P. J. Eenigenburg

A class of starlike mappings of the unit disk

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DEFINITION. Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) be univalent in the open unit disk \( D \). We say \( f \in S_\alpha(0 < \alpha \leq 1) \) if

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha, \ z \in D
\]

Note that if \( f \in S_\alpha \) then \( \Re(zf'(z)/f(z)) > 0 \) on \( D \); hence \( f \) is a starlike function. Singh [4] and Wright [5] have derived certain properties of the class \( S_\alpha \). In this paper we extend their results as follows. First, the boundary behavior of \( f \in S_\alpha \) is discussed. We then give the radius of convexity for the class \( S_\alpha \); for \( \alpha = 1 \) the radius has been given by Wright [5]. Finally, we give an invariance property for the class \( S_\alpha \).

**Theorem 1.** Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\alpha \). Then \( f \) maps \( D \) onto a domain whose boundary is a rectifiable Jordan curve. Furthermore, \( a_n = o(1/n) \), and this order is best possible.

**Proof.** It follows immediately from (1) that \( f' \) is bounded in \( D \); hence \( \partial f[D] \) is a rectifiable closed curve [3], and \( a_n = o(1/n) \). Univalency of \( f \) on \( \overline{D} \) is easily verified by a contradiction argument. Finally, let \( \{k(n)\} \) be any sequence of positive numbers which converges to 0 as \( n \to \infty \). Then there exists a subsequence \( \{k(n_j)\}, n_1 \geq 2 \), such that \( \sum_{j=1}^{\infty} k(n_j) \leq \alpha \). Define

\[
a_n = \begin{cases} k(n_j) & n = n_j, j = 1, 2, \ldots \vphantom{\sum_{n=2}^{\infty}} \\ n_j & \text{otherwise} \end{cases}
\]

Then,

\[
\sum_{n=2}^{\infty} (n+\alpha-1)|a_n| \leq \sum_{n=2}^{\infty} n|a_n| = \sum_{j=1}^{\infty} n_j \frac{k(n_j)}{n_j} \leq \alpha.
\]

By a theorem of Merkes, Scott, and Robertson [1], \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_\alpha \). Since \( na_n = k(n) \) for infinitely many \( n \), the proof is complete.
THEOREM 2. If $f \in S_\alpha$ then $f$ maps $|z| < r(\alpha)$ onto a convex domain, where

$$ r(\alpha) = \begin{cases} \frac{3 - \sqrt{5}}{2\alpha} & \text{if } \alpha_0 \leq \alpha \leq 1 \\ \frac{2(1+\alpha^2) - 3\alpha - 2(1-\alpha)\sqrt{\alpha^2 + 4\alpha + 1}}{\alpha(4\alpha - 5)} & \text{if } 0 < \alpha \leq \alpha_0 \end{cases} $$

and

$$ \alpha_0 = \frac{3 - \sqrt{5} + 2\sqrt{3(7 - 3\sqrt{5})}}{2\sqrt{5}} \approx 0.589 $$

PROOF. Since $f \in S_\alpha$ there exists $\phi, |\phi| \leq \alpha$ in $D$, such that $zf'(z)/f(z) = 1 + z\phi(z)$. Differentiation yields

$$ 1 + \frac{zf''(z)}{f'(z)} = 1 + z\phi(z) + z \left( \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right) $$

It is known [2] that

$$ \left| \frac{z\phi'(z) + \phi(z)}{1 + z\phi(z)} \right| \leq \frac{(|z| + |\phi(z)|)(|z| - |z\phi(z)|)}{(1 - |z\phi(z)|)(1 - |z|^2)} $$

From (4) and (5) it follows that $Re(1 + zf''(z)/f'(z)) \geq 0$ provided

$$ 1 - |z\phi(z)| - |z| \geq 0 $$

We write $|z| = \alpha, |\phi(z)| = x, t = ax$ and define $G(t) \equiv t^2(1 - a^2 + 1/\alpha) - 3t(1-a^2) + 1-a^2 - a^2\alpha$. Since (6) holds if and only if $G(t) \geq 0$, we must determine the largest value of $a$ for which $G(t) \geq 0$ on $[0, ax]$. Then $f$ will map $|z| < \alpha$ onto a convex domain. Note that $G(t)$ has its minimum where $t = t^* \equiv \frac{1}{2}(1-a^2)(1-a^2+1/\alpha)^{-1}$.

CASE A ($ax \leq t^*$). $G(ax) = (1-a^2)(a^2\alpha^2 - 3a\alpha + 1) \geq 0$ provided $a \leq (3 - \sqrt{5})/2\alpha$. Since $G(t)$ is decreasing on $[0, ax], f$ maps $|z| < (3 - \sqrt{5})/2\alpha$ onto a convex domain provided $ax = (3 - \sqrt{5})/2 \leq t^*$. This restraint requires that $x \in [x_0, 1]$ where $x_0$ is given by (3). The function $f_\alpha(z) = ze^{\alpha z}, \alpha \leq \alpha \leq 1$, shows that the number $r(\alpha) = (3 - \sqrt{5})/2\alpha$ is sharp.

CASE B ($t^* \leq ax$). We assume $0 < \alpha < \alpha_0$. The minimum value of $G(t)$ on $[0, ax]$ is $G(t^*)$; and $G(t^*) \geq 0$ if

$$ a^4(-5 + 4x) + a^2 \left( 6 - \frac{4}{\alpha} - 4x \right) - 5 + \frac{4}{\alpha} \geq 0. $$
Since (7) holds for $0 \leq a \leq r(\alpha)$, where $r(\alpha)$ is given by (2), it follows that $f$ maps $|z| < r(\alpha)$ onto a convex domain if $r^* \leq \alpha r(\alpha)$. A tedious calculation shows this restraint to be satisfied for $0 < \alpha < \alpha_0$. We now construct a function to show that $r(\alpha)$ is best possible. Fix $\alpha$ in $(0, \alpha_0)$ and let $a = r(\alpha)$. Set $\beta \equiv \left[2 - 3\alpha - a\alpha^2(3 - 2\alpha)\right][2a(\alpha - 1)^2]^{-1}$. Define $f$ by $f(z) = z \exp \left[\alpha \int_0^z (\beta - t)/(1 - \beta t) \, dt\right]$. By a theorem of Wright [5], $f \in S_\alpha$ provided $-1 \leq \beta \leq 1$. For the present suppose this has been done. Now, $f$ will not be convex in $|z| < r$, $r > a$, if $1 + zf''(z)/f'(z) = 0$ at $z = a$, or, equivalently, if $\beta$ is a root of $P(s)$, where

$$P(s) = s^2[a^2(\alpha^2 - 1)^2] + s[a(3\alpha^2 - 2) - \alpha a^3(3 - 2\alpha)] + 1 - 4\alpha a^2 + \alpha^2 a^4.$$ 

Since $a = r(\alpha)$ is a root of the left-hand side of (7), the definition of $\beta$ implies that $P(\beta) = 0$. In fact $P(s)$ has a double root at $s = \beta$. It follows that $\beta^2 \leq 1$ provided

$$a^2 \geq \frac{(1 + \alpha)^2 - \sqrt{(1 + \alpha)^4 - 4\alpha^2}}{2\alpha^2}.$$ 

Now, $a = r(\alpha)$ is the only root of the left-hand side of (7) which lies in $[0, 1]$. Thus, (8) holds since substitution of its right-hand side for $a^2$ in the left-hand side of (7) preserves the inequality in (7). Hence, $-1 \leq \beta \leq 1$, and the proof is complete.

**Theorem 3.** If $f(z) = z + \sum_{n=2}^{\infty} \alpha_n z^n \in S_\alpha$ then for each $\lambda$, $0 < \lambda < 1$, $h_\lambda(z) = z + \sum_{n=2}^{\infty} \lambda \alpha_n z^n \in S_\alpha$.

**Proof** Since $h_\lambda(z) = \lambda f(z) + (1 - \lambda)z$, we have

$$\frac{zh_\lambda'(z)}{h_\lambda(z)} - 1 = \left[zf'(z) - 1\right]\left[1 + \frac{1 - \lambda}{\lambda} \frac{z}{f(z)}\right]^{-1}.$$

By a theorem of Wright [5], there exists $\phi$, $|\phi| \leq 1$ in $D$, such that $f(z) = z \exp[\alpha \int_0^z \phi(t) \, dt]$. Thus $\text{Re}(f(z)/z) > 0$ and so $|1 + z(1 - \lambda)(\lambda f(z))^{-1}| > 1$ for $z \in D$. It follows from (9) that $h_\lambda \in S_\alpha$.

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(Oblatum 14-V-1971)

Department of Mathematics
Western Michigan University
Kalamazoo, Michigan 49001
U.S.A.