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## SMOOTHNESS AND REGULARITY

by

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Some applications of the cotangent complex to smoothness and regularity are given; in particular, the proof of a criterion for formal smoothness which was conjectured in [8] (see 2.1), and some generalisations of this criterion for the non-noetherian and noetherian cases (2.2, 2.6, 2.7). Also considered is the descent of formal smoothness.

0. All the rings considered are commutative with unity; the topologies are linear. The definitions and notations used are as in EGA,  $\mathbf{O}_{IV}$ , §§ 19-20, and [1]. The following facts about the cotangent complex will be needed:

Let  $A \rightarrow B$  be a morphism of rings; to any  $B$ -module  $M$  are associated the  $B$ -modules  $H_i(A, B, M)$ ,  $H^i(A, B, M)$ . (For the definitions, see: [8] for  $i = 0, 1$ ; [9] for  $i = 0, 1, 2$ ; [1] or [14] for  $i \geq 0$ . At least for  $i = 0, 1$ , the various definitions give isomorphic modules. This follows from the properties 0.1-0.4 below ([6], 3.5).)  $\{H_i(\text{resp. } H^i), i \geq 0\}$  is a (co)homological functor and has the properties.

0.1.  $H_0(A, B, M) = \Omega_{B/A} \otimes_B M$  (where  $\Omega_{B/A}$  is the module of  $A$ -differentials in  $B$ );  $H^0(A, B, M) = \text{Der}_A(B, M)$  (= the module of  $A$ -derivations of  $B$  in  $M$ ; see [9], 2.3).

0.2. If  $A \rightarrow B$  is surjective with kernel  $\mathfrak{b}$ , then  $H_1(A, B, M) = \mathfrak{b}/\mathfrak{b}^2 \otimes_B M$  and  $H^1(A, B, M) = \text{Hom}_B(\mathfrak{b}/\mathfrak{b}^2, M)$  ([9], 3.1.2).

0.3. If  $B$  is a polynomialring over  $A$ , then  $H_i(A, B, \cdot) = 0 = H^i(A, B, \cdot)$  for  $i \geq 1$  ([9], 3.1.1 or [1], 16.3).

0.4. If  $A \rightarrow B \rightarrow C$  are morphisms of rings and  $M$  is a  $C$ -module, then the sequence

$$\begin{aligned} \cdots \rightarrow H_i(A, B, M) \rightarrow H_i(A, C, M) \rightarrow H_i(B, C, M) \rightarrow H_{i-1}(A, B, M) \\ \rightarrow \cdots \rightarrow H_0(B, C, M) \rightarrow 0 \end{aligned}$$

is exact ([9], 2.3.5 or [1], 18.2), and similarly for  $H^i$ , with arrows reversed.

0.5. If  $B$  is an  $A$ -algebra and  $S$  a multiplicatively closed system in  $B$ , then the canonical morphism  $H_i(A, B, M) \otimes_B S^{-1}B \xrightarrow{\sim} H_i(A, S^{-1}B, S^{-1}M)$  is an isomorphism ([9], 2.3.4. or [1], 16).

0.6. If  $A \rightarrow B \rightarrow C$  are morphisms of rings and  $M$  a flat  $C$ -module, then the canonical morphism  $H_i(A, B, C) \otimes_C M \xrightarrow{\sim} H_i(A, B, M)$  is an isomorphism.

0.7. Let  $A' \xleftarrow{v} A \xrightarrow{u} B$  be morphisms of rings, where  $u$  or  $v$  is flat,  $B' = A' \otimes_A B$ , and  $M$  a  $B'$ -module. Then the canonical morphism  $H_i(A, B, M) \xrightarrow{\sim} H_i(A', B', M)$  is an isomorphism ([9], 2.3.2. or [1], 19.2).

0.8. If  $A \rightarrow B$  are fields, then  $H_i(A, B, \cdot) = 0 = H^i(A, B, \cdot)$  for  $i \geq 2$  ([9], 3.5 or [1], 22.2) and  $A \rightarrow B$  is separable iff it is formally smooth (EGA,  $O_{IV}$ , 19.6.1 or 9, 3.5).

0.9. Let  $A$  be a local, noetherian ring,  $B$  its residue class field and  $K$  a field which is an extension of  $B$ . Then the following three statements are equivalent:  $A$  is regular:  $H_2(A, K, K)$  is zero;  $H_i(A, K, K) = 0$  for  $i \geq 2$ . This follows from [9], 3.2.1 or [1], [1], 27.1 and 27.2, using 0.6 and 0.8.

The following criteria (0.10 and 1.1) are essentially the discrete and non-discrete forms of the Jacobian criterion of smoothness (EGA,  $O_{IV}$ , 22.6.1 and 22.6.2).

0.10. A morphism of rings  $A \rightarrow B$  is formally smooth in the discrete topologies if and only if  $\Omega_{B/A}$  is a projective  $B$ -module and  $H_1(A, B, B) = 0$  ([8], 9.5.7 or [9], 3.1.3).

1.1. Let  $A \rightarrow B$  be a morphism of topological rings: the topology of  $B$  is  $\mathfrak{c}$ -adic for some ideal  $\mathfrak{c}$  in  $B$ , and that of  $A$  is also adic. Let  $C = B/\mathfrak{c}$ . If  $A \rightarrow B$  is formally smooth then  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module and  $H_1(A, B, C) = 0$ : the converse is also true if  $B$  is a noetherian ring.

1.2. For  $A$  and  $B$  noetherian, 1.1 is proved in [2], 5.4. The proof of 1.1 in general is based on the same ideas. Let  $A \rightarrow B$  be formally smooth. Then  $\Omega_{B/A}$  is a formally projective  $B$ -module (EGA,  $O_{IV}$ , 20.4.9), hence  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module (II, IX, 1.25). Let  $R$  be a polynomialring over  $A$ , and  $R \rightarrow B$  a surjection of  $A$ -algebras with kernel  $\mathfrak{b}$ . Then the following sequence is exact (0.4 and 0.1–0.3).

$$(1.2.1) \quad 0 \rightarrow H_1(A, B, C) \rightarrow \begin{matrix} b/b^2 \\ B \end{matrix} \otimes_C \begin{matrix} C \\ \delta_{B/R/A} \otimes C \end{matrix} \longrightarrow \begin{matrix} \Omega_{R/A} \\ \otimes C \end{matrix} \rightarrow \begin{matrix} \Omega_{B/A} \\ \otimes C \end{matrix} \rightarrow 0.$$

But  $\delta_{B/R/A}$  is formally left invertible (EGA,  $O_{IV}$ , 20.7.8 and 19.4.4), so  $\delta_{B/R/A} \otimes_B C$  is injective, and  $H_1(A, B, C) = 0$ .

Now, let  $B$  be noetherian,  $\Omega_{B/A} \otimes_B C$  be a projective  $C$ -module, and  $H_1(A, B, C) = 0$ . Let  $R$  and  $\mathfrak{b}$  be as above. Then ([9], 3.1.2) and (1.2.1) imply that  $H^1(A, B, M) = 0$  for any  $C$ -module  $M$ , and hence for any discrete  $B$ -module  $M$  with open annihilator. Let  $A_d, B_d$  denote the rings  $A, B$  with the discrete topology, and  $A_t, B_t$  the rings  $A, B$  with the given topologies. Then  $H^1(A, B, M) = 0$  means  $H^1(A_d, B_d, M) = 0$ .  $A_d \rightarrow B_d \rightarrow B_t$  and  $M$  give the exact sequence ([2], 2 or EGA, O<sub>IV</sub>, 20.3.7)

$$\begin{aligned} 0 \rightarrow H_t^0(A_d, B_t, M) \xrightarrow{u} H_t^0(A_d, B_d, M) \rightarrow H_t^1(B_d, B_t, M) \xrightarrow{v} \\ H_t^1(A_d, B_t, M) \rightarrow H^1(A_d, B_d, M) = 0 \end{aligned}$$

But  $u$  is an isomorphism (EGA, O<sub>IV</sub>, 20.3.3), so  $v$  also is.  $B$  is noetherian, hence  $H_t^1(B_d, B_t, M) = 0$  ([2], 5.1): it follows that  $H_t^1(A_d, B_t, M) = 0$ . Then  $A_d \rightarrow A_t \rightarrow B_t$  and  $M$  give the exact sequence

$$H_t^0(A_d, A_t, M) \rightarrow H_t^1(A_t, B_t, M) \rightarrow H_t^1(A_d, B_t, M),$$

where the first and third terms are zero. The formal smoothness of  $A_t \rightarrow B_t$  now follows (EGA, O<sub>IV</sub>, 19.4.4).

1.3. REMARKS. (i) I do not know if the converse part of 1.1 remains valid for  $B$  a non-noetherian ring.

(ii) The criterion 1.1 can be reformulated as follows: if  $B$  is noetherian, then  $A \rightarrow B$  is formally smooth iff  $\Omega_{B/A}$  is a formally projective  $B$ -module and  $H_1(A, B, C) = 0$ . The results of N. Radu ([11], [12], [13]) shows that the condition  $H_1(A, B, C) = 0$  is superfluous in this form of 1.1 if  $B$  is a laskerian local ring with the  $\mathfrak{m}$ -adic topology ( $\mathfrak{m}$  the maximal ideal of  $B$ ), and  $A$  is a field of characteristic zero (or arbitrary characteristic if  $B$  is noetherian).

2.0. It is known that, if  $A \rightarrow B$  is a local morphism of local noetherian rings, formally smooth in the topologies given by the maximal ideals, then  $A$  is regular if and only if  $B$  is regular.

PROOF. Let  $K$  be the residue class field of  $B$ , then the maps  $A \rightarrow B \rightarrow K$  give the exact sequence (0.4):

$$H_2(A, B, K) \rightarrow H_2(A, K, K) \rightarrow H_2(B, K, K) \rightarrow H_1(A, B, K).$$

But  $H_1(A, B, K) = 0$  (1.1) and  $H_2(A, B, K) = 0$  ([4], corollary). Now apply 0.9.

2.1. In ([8], 9.6), the following criterion for formal smoothness is stated and partially proved:

**THEOREM.** *Let  $A \rightarrow B \rightarrow C$  be local homomorphisms of local noetherian rings,  $A$  and  $C$  regular,  $B \rightarrow C$  surjective, so that  $C = B/\mathfrak{c}$ ,  $\mathfrak{c}$  an ideal of  $B$ . Finally, let  $B$  be a localisation of a finitely generated  $A$ -algebra. Then  $B$  is*

formally smooth over  $A$  if and only if the following three conditions are satisfied:

- a)  $B$  is regular, i.e.  $\mathfrak{c}$  is a regular ideal;
- b)  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module;
- c) The echaracteristic homomorphism

$$N_{C/A} \rightarrow \mathfrak{c}/\mathfrak{c}^2$$

is injective.

(The morphism from c) is  $H_1(A, C, C) \rightarrow H_1(B, C, C)$  (0.2)

In [8], it was proved that the conditions are necessary. The sufficiency, however, was only proved for  $A$  a field and the sufficiency in general conjectured. I gave generalisations of this criterion for the non-noetherian and noetherian case ([5]). For the noetherian case I use essentially (1.1), but this does not work in the non-noetherian case. A direct proof for 2.1 is as follows:

Let  $A \rightarrow B$  be formally smooth (here the topology is arbitrary, cf. EGA, O<sub>IV</sub>, 22.6.4). Then  $B$  is regular (by 2.0),  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module and  $H_1(A, B, C) = 0$  (by 0.10). From the exact sequence (0.4),

$$H_2(B, C, C) \rightarrow H_1(A, B, C) \rightarrow H_1(A, C, C) \xrightarrow{f} H_1(B, C, C) = \mathfrak{c}/\mathfrak{c}^2,$$

it results that  $f$  is injective.

Let a), b), c) be satisfied. Then  $H_2(B, C, C) = 0$  ([9], 3.2.1) since  $\mathfrak{c}$  is generated by a  $B$ -regular sequence. Hence, by the above sequence and c),  $H_1(A, B, C) = 0$ ; this and b) imply that  $B$  is a formally smooth  $A$ -algebra for the  $\mathfrak{c}$ -adic topology (1.1).

We now turn to the non-noetherian case. Let  $A \rightarrow A' \xrightarrow{u} B \xrightarrow{v} C$  be morphisms of rings,  $u$  and  $v$  surjective,  $\mathfrak{b} = \text{Ker } u$ ,  $\mathfrak{c} = \text{Ker } v$ .

**2.2. THEOREM.** *Suppose that  $A'$  is a formally smooth  $A$ -algebra (in the  $\mathfrak{b}$ -adic topology) and that  $\mathfrak{b}/\mathfrak{b}^2$  is  $\mathfrak{c}$ -separated (or is a  $B$ -module of finite type with  $B$  local). Then  $A \rightarrow B$  is formally smooth (in the discrete topologies) if and only if  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module and  $H_1(A, B, C) = 0$ .*

**PROOF.** The necessity results from (0.10).

For the converse two facts are necessary.

(2.2.1) *If  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module and  $H_1(A, B, C) = 0$  then*

$$\delta_{B/A/A} \otimes_B C : \mathfrak{b}/\mathfrak{b}^2 \otimes_B C \rightarrow \Omega_{A'/A} \otimes_{A'} C$$

is left invertible. For  $A \rightarrow A' \rightarrow B$  and  $C$  give the exact sequence

$$0 = H_1(A, B, C) \rightarrow H_1(A', B, C) \xrightarrow{\delta_{B/A'/A} \otimes_B C} H_0(A, A', C) \rightarrow H_0(A, B, C) \rightarrow 0,$$

by 0.4 and 0.1, 0.2. Then 2.2.1 results from the projectivity of  $\Omega_{B/A} \otimes_B C = H_0(A, B, C)$ .

(2.2.2) *Let  $h : M \rightarrow N$  be a morphism of  $B$ -modules,  $\mathfrak{c}$  an ideal of  $B$ ,  $M/\mathfrak{c}M$   $\mathfrak{c}$ -separated  $B$ -module, and  $N$  a projective  $B$ -module. If  $h_1 = h \otimes_B B/\mathfrak{c} : M/\mathfrak{c}M \rightarrow N/\mathfrak{c}N$  is left invertible, then  $h$  is also left invertible.* To see this, let  $g' : N/\mathfrak{c}N \rightarrow M/\mathfrak{c}M$  be a left inverse for  $h_1$ . Since  $N$  is projective, there is a morphism  $g : N \rightarrow M$ , such that the composition  $N \xrightarrow{g} M \rightarrow M/\mathfrak{c}M$  equals  $N \rightarrow N/\mathfrak{c}N \xrightarrow{g'} M/\mathfrak{c}M$ . It is obvious that  $g_1 = g'$ , where the subscript 1 means  $\otimes_B B/\mathfrak{c}$ , i.e. that  $(gh)_1 = 1$ . It follows that  $(gh)_r = gh \otimes_B B/\mathfrak{c}^r$  is equal to 1 (see the proof of IL, XII, 2, 2, 1). Let  $x \in M$ ; then  $x - (gh)(x) \in \mathfrak{c}^r M$  for any  $r \geq 1$ , and so  $gh = 1$ .

Now let  $\Omega_{B/A} \otimes_B C$  be a projective  $B$ -module and  $H_1(A, B, C) = 0$ . Then  $\delta_{B/A'/A} \otimes_B C$  is left invertible (2.2.1); hence  $\delta_{B/A'/A}$  is also left invertible (for  $\Omega_{A'/A} \otimes_A B$  is a projective  $B$ -module by 0.10. Now, if  $\mathfrak{b}/\mathfrak{b}^2$  is  $\mathfrak{c}$ -separated, apply 2.2.2; otherwise apply EGA, O<sub>IV</sub>, 19.1.12). Hence  $B$  is a formally smooth  $A$ -algebra (EGA, O<sub>IV</sub>, 20.5.12).

2.2.3. REMARKS. (i) The hypotheses of (2.2) are fulfilled if  $B$  is an  $A$ -algebra essentially of finite presentation and  $C$  any quotient ring of  $B$ .

(ii) Under the hypotheses of 2.2, assume  $H_2(B, C, C) = 0$  (conditions for this are given in [3], 5.1, or [9], 3.2.1); then  $A \rightarrow B$  is formally smooth if and only if  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module and  $H_1(A, B, C) \rightarrow \mathfrak{c}/\mathfrak{c}^2$  is injective. (This shows that for the ‘only if’ part of 2.1, sufficient hypotheses on  $A \rightarrow B$  are that  $B$  be noetherian and an  $A$ -algebra essentially of finite presentation.)

2.3 COROLLARY. *Let  $Z \xrightarrow{i} Y \xrightarrow{h} X$  be morphisms of schemes,  $i$  being an immersion and  $h$  being locally of finite presentation. Then  $h$  is smooth in a neighbourhood of  $Z$  in  $Y$  if and only if  $H_0(X, Y, \mathcal{O}_Z)$  is a flat  $\mathcal{O}_Z$ -Module and  $H_1(X, Y, \mathcal{O}_Z) = 0$ .*

PROOF. Let  $z \in Z$ ,  $y = i(z)$  and  $x = h(y)$ . Then

$$H_i(X, Y, \mathcal{O}_Z)_z = H_i(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}, \mathcal{O}_{Z,z}), \text{ by 0.5}$$

Since  $H_0(\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}, \mathcal{O}_{Z,z})$  is an  $\mathcal{O}_{Z,z}$ -module of finite presentation, flat means projective. Then  $h_y : \mathcal{O}_{X,x} \rightarrow \mathcal{O}_{Y,y}$  is smooth for any  $z \in Z$  (cf. 2.2.3 and 2.2). But this is an open property (EGA, IV, 17.5). Hence  $h$  is locally smooth, i.e. it is smooth ([8] 9.5.6).

2.4. For the non-discrete topologies, 2.2 takes the following form:

PROPOSITION. *Let  $\mathfrak{a}$  be an ideal of  $A$  s.t.  $\mathfrak{m} = u(\mathfrak{a}) \supset \mathfrak{c}$ . Suppose that  $A \rightarrow A'$  is formally smooth in the  $\mathfrak{a}$ -adic topology and that the topology of  $\mathfrak{b}/\mathfrak{b}^2$  induced by  $\mathfrak{b}$  is  $\mathfrak{m}$ -adic. If  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module and*

$H_1(A, B, C) = 0$ , then  $A \rightarrow B$  is formally smooth in the  $\mathfrak{m}$ -adic topology; hence  $\Omega_{B/A}$  is a formally projective  $B$ -module.

PROOF. From 2.2.1 it results that  $\delta_{B/A'/A} \otimes_B C$  (and hence also  $\delta_{B/A'/A} \otimes_B K$ , where  $K = B/\mathfrak{m}$ ) is left invertible.  $\Omega_{A'/A}$  is a formally projective  $A$ -module in the  $\mathfrak{a}$ -adic topology (0.10), so  $\Omega_{A'/A} \otimes_A B$  is a formally projective  $B$ -module for the  $\mathfrak{m}$ -adic topology; hence  $\delta_{B/A'/A}$  is formally left invertible (EGA,  $O_{IV}$ , 19.1.9). Now the jacobian criterion of smoothness (EGA,  $O_{IV}$ , 22, 5, 1) says that  $A \rightarrow B$  is formally smooth in the  $\mathfrak{m}$ -adic topology.

Generalisations for the noetherian case. Let  $A \xrightarrow{u} B \xrightarrow{v} C$  be morphisms of rings,  $B$  and  $C$  noetherian.

2.5. PROPOSITION. Let  $\mathfrak{c} = \text{Ker } v$ ,  $\mathfrak{d} \supset \mathfrak{c}$  an ideal of  $B$ .  $D' = B/\mathfrak{d}$ , and  $D$  a  $(C/\mathfrak{d}C)$ -algebra. Suppose also that the topology of  $B$  is  $\mathfrak{d}$ -adic, that of  $A$  is  $(\mathfrak{d} \cap A)$ -adic and that of  $C$  is  $\mathfrak{d}C$ -adic.

i) If  $\mathfrak{d} \subset R(B)$  (= the Jacobson radical of  $B$ ),  $A$  is regular, and  $A \rightarrow B$  is formally smooth, then  $B$  is regular,  $\Omega_{B/A} \otimes_B D$  is a projective  $D$ -module and  $f: H_1(A, C, D) \rightarrow H_1(B, C, D)$  is injective.

ii) Let  $D' \rightarrow D$  be faithfully flat. Let  $H_2(B, C, D) = 0$  (e.g. if  $B \rightarrow C$  is a Koszul morphism ([9], 3.2.2); in particular, if  $v$  is surjective and  $\mathfrak{c}$  generated by a regular sequence ([9], 3.2.1)). If  $\Omega_{B/A} \otimes_B D$  is a projective  $D$ -module and  $f$  is injective, then  $A \rightarrow B$  is formally smooth.

iii) Let  $D' \rightarrow D$  be faithfully flat,  $H_2(B, C, D) = 0$  and  $A \rightarrow C$  formally smooth. If  $\mathfrak{d}$  is maximal or  $B \rightarrow C$  is formally étale, then  $A \rightarrow B$  is formally smooth.

PROOF.  $A \rightarrow B \rightarrow C$  and  $D$  give the exact sequence

$$(2.5.1) \quad H_2(B, C, D) \rightarrow H_1(A, B, D) \rightarrow H_1(A, C, D) \xrightarrow{f} H_1(B, C, D) \rightarrow \Omega_{B/A} \otimes_B D \rightarrow \Omega_{C/A} \otimes_C D \rightarrow \Omega_{C/B} \otimes_C D \rightarrow 0$$

i) From 1.1, it results that  $\Omega_{B/A} \otimes_B D'$  is a projective  $D'$ -module and  $H_1(A, B, D') = 0$ . Then  $\Omega_{B/A} \otimes_B D$  is a projective  $D$ -module and  $H_1(A, B, D) = 0$ . Indeed, let  $R$  be a ring of polynomials over  $A$ , and  $R \rightarrow B$  a surjection of  $A$ -algebras with kernel  $\mathfrak{b}$ . Then  $A \rightarrow R \rightarrow B$  and  $D', D$  give the exact sequence (0.4, 0.1–0.3).

$$0 \rightarrow H_1(A, B, D') \rightarrow b/b^2 \otimes_B D' \rightarrow \Omega_{R/A} \otimes_R D' \rightarrow \Omega_{B/A} \otimes_B D' \rightarrow 0$$

$$0 \rightarrow H_1(A, B, D) \rightarrow b/b^2 \otimes_B D \rightarrow \Omega_{R/A} \otimes_R D \rightarrow \Omega_{B/A} \otimes_B D \rightarrow 0$$

since  $H_1(A, B, D') = 0$  and  $\Omega_{B/A} \otimes_B D'$  is projective, it follows that  $H_1(A, B, D) = 0$ . It follows immediately that  $f$  is injective (2.5.1).

Let  $\mathfrak{d} \subset R(B)$ . If  $\mathfrak{m}$  is a maximal ideal of  $B$  and  $\mathfrak{n} = A \cap \mathfrak{m}$  then

$A_n \rightarrow B_m$  is formally smooth in the radicial topologies. Since  $A_n$  is regular, it follows from 2.0 that  $B_m$  also is.

ii) We have that  $H_1(A, B, D) = H_1(A, B, D') \otimes_{D'} D$ , and  $\Omega_{B/A} \otimes_B D = (\Omega_{B/A} \otimes_B D') \otimes_{D'} D$ . (0.6)

From the fact that  $f$  is injective and  $H_2(B, C, D) = 0$ , it results that  $H_1(A, B, D) = 0$ . Hence  $H_1(A, B, D') = 0$ . Since  $D' \rightarrow D$  is faithfully flat and  $\Omega_{B/A} \otimes_B D$  is  $D$ -projective,  $\Omega_{B/A} \otimes_B D'$  is a projective  $D'$ -module ([15]). Hence  $A \rightarrow B$  is formally smooth (1.1).

iii) From the formal smoothness of  $A \rightarrow C$ , it results that  $\Omega_{C/A} \otimes_C D$  is a projective  $D$ -module and  $H_1(A, C, D) = 0$  (1.1). Since  $H_2(B, C, D) = 0$ , we find from 2.5.1 that  $H_1(A, B, D) = 0$ . But  $H_1(A, B, D) = H_1(A, B, D') \otimes_{D'} D$ , hence  $H_1(A, B, D') = 0$ .

Let  $\mathfrak{d}$  be maximal, i.e.  $D'$  a field; then  $A \rightarrow B$  is formally smooth, because of 1.1.

Let  $B \rightarrow C$  be formally étale; then  $H_1(B, C, D) = 0$  (1.1) and  $\hat{\Omega}_{C/B} = 0$  (and also  $H_2(B, C, D) = 0$ , if  $B, C$  are local ([4])). Hence  $\Omega_{C/B} \otimes_B D = 0$ . Now from 2.5.1 it results that  $\Omega_{B/A} \otimes_B D = \Omega_{C/A} \otimes_C D$ ; hence  $\Omega_{B/A} \otimes_B D'$  is a projective  $D'$ -module. Consequently,  $A \rightarrow B$  is formally smooth (1.1). In particular, we obtain:

2.6. COROLLARY. *Let  $A, B, C, u$  and  $v$  be local and  $L$  be the residue class field of  $C$ ; the topologies are adic and given by the maximal ideals.*

i) *If  $A$  is regular and  $A \rightarrow B$  formally smooth, then  $B$  is regular and  $f: H_1(A, C, L) \rightarrow H_1(B, C, L)$  is injective.*

ii) *If  $H_2(B, C, L) = 0$  (e.g. if  $B \rightarrow C$  is Koszul, or if  $B$  is regular and  $H_3(C, L, L) = 0$  (use 0.4 and 0.9). This last occurs, for instance, if  $C$  is regular, by 0.9), and if  $f$  is injective, then  $A \rightarrow B$  is formally smooth.*

iii) *If  $H_2(B, C, L) = 0$  and  $A \rightarrow C$  is formally smooth, then  $A \rightarrow B$  is formally smooth.*

2.7. COROLLARY. *Let  $A, B, C, u$  and  $v$  be local,  $B \rightarrow C$  surjective with kernel  $\mathfrak{c}$ ,  $\mathfrak{d} \supset \mathfrak{c}$  an ideal of  $B$  and  $D = B/\mathfrak{d}$ . The topology of  $B$  is  $\mathfrak{d}$ -adic, that of  $A$  is  $(\mathfrak{d} \cap A)$ -adic.*

i) *If  $A$  is regular and  $A \rightarrow B$  is formally smooth, then  $B$  is regular,  $\Omega_{B/A} \otimes_B D$  is a projective  $D$ -module and  $f: H_1(A, C, D) \rightarrow \mathfrak{c}/\mathfrak{d}\mathfrak{c}$  is injective.*

ii) *If  $\mathfrak{c}$  is generated by a  $B$ -regular sequence (e.g. for  $B$  and  $C$  regular), and if  $\Omega_{B/A} \otimes_B D$  is a projective  $D$ -module and  $f$  is injective, then  $A \rightarrow B$  is formally smooth.*

3.0. In EGA, O<sub>IV</sub>, 19.7.1 the following smoothness criterion is given: Let  $A \rightarrow B$  be a local morphism of local noetherian rings and let  $k$  be the residue class field of  $A$ ; the topologies are adic and given by the maximal ideals. Then  $A \rightarrow B$  is formally smooth if and only if  $A \rightarrow B$  is flat and  $k \rightarrow k \otimes_A B$  is formally smooth.



The following counter-example given by *N. Radu* shows that  $B$  must be noetherian for this criterion to be valid.

(3.0.1) *Let  $k$  be a perfect field and  $B$  a  $k$ -algebra which is a non-discrete valuation ring of dimension 1; let  $\mathfrak{m}$  be the maximal ideal of  $B$ , and  $K = B/\mathfrak{m}$ . Then  $B \rightarrow K$  is formally étale.*

Indeed,  $\mathfrak{m} = \mathfrak{m}^2$ ;  $k \rightarrow B \rightarrow K$  and  $K$  give the exact sequence (0.4):

$$0 = \mathfrak{m}/\mathfrak{m}^2 \rightarrow \Omega_{B/k} \otimes_B K \xrightarrow{v_{K/B/k}} \Omega_{K/k} \rightarrow 0.$$

Hence  $v_{K/B/k}$  is left invertible; but this means that  $K$  is a formally smooth  $B$ -algebra with respect to  $k$  (EGA,  $O_{IV}$ , 20.5.7). On the other hand,  $K$  is a formally smooth  $k$ -algebra (EGA,  $O_{IV}$ , 19.6.1); hence  $B \rightarrow K$  is formally smooth.

(A purely homological proof of the above criterion is given in [4].)

The following results concern the descent of formal smoothness; from (3.1) results the ‘only if’ part of 3.0.

3.1. THEOREM. *Let  $A' \xrightarrow{v} A \xrightarrow{u} B$  be ring-morphisms with  $B$  noetherian. Let  $B' = A' \otimes_A B$ . Let  $B$  be local with maximal ideal  $\mathfrak{m}$ , and  $\mathfrak{q}$  be a prime ideal of  $B'$  s.t.  $\mathfrak{q} \cap B = \mathfrak{m}$ . The topologies of  $B$  and  $B'_\mathfrak{q}$  are adic and given by the maximal ideals. Suppose that  $u$  or  $v$  is flat. Then  $A' \rightarrow B'_\mathfrak{q}$  is formally smooth if and only if  $A \rightarrow B$  is formally smooth.*

PROOF. Let  $k = B/\mathfrak{m}$  and  $K = B'_\mathfrak{q}/\mathfrak{q}B$ . Then

$$H_1(A, B, k) \otimes_k K = H_1(A, B, K) = H_1(A', B'_\mathfrak{q}, K) = H_1(A', B'_\mathfrak{q}, K)$$

(by 0.6, 0.7 and 0.5). Now apply (1.1).

3.2. PROPOSITION. *Let  $A \rightarrow B$  be a morphism of topological rings,  $B$  noetherian, and  $\mathfrak{a} \subset A$ ,  $\mathfrak{b} \subset B$  ideals with  $\mathfrak{a}B \subset \mathfrak{b}$ , such that the topology of  $A$  is  $\mathfrak{a}$ -adic and that of  $B$ ,  $\mathfrak{b}$ -adic. Then, if (1) or (2) holds,  $A \rightarrow B$  is formally smooth if  $A' \rightarrow B'$  is.*

(1)  $A' = A/\mathfrak{a}$ ,  $B' = B/\mathfrak{b}$ ,  $A \rightarrow B$  flat.

(2)  $A'$  a faithfully flat  $A$ -algebra, with the  $(\mathfrak{a}A)$ -adic topology, and  $B' = A' \otimes_A B$  (which has the  $(\mathfrak{b}B)$ -adic topology).

Moreover, in (2),  $A \rightarrow B$  is formally étale if  $A' \rightarrow B'$  is.

PROOF. Let  $C = B/\mathfrak{b}$  and  $C' = B'/\mathfrak{b}B'$ ; then  $C \rightarrow C'$  is faithfully flat. Hence,  $\Omega_{B/A} \otimes_B C = \Omega_{B'/A'}(0.7)$  so that

$$(\Omega_{B/A} \otimes_B C) \otimes_C C' = \Omega_{B'/A'} \otimes_{B'} C',$$

and

$$H_1(A, B, C) \otimes_C C' = H_1(A, B, C') = H_1(A', B', C')$$

(0.6 and 0.7).

i) Let be (1). From 1.1 it results that  $\Omega_{B'/A'} \otimes_{B'} C (= \Omega_{B/A} \otimes_B C)$  is a projective  $C$ -module and  $H_1(A, B, C) = H_1(A', B', C) = 0$ ; now apply 1.1 again.

ii) Let be (2). Let  $A' \rightarrow B'$  be formally smooth; then  $\Omega_{B'/A'} \otimes_{B'} C'$  is a projective  $C'$ -module and  $H_1(A', B', C') = 0$  (1.1). Hence, by the above equalities,  $\Omega_{B/A} \otimes_B C$  is a projective  $C$ -module ([15]) and  $H_1(A, B, C) = 0$ ; then  $A \rightarrow B$  is formally smooth (1.1)

Let  $A' \rightarrow B'$  be formally étale, so  $\hat{\Omega}_{B'/A'} = 0$   $H_1(A', B', C') = 0$  (0.10 and 1.1). Hence, as above,  $\Omega_{B/A} \otimes_B C = 0$  and  $H_1(A, B, C) = 0$ . Let  $M'_n = \Omega_{B'/A'} \otimes_{B'} B'/\mathfrak{b}^n B'$  and  $M_n = \Omega_{B/A} \otimes_B B/\mathfrak{b}^n$ ; then  $M'_n = M_n \otimes_{B/\mathfrak{b}^n} B'/\mathfrak{b}^n B'$  (0.7). Then  $\hat{\Omega}_{B'/A'} = 0$  gives  $M'_n = 0$ , but  $B/\mathfrak{b}^n \rightarrow B'/\mathfrak{b}^n B'$  is faithfully flat, and so  $M_n = 0$ . Hence  $\hat{\Omega}_{B/A} = 0$ . Now use 0.10 and 1.1. (Observe that 3.1, 3.2 are formally very similar and probably both follow from a more general statement.)

3.3. REMARK. i) *Let  $A' \leftarrow A \rightarrow B$  be morphisms of rings,  $B' = A' \otimes_A B$  and  $A \rightarrow A'$  faithfully flat; the topologies are discrete. Then  $A' \rightarrow B'$  is formally smooth (resp. étale) iff  $A \rightarrow B$  is formally smooth (resp. étale).*

Indeed  $\Omega_{B/A} \otimes_B B' = \Omega_{B'/A'}$  (0.7) and  $H_1(A, B, B) \otimes_B B' = H_1(A', B, B') = H_1(A', B', B')$ , by 0.6 and 0.7. Now apply 0.10.

ii) *Let  $X' \rightarrow X \leftarrow Y$  be morphisms of schemes,  $Y' = X' \times_X Y$ , and  $X' \rightarrow Y'$  faithfully flat. If  $Y' \rightarrow X'$  is formally étale (resp. locally formally smooth), then  $Y \rightarrow X$  is formally étale (resp. locally formally smooth).*

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