h_1$ of $X_1$ onto $X_1 \times [0, 1]$ taking $C$ into $X_1 \{0, 0, \cdots \}$. Then $h_1 \times id : X_1 \times [0, 1] \to X_1 \times [0, 1] \times [0, 1]$ is a homeomorphism which takes $C \times [0, 1]$ into $X_1 \times \{(0, 0, \cdots)\} \times [0, 1]$. Let

$$h_2 : X_1 \times [0, 1] \to X_1 \times [0, 1]$$

be a homeomorphism in which $[0,1]$ is factored back into $X_1$. Then $h_2 \circ (h_1 \times id) : X_1 \times [0, 1] \to X_1 \times [0, 1]$ is a homeomorphism taking $C \times [0, 1]$ into $X_1 \times \{(0, 0, \cdots)\}$, and by Lemma 3.8 it follows that

$$h_2 \circ (h_1 \times id)(C \times [0, 1])$$

is a $Z$-set in $X_1 \times [0, 1]$. Thus $C \times [0, 1]$ is a $Z$-set in $X_1 \times [0, 1]$.

Let $A = (X \times \{1\}) \cup (F \times [0, 1])$ and $B = (X \times \{1\}) \cup (F \times [\frac{1}{2}, 1])$ be subsets of $X \times [0, 1]$. Since $A$ and $B$ are $Z$-sets in $X \times [0, 1]$ we can use Lemma 3.7 to get a homeomorphism $f : X \times [0, 1] \to X \times [0, 1]$ satisfying $f(A) = B$ and $f|X \times \{1\} = id$. It follows from [3] that we can additionally choose $f$ to be isotopic to $id_{X \times [0, 1]}$ (with each level fixed on $X \times \{1\}$). Therefore $f|X \times [0, 1]$ gives a homeomorphism of $X \times [0, 1]$ onto itself which is homotopic (in $X \times [0, 1]$) to $id_{X \times [0, 1]}$.

Let $h_t : [0,1] \to [0,1]$ be a homotopy which satisfies the following properties:

1. $h_0 = id$,
2. $h_t([\frac{1}{2}, 1]) = \{1\}$,
3. $h_t|[0, \frac{1}{2}]$ is a homeomorphism of $[0, \frac{1}{2}]$ onto $[0, 1]$,
4. $h_t : [0, 1] \to [0, 1]$ is a homeomorphism for all $t \neq 1$.

Define a continuous function $g : X \times [0, 1] \to X \times [0, 1]$ as follows: for each $x \in X$ and $y \in [0, 1]$, let $g(x, y) = (x, h_t(y))$, where $t = 1/(1 + d(x, F))$. Clearly $g|X \times [0, 1] \setminus B$ gives a homeomorphism of $(X \times [0, 1] \setminus B$ onto $X \times [0, 1]$ which is homotopic to the inclusion of $(X \times [0, 1] \setminus B$ in $X \times [0, 1]$. Then $g \circ f|\{(X \setminus F) \times [0, 1)$ gives a homomorphism of $(X \setminus F) \times [0, 1)$ onto $X \times [0, 1]$ which is homotopic to the inclusion of $(X \setminus F) \times [0, 1)$ in $X \times [0, 1]$.

We will also need the following result.
LEMMA 6.2. Let $X$ be a $Q$-manifold, $P$ be a polyhedron, and let $f : P \times (I^{\infty} \setminus W_1^+ \to X$ be a homotopy equivalence. Then there exists an open embedding $g : P \times (I^{\infty} \setminus W_1^+) \to X$ such that $g$ is homotopic to $f$ and $X \setminus g(P \times (I^{\infty} \setminus W_1^+))$ is a Z-set in $X$.

PROOF. It follows routinely from the coordinate structure of $I^{\infty}$ that there is a homeomorphism of $I^{\infty} \times I^{\infty}$ onto $I^{\infty}$ which is homotopic to the projection of $I^{\infty} \times I^{\infty}$ onto the first factor. Since $X \times I^{\infty} \cong X$, it follows that there is a homeomorphism $\beta : X \times I^{\infty} \to X$ which is homotopic to $\pi_x$, the projection of $X \times I^{\infty}$ onto $X$. Define $f' : P \to X$ by $f'(x) = f(x, (0, 0, \cdots))$, for all $x \in P$. Then $f'$ is also a homotopy equivalence.

It follows from [15] that $P \times s$ is an $F$-manifold and it follows routinely from the definition that $X \times s$ is an $F$-manifold. Note that

$$f' \times \text{id}_s : P \times s \to X \times s$$

is a homotopy equivalence. Thus $f' \times \text{id}_s$ is homotopic to a homeomorphism $\alpha : P \times s \to X \times s$ (see [10]).

Now $P \times \Sigma$ is a cap-set for $P \times s$ (see [6]) and therefore $\alpha(P \times \Sigma)$ is a cap-set for $X \times I^{\infty}$ (since $X \times I^{\infty}$ can be deformed into $X \times s$ with ‘small’ motions). Hence $\beta \circ \alpha(P \times \Sigma)$ is a cap-set for $X$. As in the proof of Theorem 2 let $F_1 = \text{Cl}(\varphi(P \times \Sigma_2)) \setminus \varphi(P \times \Sigma_2)$, where $\varphi = \beta \circ \alpha|P \times \Sigma$, and let $h : P \times I^{\infty} \to X \setminus F_1$ be a closed embedding such that

$$h(x, (0, 0, \cdots)) = \varphi(x, (0, 0, \cdots)),$$

for all $x \in P$, and $\text{Bd}(h(P \times I^{\infty})) = h(P \times W_1^+)$. In the proof of Theorem 2 a homeomorphism $g' : h(P \times (I^{\infty} \setminus W_1^+)) \to X \setminus F$ was constructed, where $F$ is a Z-set in $X$ containing $F_1$. Moreover it is clear from the construction given there that $g'$ is homotopic to the inclusion of $h(P \times (I^{\infty} \setminus W_1^+))$ in $X$. Thus $g = g' \circ h|P \times (I^{\infty} \setminus W_1^+)$ gives an open embedding of $P \times (I^{\infty} \setminus W_1^+)$ in $X$ whose complement is a Z-set in $X$. Moreover $g$ is homotopic to $h' = h|P \times (I^{\infty} \setminus W_1^+)$. All that is left to do is prove that $h'$ is homotopic to $f$.

To this end let $r : P \times (I^{\infty} \setminus W_1^+) \to P \times \{(0, 0, \cdots)\}$ be given by $r(x, t) = (x, (0, 0, \cdots))$, for all $x \in P$ and $t \in I^{\infty} \setminus W_1^+$. It is clear that $h'$ is homotopic to $h' \circ r$ and $h' \circ r = \beta \circ \alpha \circ r$. Since $\alpha$ is homotopic to $f' \times \text{id}_s$ it follows that $\beta \circ \alpha \circ r$ is homotopic to $\beta \circ (f' \times \text{id}_s) \circ r$. But $\beta \circ (f' \times \text{id}_s) \circ r$ is homotopic to $\pi_x \circ (f' \times \text{id}_s) \circ r$. But $\pi_x \circ (f' \times \text{id}_s) \circ r = f \circ r$, and since $r$ is homotopic to $\text{id}_{P \times (I^{\infty} \setminus W_1^+)}$ it follows that $f \circ r$ is homotopic to $f$.

PROOFS OF THEOREMS 3 AND 5.

Let $f : X \to Y$ be a homotopy equivalence, where $X$ and $Y$ are $Q$-
manifolds. Let $P$ be a polyhedron for which there exists a homotopy equivalence $g : P \times (I^\infty \setminus W_1^+) \to X$. Using Lemma 6.2 we see that $g$ is homotopic to a homeomorphism $\alpha : P \times (I^\infty \setminus W_1^+) \to X \setminus F_1$, where $F_1 \subset X$ is a $Z$-set. Also $f \circ g$ is homotopic to a homeomorphism $\beta : P \times (I^\infty \setminus W_1^+) \to Y \setminus F_2$, where $F_2 \subset Y$ is a $Z$-set. Using Lemma 6.1 it follows that $\alpha \times \text{id} : (P \times (I^\infty \setminus W_1^+)) \times [0, 1) \to (X \setminus F_1) \times [0, 1)$ is homotopic to a homeomorphism $\gamma : (P \times (I^\infty \setminus W_1^+)) \times [0, 1) \to X \times [0, 1)$, with the homotopy taking place in $X \times [0, 1)$. Similarly $\beta \times \text{id}$ is homotopic to a homeomorphism $\delta : (P \times (I^\infty \setminus W_1^+)) \times [0, 1) \to Y \times [0, 1)$, with the homotopy taking place in $Y \times [0, 1)$.

In order to see that $X \times [0, 1) \cong P \times (I^\infty \setminus \{\text{point}\})$ note that $\gamma^{-1}$ gives a homeomorphism of $X \times [0, 1)$ onto $P \times (I^\infty \setminus W_1^+) \times [0, 1)$. Since $I^\infty \setminus W_1^+ = [-1, 1) \times \Pi_{i=2}^\infty I_i$ and since $[-1, 1) \times [0, 1)$ is obviously homeomorphic to $[-1, 1) \times [0, 1)$, we have $X \times [0, 1) \cong P \times (I^\infty \setminus W_1^+)$. To finish the proof of Theorem 3 all we need do is note that $I^\infty \setminus W_1^+ \cong I^\infty \setminus \{\text{point}\}$.

For the proof of Theorem 5 note that $\delta \circ \gamma^{-1} : X \times [0, 1) \to Y \times [0, 1)$ is a homeomorphism. All that remains to be done is prove that $\delta \circ \gamma^{-1}$ is homotopic to $f \times \text{id}$, or equivalently, to prove that $\delta$ is homotopic to $(f \times \text{id}) \circ \gamma$. But $\delta$ is homotopic to $\beta \times \text{id}$, which in turn is homotopic to $(f \circ g) \times \text{id} = (f \times \text{id}) \circ (g \times \text{id})$. Since $g \times \text{id}$ is homotopic to $\alpha \times \text{id}$, and $\alpha \times \text{id}$ is homotopic to $\gamma$, we are done.

**Proof of Corollary 1.**

Choose any polyhedron $P$ for which $P \sim X$ and use Theorem 3 to get $X \times [0, 1) \cong P \times (I^\infty \setminus \{\text{point}\})$. Now $I^\infty \setminus \{\text{point}\} \cong I^\infty \times [0, 1)$, hence $P \times (I^\infty \setminus \{\text{point}\}) \cong (P \times [0, 1)) \times I^\infty$. But $P \times [0, 1)$ can obviously be triangulated by a complex.

**Proof of Corollary 2.**

Apply Theorem 3.

**Proof of Corollary 3.**

Apply Theorem 3.

**Proof of Theorem 4.**

Let $Y = X \times s$, which is obviously an $F$-manifold satisfying $Y \sim X$. Using Henderson's open embedding theorem let $g : Y \to s$ be an open embedding. Let $U$ be an open subset of $I^\infty$ for which $U \cap s = g(Y)$. Then $U$ is a $Q$-manifold, and as $U \cap B(I^\infty)$ is obviously a cap-set for $U$, it follows from Lemma 3.6 that $U \sim g(Y)$. Thus $X \sim U$. Using Corollary 2 we have $X \times [0, 1) \cong U \times [0, 1)$, and using the fact that $U \times [0, 1] \cong U$ we have $U \times [0, 1) \cong U \setminus F$, for some closed subset $F$ of $U$. Thus $X \times [0, 1) \cong U \setminus F$, which is open in $I^\infty$. 

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PROOF OF COROLLARY 4.
Let \( f : X \to X \times [0, 1] \) be a homeomorphism and put
\[
U = f^{-1}(X \times [0, 1]), \quad V = f^{-1}(X \times (0, 1]).
\]

7. Proofs of Theorem 6, its Corollary, and Theorem 7

The following result will be used in the proof of Theorem 6.

**Lemma 7.1.** Let \( X \) be a compact \( Q \)-manifold and assume that \( X \sim P \), for some compact polyhedron \( P \). Then there is a copy \( P' \) of \( P \) in \( X \) which is a \( Z \)-set and a pseudo-isotopy \( h_t : X \to X \) which satisfies the following properties.

1. \( h_0 = \text{id} \),
2. \( h_1(X) = P' \),
3. \( h_t|P' = \text{id} \) for all \( t \), and
4. \( h_t : X \to X \) is an embedding for all \( t \neq 1 \).

**Proof.** Let \( f : X \to X \times I^\infty \) be a homeomorphism. Since \( X \times s \) is an \( F \)-manifold and \( X \times s \sim P \), it follows that there is a homeomorphism \( \varphi : P \times s \to X \times s \). Using the fact that \( \varphi(P \times \{(0, 0, \cdots)\}) \) is a compact subset of \( X \times s \), it is clear that there is an isotopy \( f_t : X \times I^\infty \to X \times I^\infty \) such that \( f_0 = \text{id} \), \( f_1(X \times I^\infty) \subset X \times s \), and \( f_t|\varphi(P \times \{(0, 0, \cdots)\}) = \text{id} \), for all \( t \).

One can obviously get a pseudo-isotopy \( g_t : \varphi(P \times s) \to \varphi(P \times s) \) such that \( g_0 = \text{id} \), \( g_t \circ \varphi(P \times s) = \varphi(P \times \{(0, 0, \cdots)\}) \), \( g_t \) is an embedding for all \( t \neq 1 \), and \( g_t|\varphi(P \times \{(0, 0, \cdots)\}) = \text{id} \), for all \( t \). Then let \( h'_t : X \times I^\infty \to X \times I^\infty \) be defined by
\[
h'_t(x) = \begin{cases} 
f_{2t}(x), & \text{for } 0 \leq t \leq \frac{1}{2} 
g_{2t-1} \circ f_1(x), & \text{for } \frac{1}{2} \leq t \leq 1 
\end{cases}
\]

Obviously \( h'_t \) is a pseudo-isotopy satisfying
\[
h'_0 = \text{id}, \quad h'_1(X \times I^\infty) = \varphi(P \times \{(0, 0, \cdots)\}), \quad h'_t|\varphi(P \times \{(0, 0, \cdots)\}) = \text{id}
\]
for all \( t \), and \( h'_t \) is an embedding for all \( t \neq 1 \). Then let
\[
P' = f^{-1} \circ \varphi(P \times \{(0, 0, \cdots)\})
\]
and let \( h_t : X \to X \) be defined by \( h_t(x) = f^{-1} \circ h'_t \circ f(x) \).

**Proof of Theorem 6.**
Using Theorem 3 and the fact that \( X \simeq X \times [0, 1] \), there is a copy \( X' \) of \( X \) in \( X \) which is a \( Z \)-set and there is a homeomorphism
\[
f : P \times (I^\infty \setminus W_1^+) \to X \setminus X'.
\]
Using Lemma 7.1 let $P'$ be a copy of $P$ in $X'$ and let $h_t : X' \to X'$ be a pseudo-isotopy satisfying $h_0 = \text{id}$, $h_1(X') = P'$, $h_t$ is an embedding for all $t \neq 1$, and $h_t|P' = \text{id}$ for all $t$. Since $P' \subset X'$ it easily follows that $P'$ is a Z-set in $X$.

Let $\{U_i\}_{i=1}^\infty$ be any collection of open subsets of $X$ such that $\bigcap_{i=1}^\infty U_i = P'$ and $X' \subset U_1$. Using the compactness of $P$ and $X$ we can find a number $t_1 \in (-1, 1)$ such that $f(P \times [t_1, 1] \times \prod_{i=2}^\infty I_i) \subset U_1$. Let $V_1 = X \setminus f(P \times [-1, t_1] \times \prod_{i=2}^\infty I_i)$, which is an open set containing $X'$. By choosing $t \in (0, 1)$ sufficiently close to 1 we have an embedding $h_t : X' \to X' \cap U_2$ which is properly homotopic to the identity, where the image of the proper homotopy is entirely contained in $X'$. Moreover this proper homotopy is limited by some open cover of $V_1$ which is normal with respect to $X \setminus V_1$. Thus we can apply Lemma 3.7 to extend $h_t$ to a homeomorphism $g_1 : X \to X$ which satisfies

$$g_1|f(P \times [-1, t_1] \times \prod_{i=2}^\infty I_i) = \text{id},$$

$g_1|P' = \text{id}$, and $g_1(X') \subset U_2$.

Now choose $t_2 \in (t_1, 1)$ such that $g_1 \circ f(P \times [t_2, 1] \times \prod_{i=2}^\infty I_i) \subset U_2$ and use the above techniques to construct a homeomorphism $g_2 : X \to X$ satisfying $g_2|g_1 \circ f(P \times [-1, t_2] \times \prod_{i=2}^\infty I_i) = \text{id}$, $g_2|P' = \text{id}$, and

$$g_2 \circ g_1(X') \subset U_3.$$

It is clear that we can continue this process to obtain homeomorphisms $\{g_i\}_{i=1}^\infty$ of $X$ onto itself and numbers $t_1 < t_2 < \cdots < 1$ limiting to 1 such that

$$g_{i+1}g_i \circ \cdots \circ g_1 \circ f(P \times [-1, t_{i+1}] \times \prod_{i=2}^\infty I_i) = \text{id},$$

$g_1 \circ \cdots \circ g_1(X') \subset U_{i+1}$, and $g_i|P' = \text{id}$, for all $i$. Then define $g : P \times (I^\infty \setminus W^n_+) \to X \setminus P'$ by $g(x) = \lim g_1 \circ \cdots \circ g_1 \circ f(x)$. Clearly $g$ is a homeomorphism which is what we wanted.

**Proof of Corollary 5.**

It follows from [12] that any homotopically trivial metric ANR is contractible. Thus $X$ must be a compact contractible $Q$-manifold, hence it has the homotopy type of a point. It follows from Theorem 6 that $X \setminus \{\text{point}\} \cong I^\infty \setminus \{\text{point}\}$, thus $X \cong I^\infty$.

We will need the following result for the proof of Theorem 7.

**Lemma 7.2.** Let $X$ be a compact $Q$-manifold for which $X \sim P$, for some compact polyhedron $P$. Then there is an embedding $h : P \times I^\infty \to X$ such that $\text{Bd}(h(P \times I^\infty)) = h(P \times W^n_+)$ and there is a strong deformation retraction of $X$ onto $h(P \times W^n_-)$.

**Proof.** Let $\varphi : P \times s \to X \times s$ be a homeomorphism and let
be an embedding such that $h'(x, (0, 0, \cdots)) = \varphi(x, (0, 0, \cdots))$, for all $x \in P$, and \(\text{Bd}(h'(P \times I^\infty)) = h'(P \times W_1^+).\) Now $h'(P \times W_1^-)$ is a Z-set in $X \times I^\infty$, thus Lemma 3.7 implies that there is a homeomorphism $f : X \times I^\infty \to X \times I^\infty$ for which $f \circ h'(P \times W_1^-) = \varphi(P \times \Sigma_1)$.

Using an argument similar to that used in the proof of Lemma 7.1, there is a strong deformation retraction $h_t$ of $X \times I^\infty$ onto $\varphi(P \times \Sigma_1)$. Thus $f^{-1} \circ h_t \circ f$ gives a strong deformation retraction of $X \times I^\infty$ onto $h'(P \times W_1^-)$. Using the fact that $X \cong X \times I^\infty$ we can easily transfer this information back to $X$.

**Proof of Theorem 7.**

The procedure will be to attach a copy of $I^\infty$ to $X$ so that the resulting space is a compact contractible $Q$-manifold.

Assume that $\dim(P) \leq n$ and consider $P$ as linearly embedded in the $(2n + 1)$-cell $\prod_{i=1}^{n+1}I_i$. Let $f : P \times [1, 2] \times I^\infty \to X$ be an embedding such that \(\text{Bd}(f(P \times [1, 2] \times I^\infty)) = f(P \times \{2\} \times I^\infty),\) where we consider $P \times [1, 2] \subset E^{2n+2}$

($(2n+2)$-dimensional Euclidean space), and for which there is a strong deformation retraction of $X$ onto $f(P \times \{1\} \times I^\infty)$.

Let $X^*$ be the space constructed by attaching $(\prod_{i=1}^{n+2}I_i) \times I^\infty$ to $X$, with the attaching map being $f|P \times \{1\} \times I^\infty$. To show that $X^*$ is a $Q$-manifold all we have to do is check at $f(P \times \{1\} \times I^\infty)$. We know from [15] that the product of any polyhedron with $I^\infty$ gives a $Q$-manifold. Since there is obviously a neighborhood of $f(P \times \{1\} \times I^\infty)$ in $X^*$ which is homeomorphic to $[(\prod_{i=1}^{n+2}I_i) \cup (P \times [1,2])] \times I^\infty$, we conclude that $X^*$ is a compact $Q$-manifold.

To see that $X^*$ is contractible we note that there is a strong deformation retraction of $X^*$ onto the attached copy of $(\prod_{i=1}^{n+2}I_i) \times I^\infty$ in $X^*$. Thus it follows that $X^*$ is contractible, hence $X^* \cong I^\infty$ by Corollary 5. The proof of the theorem is now complete.

**8. Proof of Theorem 8**

We will need the following preliminary result. A proof can easily be constructed using techniques similar to those used to establish Lemma 3.1 of [4]. For this reason we do not give a proof.

**Lemma 8.1.** Let $J^\infty$ be a copy of $I^\infty$. There is a continuous function $g : I^\infty \times [1, \infty) \to I^\infty \times J^\infty$ which satisfies the following properties.
(1) for $n$ an integer and $n \leq u < n+1$, $g_u$ is a homeomorphism of $I^\infty$ onto $(I_1 \times \cdots \times I_n \times [n-u, u-n] \times \{(0, 0, \cdots)\}) \times J^{\infty}$, where $g_u$ is defined by $g_u(x) = g(x, u)$, for all $x \in I^\infty$, and

(2) for $u \in [1, \infty)$ and $n \leq u$ (n an integer),

$$
\pi_n \circ \pi_{I^\infty} \circ g_u(x_i) = (x_1, \cdots, x_n),
$$

for all $(x_i) \in I^\infty$.

We will need one more preliminary result before we establish Theorem 8. We will need a definition first.

Let $G$ be an open subset of $I^\infty$. A continuous function $\varphi : G \to [1, \infty)$ is said to have the local product property with respect to $G$ provided that for each $x \in G$ there is an integer $m(x) \leq \varphi(x)$ such that the following properties are satisfied.

(1) for all $x = (x_i) \in G$, $\{(x_1, \cdots, x_{m(x)})\} \times \prod_{i=m(x)+1}^{\infty} I_i \subset G$

(2) for all $x = (x_i) \in G$ and $(y_{m(x)+1}, y_{m(x)+2}, \cdots) \in 

\prod_{i=m(x)+1}^{\infty} I_i$, $\varphi((x_i)) = \varphi(x_1, \cdots, x_{m(x)}, y_{m(x)+1}, y_{m(x)+2}, \cdots)$, and

(3) $\varphi$ is unbounded near $I^\infty \setminus G$, i.e. for each $x \in \text{Bd}(G)$ and each integer $n > 0$, there is an open set $U$ containing $x$ such that $\varphi(G \cap U) \subset [n, \infty)$.

**Lemma 8.2.** Let $G$ be an open subset of $I^\infty$ and assume that there is a continuous function $\varphi : G \to [1, \infty)$ which has the local product property with respect to $G$. Let $\alpha : E^1 \to E^1$ (where $E^1$ is the real line) be defined by $\alpha(x) = x$, for $x \geq 0$, and $\alpha(x) = 0$, for $x \leq 0$. Then $G \cong G(\varphi) \times J^{\infty}$, where

$$
G(\varphi) = \{(x_i) \in G| |x_i| \leq \alpha(\varphi(x)-(i-1)), \text{ for all } i \geq 1\}.
$$

**Proof.** Let $g : I^\infty \times [1, \infty) \to I^\infty \times J^{\infty}$ be the continuous function of Lemma 8.1. For each $x \in G$ let $h(x) = g(x, \varphi(x))$, which gives a homeomorphism of $G$ onto $G(\varphi) \times J^{\infty}$. The details of the argument are elementary.

**Proof of Theorem 8.**

Using a standard technique (for example see Lemma 6.1 of [6]) there is a countable star-finite collection $\mathcal{U}$ of basic open subsets of $I^\infty$ such that $G = \bigcup \{U|U \in \mathcal{U}\}$ and $\text{Cl}(U) \subset G$, for all $U \in \mathcal{U}$. (An open subset of $I^\infty$ is basic provided that its closure is a basic closed set). It is clear that by subdividing $\{\text{Cl}(U)|U \in \mathcal{U}\}$ we can get a countable star-finite collection
\( \mathcal{F} \) of basic closed subsets of \( I^\infty \) such that (1) \( G = \bigcup \{ F | F \in \mathcal{F} \} \), (2) for each \( F \in \mathcal{F} \), \( \text{Int}(F) \) is a non-null basic open subset of \( I^\infty \), and (3) if \( F_1, F_2 \in \mathcal{F} \) and \( F_1 \neq F_2 \), then \( F_1 \cap F_2 \) lies in an endslice of each.

Without loss of generality we may assume that \( G \) is connected. Thus we can order \( \mathcal{F} \) as \( \{ F_i \}_{i=1}^\infty \) so that

\[
\text{St}(F_1, \mathcal{F}) = F_1 \cup F_2 \cup \cdots \cup F_{i(1)} \\
\text{St}^2(F_1, \mathcal{F}) = F_1 \cup F_2 \cup \cdots \cup F_{i(1)} \cup F_{i(1)+1} \cup \cdots \cup F_{i(2)} \\
\vdots
\]

where \( 1 = i(0) < i(1) < \cdots \) and \( \text{St}^n(F_1, \mathcal{F}) \) has the usual meaning.

For each \( j > 0 \) let \( m(j) \) denote a positive integer such that \( F_j = A_j \times \prod_{i=m(j)+1}^\infty I_i \), where \( A_j \) is a basic closed subset of \( \prod_{i=m(j)+1}^\infty I_i \). By subdividing \( \{ F_i \}_{i=1}^\infty \) sufficiently (if necessary) we can choose \( \{ m(j) \}_{j=1}^\infty \) so that \( m(j) = m(i(k)) + 1 \), for all \( j \) satisfying \( i(k) + 1 \leq j \leq i(k+1) \).

For each \( j > 0 \) let \( R_j = (A_j \times I_{m(j)+1}) \times \{ 0, 0, \cdots \} \).

Then \( \{ R_j \}_{j=1}^\infty \) is a locally-finite collection of finite-dimensional cells in \( G \). It is clear that we can define a piecewise linear function \( \varphi' : \bigcup_{j=1}^\infty R_j \to [1, \infty) \) which satisfies

1. \( \varphi'(x) = m(1) + 2 \), for all \( x \in R_1 \),
2. \( m(1) + j + 1 < \varphi'(x) \leq m(1) + j + 2 \), for all integers \( j \geq 1 \) and \( x \in \left( \bigcup_{i=i(j-1)+1}^{i(j)} R_i \right) \setminus \bigcup_{i=i(j-1)+1}^{i(j)-1} R_i \), and
3. \( \varphi'(x) = m(1) + j + 2 \), for all \( x \in \left( \bigcup_{i=i(j-1)+1}^{i(j)} R_i \right) \cap \left( \bigcup_{i=i(j)+1}^\infty R_i \right) \).

Then extend \( \varphi' \) to a continuous function \( \varphi : G \to [1, \infty) \) by defining \( \varphi((x_i)) = \varphi'(x_1, \cdots, x_{m(j)+1}, 0, 0, \cdots) \), for all \( (x_i) \in F_j \). It is clear that \( \varphi \) has the local product property with respect to \( G \). Using Lemma 8.2 we find that \( G \cong G(\varphi) \times J^{\infty} \). If we can prove that \( G(\varphi) \) can be triangulated by a complex, then we will be done.

We have chosen \( \{ F_i \}_{i=1}^\infty \) so that for the corresponding \( \{ R_i \}_{i=1}^\infty \), \( R_i \cap R_j \) lies in a face of each, for \( i \neq j \). It is obvious that we could have chosen \( \{ F_i \}_{i=1}^\infty \) so that if \( i > j \), then \( R_i \cap R_j \) is exactly a face of \( R_i \). This will aid in an inductive triangulation of \( G(\varphi) \). The details of the triangulation are tedious, but elementary. Accordingly we only sketch the details.

There is obviously a triangulation \( \Delta' \) of \( R_1 \) such that for each \( i \), with \( 1 < i \leq i(1) \), \( R_i \cap R_1 \) is triangulated by a subcomplex of \( \Delta' \). We can extend \( \Delta' \) to a triangulation \( \Delta_1 \) of

\[
B_1 = \{ (x_i) \in F_1 | |x_i| \leq \alpha(\varphi((x_i)) - (i-1)) \}, \text{ for all } i \geq 1
\]

so that for \( 1 < i \leq i(1) \), \( R_i \cap B_1 \) is triangulated by a subcomplex of \( \Delta_1 \).

We have chosen \( \{ R_i \}_{i=1}^\infty \) so that for each \( i > 0 \), \( R_{i+1} \cap (R_1 \cup \cdots \cup R_i) \)
is a union of faces of $R_{i+1}$. Using this fact and an inductive procedure on $\{R_2, \cdots, R_{i(1)}\}$ we can extend $A_1$ to a triangulation $A'_2$ of $B_1 \cup (R_2 \cup \cdots \cup R_{i(1)})$ so that if $i(1) < i \leq i(2)$, then $R_i \cap (B_1 \cup (R_2 \cup \cdots \cup R_{i(1)}))$ is triangulated by a subcomplex of $A'_2$. Put

$$B_2 = \{(x_i) \in F_1 \cup \cdots \cup F_{i(1)} | x_i | \leq \alpha(\varphi((x_i))-(i-1)), \text{ for all } i \geq 1\}$$

and extend $A'_2$ to a triangulation $A_2$ of $B_2$ so that for $i(1) < i \leq i(2)$, $R_i \cap B_2$ is triangulated by a subcomplex of $A_2$. It is clear that we can inductively continue this process to obtain our desired triangulation.

9. Proofs of Theorems 9 and 10

The following lemma is a basic separation result which will be needed in the proofs of Theorems 9 and 10.

**Lemma 9.1.** Let $X$ be a metric ANR, $A$ be a closed subset of $X$ which is an ANR and for which the inclusion map $i : A \to X$ is a homotopy equivalence, and let $h : A \times (-1, 1) \to X$ be an open embedding such that $h(x, 0) = x$, for all $x \in A$. Then we can write $X \setminus A = U \cup V$, where $U$ and $V$ are disjoint open subsets of $X$ satisfying $h(A \times (0, 1)) \subset U$ and $h(A \times (-1, 0)) \subset V$. Moreover, there are strong deformation retractions of $\text{Cl}(U)$ and $\text{Cl}(V)$ onto $A$.

**Proof.** The proof of the existence of disjoint open subsets $U$, $V$ of $X$ satisfying $X \setminus A = U \cup V$, $h(A \times (0, 1)) \subset U$, and $h(A \times (-1, 0)) \subset V$ is straightforward. We merely remark that in the case $A$ is connected the desired separation follows immediately from the reduced Mayer-Vietoris sequence of the excisive couple $\{h(A \times (-1, 1)), X \setminus A\}$. In case $A$ is not connected one can do a standard argument on the components of $A$.

The inclusion map $i : A \to X$ being a homotopy equivalence means that $A$ is a weak deformation retract of $X$. Since $A$ and $X$ are ANR’s it follows that $A$ is a strong deformation retract of $X$ (see [14], page 31). Let $f_t : X \to X$ be a strong deformation retraction of $X$ onto $A$, where $f_0 = \text{id}$ and $f_1$ is a retraction of $X$ onto $A$.

Let $g : X \to X$ be defined by

$$g(x) = \begin{cases} x, & \text{for } x \in \text{Cl}(U) \\ f_t(x), & \text{for } x \in \text{Cl}(V), \end{cases}$$

which is clearly continuous. Define $h_t = g \circ f_t$, for all $r \in [0,1]$. It is clear that $h_t(\text{Cl}(U)) \subset \text{Cl}(U)$, for all $t$. Thus $h_t|\text{Cl}(U)$ defines a strong defor-
We will now give a proof of Theorem 9. For its proof we will use Lemma 9.1 and some of the results that have been established for $Q$-manifolds in this paper. We will not prove Theorem 10, since similar results for $F$-manifolds that have been established elsewhere will permit a proof similar to that given for Theorem 9.

**Proof of Theorem 9.**

Note that $X$ and $Y$ are metric ANR's and the inclusion maps $i : f(X) \rightarrow Y$, $j : g(X) \rightarrow Y$ are obviously homotopy equivalences. Thus we can apply Lemma 9.1 to obtain disjoint pairs $U_1$, $U_2$ and $V_1$, $V_2$ of open subsets of $Y$ such that the following properties are satisfied.

1. $Y \setminus f(X) = U_1 \cup U_2$ and $Y \setminus g(X) = V_1 \cup V_2$,
2. $f(X) = Cl(U_1) \cap Cl(U_2)$ and $g(X) = Cl(V_1) \cap Cl(V_2)$,
3. $f(X)$ is collared in each of $Cl(U_1)$, $Cl(U_2)$, and $g(X)$ is collared in each of $Cl(V_1)$, $Cl(V_2)$,
4. $f(X)$ is a strong deformation retract of each of $Cl(U_1)$, $Cl(U_2)$, and $g(X)$ is a strong deformation retract of each of $Cl(V_1)$, $Cl(V_2)$.

From (3) it easily follows that $Cl(U_1)$ and $Cl(V_1)$ are $Q$-manifolds. Let $r : Cl(U_1) \rightarrow f(X)$ be a retraction homotopic to id and note that the map $g \circ f^{-1} \circ r : Cl(U_1) \rightarrow Cl(V_1)$ is a homotopy equivalence. Using Theorem 6 we know that $(g \circ f^{-1} \circ r) \times id : Cl(U_1) \times [0, 1) \rightarrow Cl(V_1) \times [0, 1)$ is homotopic to a homeomorphism $h_1 : Cl(U_1) \times [0, 1) \rightarrow Cl(V_1) \times [0, 1)$.

Now $g \times id : X \times [0, 1) \rightarrow Cl(V_1) \times [0, 1)$ and $h_1 \circ (f \times id) : X \times [0, 1) \rightarrow Cl(V_1) \times [0, 1)$ are homotopic embeddings. It is easy to see that $(g \times id)(X \times [0, 1))$ and $h_1 \circ (f \times id)(X \times [0, 1))$ are $Z$-sets in $Cl(V_1) \times [0, 1)$. Using Corollary 6.1 of [3] there is a homeomorphism

$$h_2 : Cl(V_1) \times [0, 1) \rightarrow Cl(V_1) \times [0, 1)$$

which satisfies $h_2 \circ h_1 \circ (f \times id) = g \times id$. Put $h' = h_2 \circ h_1$, which is a homeomorphism of $Cl(U_1) \times [0, 1)$ onto $Cl(V_1) \times [0, 1)$ which satisfies $h' \circ (f \times id) = g \times id$.

Similarly we can obtain a homeomorphism

$$h'' : Cl(U_2) \times [0, 1) \rightarrow Cl(V_2) \times [0, 1)$$

which satisfies $h'' \circ (f \times id) = g \times id$. Then define $h : Y \times [0, 1) \rightarrow Y \times [0, 1)$ by $h|Cl(U_1) \times [0, 1) = h'$ and $h|Cl(U_2) \times [0, 1) = h''$. 

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