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FLAT MODULES IN ALGEBRAIC GEOMETRY

by

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Consider the following data:

$$(*)\qquad \begin{cases} \text{a noetherian scheme } S, \\ \text{a morphism of finite type } f\colon X \to S, \\ \text{a coherent sheaf of } 0_X\text{-modules } \mathscr{M}. \end{cases}$$

If x is a point of X and $s = f(x)$, recall that $\mathscr{M}$ is *flat* over S at the point x, if the stalk $\mathscr{M}_x$ is a flat $0_{S,s}$-module; $\mathscr{M}$ is flat over S, or is S-flat, if $\mathscr{M}$ is flat over S at every point of X.

Grothendieck has investigated, in great details, the properties of the morphism f when $\mathscr{M}$ is S-flat (EGA IV, 11 12 $\cdots$), and some of its results are now classical. For instance we have:

a) the set of points x of X where $\mathscr{M}$ is flat over S is open (EGA IV 11.1.1).

b) Suppose $\mathscr{M}$ is S-flat and supp $(\mathscr{M}) = X$. Then the morphism f is open (EGA 2.4.6). Further, if S is a domain and if the generic fibre is equidimensional of dimension n, then each fibre of f is equidimensional of dimension n (EGA IV 12.1.1.5).

In this lecture, we want to give a new approach to the problem of flatness and get structure theorems for flat modules. Much of the following theory is local on S and on X and we may assume S and X are affine schemes. Then the data (*) are equivalent to

$$(**)\qquad \begin{cases} \text{a noetherian ring } A, \\ \text{an } A\text{-algebra } B \text{ of finite type}, \\ \text{a } B\text{-module } M \text{ of finite type}. \end{cases}$$

Chapter I

Flat modules and free finite modules on smooth schemes

1. A criterion of flatness

Consider the data (*). Let x be a point of X and $s = f(x)$. We denote by $\dim_x(\mathscr{M}/S)$ the Krull-dimension of $\mathscr{M} \otimes_S k(s)$ at the point x. So, if

$\operatorname{Supp}(\mathscr{M}) = X$, $\dim_x(\mathscr{M}/S)$ is the maximum of the dimensions of the irreducible components of $X \otimes_S k(s)$ containing x. We set

$$\dim(\mathscr{M}/S) = \sup_{x \in X} \dim_x(\mathscr{M}/S) = \sup_{s \in S} \dim(\mathscr{M} \underset{S}{\otimes} k(s)).$$

If $\mathscr{M} = O_X$, we write also $\dim_x(X/S)$ and $\dim(X/S)$.

Let $f: X \to S$ be a smooth morphism of affine schemes with irreducible fibres, s a point of S, η the generic point of the fibre $X_s = X \otimes_S k(s)$, x a point of X_s. Let $\mathscr{M}$ be a coherent sheaf on X and $\mathscr{M}_s = \mathscr{M} \otimes_S k(s)$.

Then $(\mathscr{M}_s)_\eta$ is a $k(\eta)$-vectorspace of some finite dimension r. So there exists an X_s-morphism

$$\bar{u} : O^r_{X_s} \to \mathscr{M}_s,$$

which is bijective at the generic point η. If we restrict S to some suitable neighbourhood of s, we can extend $\bar{u}$ to an X-morphism

$$u : \mathscr{L} \to \mathscr{M},$$

where

$$\mathscr{L} \simeq O^r_X.$$

Note that $(\mathscr{M}_s)_\eta \simeq \mathscr{M}_\eta \otimes_S k(s)$; so the morphism

$$u_\eta \underset{O_{S,s}}{\otimes} k(s) : \mathscr{L}_\eta \underset{O_{S,s}}{\otimes} k(s) \to \mathscr{M}_\eta \underset{O_{S,s}}{\otimes} k(s)$$

is surjective, and by Nakayama's lemma,

$$u_\eta : \mathscr{L}_\eta \to \mathscr{M}_\eta$$

is surjective.

Lemma 1. *Suppose S to be local with closed point s; if x is any point of X above s, then*

$$u_\eta \text{ injective} \Leftrightarrow u_x \text{ injective} \Leftrightarrow u \text{ injective}$$

Proof: Denote by $\operatorname{Ass}(\mathscr{L})$ the set of associated primes of the 0_X-module $\mathscr{L}$. Because $\mathscr{L}$ is free, and X is S-flat we have (EGA IV 3.3.1)

$$\operatorname{Ass}(\mathscr{L}) = \bigcup_{t \in \operatorname{Ass}(S)} \operatorname{Ass}(\mathscr{L} \underset{S}{\otimes} k(t)).$$

Now, X being smooth over S with irreducible fibres, the fibres X_t are reduced, thus integral. Hence

$$\operatorname{Ass}(\mathscr{L} \underset{S}{\otimes} k(t)) = \{\eta_t\}$$

where η_t is the generic point of X_t.

Because $0_{X,\eta}$ is faithfully flat over $0_{S,s}$, the morphism $\operatorname{Spec}(0_{X,\eta}) \to S$ is surjective. This implies that each η_t is a generisation of η. And then $\operatorname{Ass}(\mathscr{L}) \subset \operatorname{Ass}(\mathscr{L}_\eta)$.

The inclusion $\mathrm{Ass}(\mathscr{L}_\eta) \subset \mathrm{Ass}(\mathscr{L}_x)$ being trivial, we conclude that $\mathrm{Ass}(\mathscr{L}_\eta) = \mathrm{Ass}(\mathscr{L}_x) = \mathrm{Ass}(\mathscr{L})$. Hence the canonical morphism $\Gamma(X, \mathscr{L}) \to \Gamma(X_\eta, \mathscr{L}_\eta)$ is injective. Let $\mathscr{R}$ be $\mathrm{Ker}(u)$. Then the canonical morphism $\Gamma(X, \mathscr{R}) \to \Gamma(X_\eta, \mathscr{R}_\eta)$ is injective, so $\mathrm{Ass}(\mathscr{R}_\eta) = \mathrm{Ass}(\mathscr{R}_x) = \mathrm{Ass}(\mathscr{R})$. And the lemma follows.

THEOREM 1. *Let $X \to S$ be a smooth morphism of affine schemes with irreducible fibres, x a point of X above s in S, $\mathscr{M}$ a coherent sheaf on X,*

$$\mathscr{L} \xrightarrow{u} \mathscr{M} \to \mathscr{P} \to 0$$

an exact sequence of 0_X-modules, such that $\mathscr{L}$ is free and $u \otimes k(s)$ is bijective at the generic point η of the fibre $X_s = X \otimes_S k(s)$. Then the following conditions are equivalent:

1) $\mathscr{M}$ is S-flat at the point x.
2) $u_\eta : \mathscr{L}_\eta \to \mathscr{M}_\eta$ is injective and $\mathscr{P}_x$ is S-flat.
3) $u_x : \mathscr{L}_x \to \mathscr{M}_x$ is injective and $\mathscr{P}_x$ is S-flat.

PROOF: The equivalence of 2) and 3) is supplied by lemma 1 (the restriction to local S being no loss of generality). Let $\mathscr{R} = \mathrm{Ker}(u)$. By Nakayama's lemma, u_η is surjective, so we have the exact sequence

$$0 \to \mathscr{R}_\eta \to \mathscr{L}_\eta \xrightarrow{u_\eta} \mathscr{M}_\eta \to 0.$$

If $\mathscr{R}_\eta = 0$, $\mathscr{M}_\eta \simeq \mathscr{L}_\eta$ is S-flat. Conversely, if $\mathscr{M}_\eta$ is S-flat, the exact sequence above remains exact after tensoring with $k(\eta)$. But $u_\eta \otimes k(\eta)$ is bijective and so $\mathscr{R}_\eta \otimes k(\eta) = 0$ and, by Nakayama's lemma again, $\mathscr{R}_\eta = 0$.

1) $\Rightarrow$ 3). If $\mathscr{M}_x$ is flat over S, then $\mathscr{M}_\eta$ is flat over S, and by the preceeding remark, u_η is injective and therefore u_x is injective (lemma 1). Now the proof of injectivity of u_x remains valid if we replace S by any closed sub-scheme and so $\mathscr{P}_x$ is S-flat.

3) $\Rightarrow$ 1), because a flat by flat extension is flat.

COROLLARY. *The module $\mathscr{M}$ is S-flat at the point x, if and only if $\mathscr{M}_\eta$ is a free* $0_{X,\eta}$-module and $\mathscr{P}_x$ is S-flat.

2. The main theorem of Zariski

Let $f: X \to S$ be a morphism of finite type, s a point of S, x an *isolated* point of the fibre $X \otimes_S k(s)$. Then, the main theorem of Zariski, in its classical form, asserts that there is an open neighbourhood U of x in X which is an open sub-scheme of a *finite* S-scheme Y (EGA III 4.4.5). Of course it is a good thing to have a finite morphism; but, in counterpart, we have to add extra points: those of $Y - U$ and, on these new points, we have very few informations. For instance, if $\mathcal{M}$ is a coherent sheaf on U, S-flat, we cannot expect to extend $\mathcal{M}$ into a coherent sheaf $\mathcal{N}$ on Y which is still S-flat. So, we shall give another formulation of the main theorem, a little more sophisticated, which avoids to add bad extra points.

a) Suppose first that S is local, *henselian*, with closed point s. Then, the finite S-scheme Y splits into its local components. The local component (V, x), which contains x is clearly included in U. So, if we replace U by V, we get an open neighbourhood of x in X, which is already finite over S.

b) In the general case, we introduce the henselisation $(\tilde{S}, \tilde{s})$ of S at the point s. Then, if $\tilde{U} = U \times_S \tilde{S}$, we can find (case a)) an open and closed sub-scheme $\tilde{V}$ of $\tilde{U}$, which contains the inverse image $\tilde{x}$ of x and is finite over $\tilde{S}$. Then $\tilde{V}$ is defined by an idempotent $\tilde{e}$ of $\Gamma(\tilde{U}, 0_{\tilde{U}})$.

It is convenient to set the following definition:

Definition 1. Let (X, x) be a pointed scheme. An *étale neighbourhood* of x in X (or of (X, x)) is a pointed scheme (X', x') with an *étale* pointed morphism $(X', x') \to (X, x)$ such that the residual extension $k(x')/k(x)$ is trivial.

We know that $(\tilde{S}, \tilde{s})$ is the inverse limit of affine étale neighbourhoods $(S_i, s_i)_{i \in I}$ of (S, s) (EGA IV 18). Let x_i be the point of $U \times_S S_i$ which has respective projections x and s_i. For i large enough, the idempotent $\tilde{e}$ comes from an idempotent e_i of $\Gamma(U_i, 0_{U_i})$. Let V_i be the corresponding component of U_i which contains x_i. Then $\tilde{V} = V_i \times_{S_i} \tilde{S}$ is finite on $\tilde{S}$ and consequently, V_i is finite on S_i for a suitable i. Suppose now V_i is finite on S_i and set $(S', s') = (S_i, s_i)$, $(X', x') = (V_i, x_i)$; we get:

Proposition 1. *Let $f : X \to S$ be a morphism of finite type, s a point of S and x an isolated point of $X \otimes_S k(s)$. Then there exists an étale neighbourhood (S', s') of (S, s), an étale neighbourhood (X', x') of (X, x) and a commutative diagram of pointed schemes*

$$\begin{array}{ccc} (X, x) & \longleftarrow & (X', x') \\ \downarrow & & \downarrow \\ (S, s) & \longleftarrow & (S', s') \end{array}$$

such that X' is finite on S' and x' is the only point of X' above s'.

3. Reduction to the smooth case

Consider the data (*).

As we are interested in the flatness of $\mathcal{M}$ over S, the structure of $\mathcal{M}$ as an 0_X-module is not essential. We shall use this remark and the main theorem of Zariski, to change X into a smooth S-scheme.

First, we may replace X by the closed sub-scheme defined by the annihilator of $\mathscr{M}$, and so assume that

$$\operatorname{Supp}(\mathscr{M}) = X$$

Let x be a point of X above s in S and let

$$n = \dim_x(\mathscr{M}/S) = \dim_x(X/S).$$

Choose a closed specialisation z of x in $X_s = X \otimes_S k(s)$. Then $\dim(0_{X_s,z}) = n$. If we replace X by a suitable affine neighbourhood of z, we may use a system of parameters of the local ring $0_{X_s,z}$ to find an S-morphism

$$v : X \to S[T_1, \cdots, T_n]$$

such that z is an isolated point of its fibre $v^{-1}(v(z))$. Thus the generisation x of z is also an isolated point of $v^{-1}(v(x))$. Now apply proposition 1: we can find a commutative diagram of pointed schemes

$$\begin{array}{ccc} (X, x) & \xleftarrow{\ h\ } & (X', x') \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ (S[T_1, \cdots, T_n], v(x)) & \xleftarrow[\ g\ ]{} & (Y, y) \end{array}$$

such that g and h are étale neighbourhoods, w is finite and x' is the only point of X' above y.

The composed morphism

$$(Y, y) \to S[T_1, \cdots, T_n] \to S$$

is smooth of relative dimension n.

Let $\mathscr{M}'$ be the inverse image of $\mathscr{M}$ over X' and $\mathscr{M} = w_*(\mathscr{M}')$; $\mathscr{N}$ is a coherent sheaf because w is finite.

Note the following equivalences:

$\mathscr{M}_x$ flat over $S \Leftrightarrow \mathscr{M}'_{x'}$ flat over S (because X' is flat over X);

$\mathscr{M}'_{x'}$ flat over $S \Leftrightarrow \mathscr{N}_y$ flat over S (because w is finite and x' is the only point of X' above y, $\mathscr{M}'_{x'}$ and $\mathscr{N}_y$ define the same $0_{S,s}$-module).

Hence, in order to study the flatness of $\mathscr{M}$ at the point x, we may replace X by Y, $\mathscr{M}$ by $\mathscr{N}$ and x by y, and we are reduced to the case where X is smooth over S of relative dimension $n = \dim_x(\mathscr{M}/S)$.

4. Relative presentation

If we are a bit more cautious in the constructions given above, we can choose Y such that the fibre $Y \otimes_S k(s)$ is irreducible. Then we can find an étale affine neighbourhood (S', s') of (S, s) and an open affine subscheme Y' of $Y \otimes_S S'$, which contains the inverse image of y and such that the fibres of the morphism $Y' \to S'$ are irreducible. Let $\mathcal{N}'$ be the inverse image of $\mathcal{N}$ on Y'. After a slight change on X' we get the following diagram

$$(***) \qquad \begin{array}{ccc} \mathcal{M}' & & \mathcal{N}' \\ (X', x') & \xrightarrow{w'} & (Y', y') \\ \varphi \swarrow & & \\ \mathcal{M} \ (X, x) & & \\ f \downarrow & & \downarrow g \\ (S, s) & \xleftarrow[\Psi]{} & (S', s') \end{array}$$

where (X', x') is an étale affine neighbourhood of (X, x), (S', s') is an étale neighbourhood of (S, s), w' is finite and x' is the only point above y', $\mathcal{M}$ is a coherent sheaf on X with Supp $(\mathcal{M}) = X$, $\mathcal{M}' = \varphi^*(\mathcal{M})$, $\mathcal{N}' = w'_*(\mathcal{M}')$, g is smooth affine with irreducible fibres of dimension $n = \dim_x(\mathcal{M}/S)$.

DEFINITION 2. Consider the data (*), and let x be a point of X above s in S. Suppose Supp $(\mathcal{M}) = X$. Then a *relative presentation* of $\mathcal{M}$ at the point x, consists of the data (***) above, together with an exact sequence of $0_{Y'}$-modules

$$\mathcal{L}' \xrightarrow{\alpha} \mathcal{N}' \to \mathcal{P}' \to 0,$$

such that $\mathcal{L}'$ is free and

$$\alpha \underset{S'}{\otimes} k(s') : \mathcal{L}' \underset{S'}{\otimes} k(s') \to \mathcal{N}' \underset{S'}{\otimes} k(s')$$

is bijective at the generic point of $Y' \otimes_{S'} k(s')$.

The introductory remarks of no 1 show, that $\mathcal{M}$ always admits a relative presentation at the point x.

5. Amplifications

1) Consider the initial data (**) and suppose M is A-flat at a point x of Spec (B). We can use a relative presentation of M at x and then apply theorem 1. In fact, by an easy induction on $n = \dim_x(M/\text{Spec}(A)$ we can prove that locally on Spec(A) and Spec (B), for the étale topology, the A-module M has a 'composition series'

$$0 \to M_n \to M_{n-1} \to \cdots \to M_0 = M,$$

such that M_i/M_{i+1} is the A-module defined by a free finite module over some algebra B_i smooth over A, with geometrically irreducible fibers of dimension i.

2) Let A be any ring, B an A-algebra of finite presentation and M a B-module. The structure theorem for flat modules, proved in the noetherian case, remains valid if M is a B-module of finite presentation and even if M is a B-module of finite type. In fact, if M is a B-module of finite type, such that M_q is A-flat for some prime ideal q of B, then necessarily, M_q is a B_q-module of finite presentation. Moreover, if the ring A is not too bad, for instance if A is a domain, then, the set of points q where the B-module M of finite type is A-flat, is an open subset of Spec (B) and if M is A-flat, M is a B-module of finite presentation. As a corollary we get: let A be a domain and B an A-algebra of finite type which is A-flat, then B is an algebra of finite presentation.

Chapter 2

Flat and projective modules

1. Introduction

Let A be a noetherian ring, B an A-algebra of finite type, M a B-module of finite type. If M is a projective A-module, then M is A-flat. The converse is not true in general: For instance, let A be a (discrete) valuation ring with quotient field K and take for B a K-algebra of finite type. Then M is K-free and hence A-flat. But, if $M \neq 0$, M is not projective as an A-module; because a submodule of a free A-module is free (Bourbaki, Alg. VII § 3 th. 1) it cannot be a K-vectorspace $\neq 0$.

In this example, Spec (B) lies entirely above the generic point η of Spec (A); consequently, an associated prime x of M cannot specialize into a point of the special fibre: and this happens to be the main obstruction for a flat A-module to be projective.

Definition 1. Consider the initial data (*). For $s \in S$ we denote by Ass $(\mathscr{M} \otimes_S k(s))$ the set of associated primes of $\mathscr{M} \otimes_S k(s)$. We set

$$\mathrm{Ass}\,(\mathscr{M}/S) = \bigcup_{s \in S} \mathrm{Ass}\,(\mathscr{M} \underset{S}{\otimes} k(s)).$$

Definition 2. The 0_X-module $\mathscr{M}$ is *S-pure* if the following condition holds:

For every s in S, if $(\tilde{S}, \tilde{s})$ denotes the henselisation of S at the point s, $\tilde{X} = X \times_S \tilde{S}$, $\tilde{\mathscr{M}} = \mathscr{M} \times_S \tilde{S}$, then every x in $\mathrm{Ass}(\tilde{\mathscr{M}}/\tilde{S})$ specializes into a point of the fibre $\tilde{X}_{\tilde{s}}$.

EXAMPLES

1) If $X \to S$ is proper, then every coherent sheaf $\mathscr{M}$ on X is S-pure.

2) If $\dim(X/S) = 0$ and X is separated over S, then 0_X is S-pure if and only if X is finite over S.

3) If $X \to S$ is flat with geometrically irreducible and reduced fibres, then 0_X is S-pure.

THEOREM 1. Consider the initial data (**); then, the flat A-module M is projective if and only if it is A-pure.

In fact we can be more precise:

a) If $\dim(M/A) = 0$, and if M is A-projective, then M certainly is a finite type A-module and so is locally free on Spec (A).

b) If $S = \text{Spec}(A)$ is connected and $\dim(M/A) \geqq 1$, then M cannot be an A-module of finite type and we can apply a result of H. Bass which asserts that the projective A-module M is in fact free. Thus we get the following corollary:

COROLLARY 1. *If M is A-flat and A-pure, M is locally (for the Zariski-topology on Spec (A)) a free module.*

PROOF OF THEOREM 1 (necessity). We suppose M to be a projective A-module and we want to show that M is A-pure. The hypothesis of projectivity is preserved by any base change $A \to A'$; hence, taking into account definition 2, it is sufficient to prove the following assertion: If moreover A is a local ring with maximal ideal m and q is any associated prime of M, then $V(q) \cap V(mB) \neq \phi$. Now if this assertion were false, we should have $q+mB = B$ and so $1 = q+h$ for some $q \in q$ and $h \in mB$. As q is an associated prime of M, there exists $0 \neq a$ in M such that $(1-h)a = qa = 0$. Consequently, by [Bourkaki, Alg. Comm. III § 3 prop. 5] M is not separated in the mB-adic topology. Hence the A-module M is not separated in the m-adic topology and M cannot be a direct factor of a free A-module.

In order to prove the sufficiency part of the theorem, we shall use a small part of new results of L. Gruson on projective modules ([3]).

2. Mittag-Leffler and projective modules

Daniel Lazard proved in [4] that every flat A-module M is the direct limit of free finite A-modules. Conversely, a direct limit of free finite modules is flat. So, without restrictive hypothesis on the flat module M, we cannot expect to have restrictive conditions on the direct system. Following Gruson, we shall introduce a restrictive condition on the direct system $(M_i)_{i \in I}$.

Definition 3. An A-module P is a *Mittag-Leffler module* (shorter: *M.L. module*) if P is the direct limit of free finite A-modules $(P_i)_{i\in I}$ such that the inverse system $(\mathrm{Hom}\,(P_i, A))_{i\in I}$ satisfies the usual Mittag-Leffler condition.

N.B. An inverse system $(Q_i)_{i\in I}$ of A-module satisfies the (usual) Mittag-Leffler condition, if $\forall i \in I$, $\exists j \in I$, $j \geqq i$ such that for $k \geqq j$ we have $\mathrm{Im}(Q_k \to Q_i) = \mathrm{Im}(Q_j \to Q_i)$.

Remarks

1) The fact that the inverse system $\mathrm{Hom}\,(P_i, A)$ satisfies the Mittag-Leffler condition, does not depend on the choice of the family of free finite modules P_i, with $\varinjlim P_i = P$.

2) For any A-module Q and any free finite A-module P_i we have a canonical isomorphism

$$\mathrm{Hom}\,(P_i, A) \underset{A}{\otimes} Q \simeq \mathrm{Hom}\,(P_i, Q).$$

So if $P = \varinjlim P_i$ is an M.L. module, then for every A-module Q, the inverse system $\mathrm{Hom}\,(P_i, Q)$ satisfies the Mittag-Leffler condition.

Examples. a) Every free module is an M.L. module.
b) A direct factor of an M.L. module is an M.L. module.
c) A projective A-module is an M.L. module.

The last assertion admits a partial converse:

Proposition 1. *Suppose A is a noetherian ring and M is of countable type (i.e. M is generated by countably many elements); then, if M is an M.L. module, M is projective.*

Proof. We can write M as a direct limit of free finite A-modules $(M_i)_{i\in I}$. As M is of coutable type and A is noetherian, we easily see that we can take I equal to the set N of natural numbers. We have to show that for every exact sequence of A-modules

$$0 \to P' \to P \to P'' \to 0,$$

the sequence

$$0 \to \mathrm{Hom}\,(M, P') \to \mathrm{Hom}\,(M, P) \to \mathrm{Hom}\,(M, P'') \to 0$$

is also exact. But we have $\mathrm{Hom}\,(M, \cdot) = \varprojlim \mathrm{Hom}\,(M_n, \cdot)$ and because M_n is a free module, we get for every n an exact sequence

(1) $\quad 0 \to \mathrm{Hom}\,(M_n, P') \to \mathrm{Hom}\,(M_n, P) \to \mathrm{Hom}\,(M_n, P'') \to 0.$

By hypothesis, the inverse countable system $(\mathrm{Hom}\,(M_n, P'))_{n\in N}$ satisfies the Mittag-Leffler condition; hence, taking the inverse limit on the exact sequences (1), we still get an exact sequence (cf. EGA 0_{III} 13.2.2).

PROPOSITION 2. *Let A be a noetherian ring, A' a faithfully, flat A-algebra M and A-module of countable type. If $M' = M \otimes_A A'$ is a projective A'-module, M is a projective A-module.*

PROOF. It is sufficient to prove that M is an M.L. module (prop. 1). Of course, M is A-flat, so M is the direct limit of free finite A-modules $(M_i)_{i \in I}$. Now observe that the property of being an M.L. module clearly is invariant under faithfully flat extension.

PROPOSITION 3. *Let M be a flat A-module. Suppose that for every free finite A-module Q and every $x \in M \otimes_A Q$, there exists a smallest submodule R of Q such that $x \in M \otimes_A R$. Then M is an M.L. module.*

PROOF. Because M is A-flat, M is a direct limit of free finite modules $(M_i)_{i \in I}$. Denote by $u_i : M_i \to M$ the canonical morphism and by $u_{ij} : M_i \to M_j$ the 'transition' morphism for $j \geqq i$. Then $u_{ij} \in \operatorname{Hom}(M_i, M_j)$ which is canonically identified with $\operatorname{Hom}(M_i, A) \otimes_A M_j$. We fix i. It is easy to see that the image of the morphism

$$\operatorname{Hom}(u_{ij}, 1_A) : \operatorname{Hom}(M_j, A) \to \operatorname{Hom}(M_i, A)$$

for $j \geqq i$ is the smallest submodule R_j of $\operatorname{Hom}(M_i, A)$ such that $u_{ij} \in R_j \otimes_A M_j$. The morphism u_i is an element of $\operatorname{Hom}(M_i, M)$, which is canonically identified with $\operatorname{Hom}(M_i, A) \otimes_A M$. By hypothesis, there exists a smallest submodule R of $\operatorname{Hom}(M_i, A)$ such that $u_i \in R \otimes_A M$. Now we look at the following commutative diagram of isomorphisms:

$$\begin{array}{ccc} \operatorname{Hom}(M_i, M) & \simeq & \operatorname{Hom}(M_i, A) \otimes_A M \\ \wr & & \wr \\ \varinjlim_j \operatorname{Hom}(M_i, M_j) & \simeq & \varinjlim_j (\operatorname{Hom}(M_i, A) \otimes M_j) \end{array}$$

Here $u_i = \varinjlim u_{ij}$ $(j \geqq i)$ is an element of $R \otimes_A M = \varinjlim (R \otimes_A M_j)$. So we can choose $j \geqq i$ such that $u_{ik} \in R \otimes_A M_k$ for every $k \geqq j$. Hence $R \supset R_k$ for $k \geqq j$. But clearly $R \subset R_k$ (for $k \geqq i$), thus $R = R_k$ for $k \geqq j$ and the inverse system $\operatorname{Hom}(M_j, A)$ satisfies the Mittag-Leffler condition.

COROLLARY 1. *Let A be a noetherian ring and n a natural number. Then the ring $B = A[[T_1, \cdots, T_n]]$ of formal series is an M.L. module.*

PROOF. Since A is noetherian, B is A-flat. If Q is a free finite A-module, then $B \otimes_A Q$ is the A-module of formal power series $Q[[T]]$ with coefficients in Q. If $x = \sum q_i T^i$ is an element of $Q[[T]]$, the submodule Q' of Q generated by the q_i is the smallest submodule of Q such that $x \in Q'[[T]]$, and we may apply proposition 3.

DEFINITION 4. Let $u : M' \to M$ be a morphism of A-modules. We say that u is *universally injective* if, for every A-module P of finite type, the morphism $u \otimes_A 1_P : M' \otimes_A P \to M \otimes_A P$ is injective.

REMARKS.

If $u : M' \to M$ is universally injective, $u \otimes_A 1_P$ is injective for every A-module P; moreover, if M is A-flat, M/M' and M' are A-flat.

COROLLARY 2 (*of proposition* 3). *Let* $u : M' \to M$ *be a universally injective morphism. If* M *satisfies the condition of proposition* 3, *then* M' *satisfies the same condition and hence is an M.L. module.*

PROOF. Firstly, we deduce from the preceding remarks that M' is A-flat. Then, let Q be a free finite A-module, x an element of $M' \otimes_A Q$ and R the smallest submodule of Q such that $u(x) \in M \otimes_A R$. It is sufficient to prove that $x \in M' \otimes_A R$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow & M' \otimes_A R & \longrightarrow & M' \otimes_A Q & \longrightarrow & M' \otimes_A Q/R \\ & \downarrow & & \downarrow & & \downarrow \\ & M \otimes_A R & \longrightarrow & M \otimes_A Q & \longrightarrow & M \otimes_A Q/R. \end{array}$$

Because M' is A-flat the upper row is exact, because u is universally injective the right vertical arrow is injective, and so $x \in M' \otimes_A R$.

AMPLIFICATIONS. Gruson proved that the condition of proposition 3 is fulfilled by every M.L. module and hence in fact characterises M.L. modules. He also proved that the projectivity of an A-module can be checked after any faithfully flat ring extension $A \to A'$.

3. End of the proof of theorem 1 (sufficiency)

For sake of brevity, we shall prove the theorem only in the case where B is a smooth A-algebra with geometrically irreducible and reduced fibres, and $M = B$. In fact this is the fundamental case: the general case is an easy consequence by using the technique below and the structure theorem for flat modules proved in chapter I.

1) The A-Algebra B is a quotient of some polynomial algebra $A[T_1, \cdots, T_n]$ and therefore, the A-module B is of countable type.

2) To prove that B is a projective A-module, we may then make a faithfully flat base change $A \to A'$ (prop. 2). Take $A' = B$; then we are reduced to the case where the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has a section (i.e. there is an A-morphism $u : B \to A$). Let I be the kernel of u. Because B is smooth over A the A-module $J = I/I^2$ is a projective module of

finite type (EGA 0_{IV} 19.5.4); hence J is locally free on Spec(A). Using proposition 2 again, we may suppose J to be free. Then the I-adic completion $\hat{B}$ of B is isomorphic to some A-algebra $A[[T_1, \cdots, T_m]]$ of formal power series (EGA 0_{IV} 19.5.4); and by prop. 3, cor. 1 $\hat{B}$ is an $M.L.$ module.

3) LEMMA. The canonical morphism $B \to \hat{B}$ is universally injective.

PROOF. Let M be an A-module of finite type. We have to prove that the morphism $M \otimes_A B \to M \otimes_A \hat{B}$ is injective. But $\hat{B}$ is B-flat and it will be sufficient to prove that Ass($M \otimes {}_A B$) is contained in the image of Spec($\hat{B}$), ([4], Ch. II prop. 3.3). Since B is A-flat with irreducible and reduced fibres, we have (EGA IV 3.3.1)

$$\mathrm{Ass}\,(M \underset{A}{\otimes} B) = \bigcup_{p \in \mathrm{Ass}\,(M)} \mathrm{Ass}\,(B \underset{A}{\otimes} k(p) = \bigcup_{p \in \mathrm{Ass}\,(M)} (pB);$$

and pB is contained in the image of Spec($\hat{B}$) since $\hat{B}$ is faithfully flat over A.

4) From the above lemma we deduce that B is an $M.L.$ module (prop. 3, cor. 2). Thus B is a projective A-module indeed (prop. 1).

4. Proposition

Let S be a noetherian scheme, X an S-scheme of finite type, $\mathscr{M}$ a coherent sheaf on X which is S-flat and S-pure, $u : \mathscr{M} \to \mathscr{N}$ a surjective morphism of coherent shesfs. Let F be the subfunctor of S defined as follows:

For any S-scheme T, T factors through F if and only if the morphism $u_T : \mathscr{M}_T \to \mathscr{N}_T$, deduced from u by the base change $T \to S$, is an isomorphism.

Then is represented by a closed subscheme of S.

PROOF. For the sake of simplicity, we suppose X to be affine over S. The assertion to be proved is local on S. So we can suppose $S =$ Spec(A), $X =$ Spec(B) and M a free A-module (th. 1 cor. 1). Let $(e_i)_{i \in I}$ be a basis for the A-module M and $(a_\lambda)_{\lambda \in \Lambda}$ a system of generators of $R = \mathrm{Ker}\, u$. Then each a_λ has coordinates $a_{\lambda i}$ with respect to the basis $(e_i)_{i \in I}$. Now it is clear, that F is represented by the closed subscheme $V(J)$ of Spec(A), where J is the ideal generated by the family $\{a_{\lambda i} | \lambda \in \Lambda,\ i \in I\}$.

Chapter 3

Universal flattening functor

1. The local case

Let S be a local, noetherian scheme with closed point s, X an S-scheme of finite type, $\mathscr{M}$ a coherent sheaf on X, and x a point of X lying over s.

THEOREM 1. *Suppose further that S is henselian. Then there exists a greatest closed subscheme $\bar{S}$ of S, such that $\bar{\mathscr{M}} = \mathscr{M} \times_S \bar{S}$ is $\bar{S}$-flat at the point x. Further, the subscheme $\bar{S}$ is universal in the following sence:*

Let T be a local S-scheme with closed point t over s; set $X_T = X \times_S T$ and $\mathscr{M}_T = \mathscr{M} \times_S T$. Then $\mathscr{M}_T$ is T-flat at any point of X_t which lies over x if and only if the morphism $T \to S$ factors through $\bar{S}$.

PROOF: We proceed by induction on $n = \dim_x(\mathscr{M}/S)$.

a) If $n < 0$, we have $\mathscr{M}_x = 0$, and we can take $\bar{S} = S$.

b) Assume $n \geqq 0$, and that the theorem holds for modules of relative dimension smaller than n. If we replace X by a suitable subscheme, we are reduced to the case where Supp $(\mathscr{M}) = X$. Then $\dim_x(X/S) = n$. Proceeding as in ch. I, § 3, we see that we may suppose that X is smooth over S, of relative dimension n, and also, since S is henselian, that X has geometrically irreducible fibres. (Chap. I, § 4). Let η be the generic point of the closed fibre. We have exact sequence of coherent sheaves on X:

$$\mathscr{L} \to \mathscr{M} \to \mathscr{P} \to 0$$

where $\mathscr{L}$ is a free 0_X-module of finite rank, and $u \otimes k(\eta)$ is bijective. We now apply theorem 1 of ch. I: Let (T, t) be a local (noetherian) scheme over (S, s), and denote by η_t the generic point of X_t. Then the inverse image $\mathscr{M}_T$ of $\mathscr{M}$ on X_T is T-flat at a point z of X_t, if and only if u_T : $\mathscr{L}_T \to \mathscr{M}_T$ is bijective at the point η_t and $\mathscr{P}_T$ is T-flat at the point z.

We have $\dim_x(\mathscr{P}/S) \leqq n-1$; hence, by the induction hypothesis, there exists a greatest closed subscheme S' of S such that $\mathscr{P} \times_S S'$ becomes S'-flat at the point x. We may thus replace S by S' and assume that $\mathscr{P}$ is S-flat at x.

Now, set $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, $\mathscr{L} = \tilde{L}$, $\mathscr{M} = \tilde{M}$, and let $\mathscr{Q}$ be the prime ideal of B corresponding to x. The A-module L is free (ch. II, th. 1, cor. 1); let $\{e_i\}_{i \in I}$ be a basis of L over A. Choose a system of generators $\{a_\lambda\}_{\lambda \in \Lambda}$ of $R = \text{Ker}(u)$, and let $\{a_{\lambda, i}\}_{i \in I}$ be the coordinates of a_λ in L. Then I claim that $\bar{S}$ is the closed subscheme $V(J)$, where J is the ideal of A generated by the family $\{a_{\lambda, i})_{\lambda \in \Lambda, i \in I}$. In fact, let J' be an ideal of A, set $A' = A/J$; then we have the following equivalences: $(M/J'M)_{\mathscr{Q}}$ is A'-flat $\Leftrightarrow$ $(u \otimes_A A')_\eta$: $(L/J'L)_\eta \to (M/J'M)_\eta$ is injective $\Leftrightarrow$ $u \otimes_A A' : L/J'L \to M/J'M$ is injective (ch. I, th. 1, lemma 1) $\Leftrightarrow$ the images of the a_λ in $L/J'L$ are zero $\Leftrightarrow$ $J \subset J'$. That proves the existence of $\bar{S}$; to see that $\bar{S}$ is universal, we proceed in the same manner.

COROLLARY 1. (*Valuative criterion of flatness* (*cf. EGA IV*, 11.8.1)).

Let S be a *reduced* noetherian scheme, $X \to S$ a morphism of finite type, $\mathscr{M}$ an 0_X-coherent sheaf. Then $\mathscr{M}$ is S-flat if and only if, for any

S-scheme T, which is the spectrum of a discrete valuation ring, $\mathcal{M} \times_S T$ is T-flat.

PROOF: Of course the necessity is clear. To prove the sufficiency, we may assume that S is local, with closed point s, and we may replace S by its henselisation which is also reduced. Choose a point x of the closed fibre X_s, and let $\bar{S}$ be the greatest closed subscheme of S such that $\mathcal{M} \times_S \bar{S}$ is $\bar{S}$-flat at the point x (th.1). We must prove that $\bar{S} = S$.

Set $S = \operatorname{Spec}(A)$, $\bar{S} = \operatorname{Spec}(A/J)$, and consider the set $\mathscr{P}_i$ of minimal primes of A. Because A is reduced, the canonical morphism

$$A \to \prod_i A/\mathscr{P}_i$$

is injective. We know that each of the local domains $A/\mathscr{P}_i$ is dominated by some discrete valuation ring R_i (EGA II, 7.1.7), consequently we get an injective morphism $A \to \prod_i R_i$. But, the universality of $\bar{S}$ (th. 1) and the assumption imply that each of the local morphisms $A \to R_i$ factors through A/J, and hence $J = 0$.

2. The global case

Consider the initial data (*). The *universal flattening functor* F of the S-module $\mathcal{M}$ is the subfunctor of the final object S defined as follows:

An S-scheme T factors through F if and only if $\mathcal{M}_T = \mathcal{M} \times_S T$ is T-flat.

THEOREM 2. *Suppose $\mathcal{M}$ is S-pure (ch. II, def. 2). Then the morphism of functors $F \to S$ is represented by a surjective monomorphism of finite type.*

For the sake of simplicity, we shall only give the details of the fundamental step of the proof, which is contained in Proposition 1 below.

Suppose $X \to S$ is a smooth morphism with geometrically irreducible fibres, and let $\mathcal{M}$ be a coherent sheaf on X. Then, if $\mathcal{M}$ is S-flat, $\mathcal{M}$ is a locally free 0_X-module at the generic point of each fibre of X over S (ch. I, th. 1). Let r be an integer, and define subfunctors F (resp. F_r) of S as follows: An S-scheme T factors through F (resp. F_r) if and only if the inverse image $\mathcal{M}_T = \mathcal{M} \times_S T$ of $\mathcal{M}$ on $X_T = X \times_S T$ is locally free (resp. locally free of rank r) at the generic point of each fibre of X_T over T.

PROPOSITION 1. i) *The functor F is the disjoint sum of the functors F_r, $r \in N$.*

ii) *The monomorphism $F_r \to S$ is an immersion.*

PROOF: i) Let $T \to S$ be a morphism which factors through F. Then $\mathcal{M}_T$ is locally free on an open set U of X_T which covers T. But the smooth morphism $X_T \to T$ is open, and hence we get a canonical splitting of T:

$$T = \coprod_{r \in N} T_r$$

such that $\mathscr{M}_{T_r}$ is locally free of fixed rank r on $U \cap X_{T_r}$. The assertion i) says nothing else.

ii) We have to prove that F_r is represented by a subscheme of S. Let s be a point of S, η_s the generic point of the fibre $X \otimes_S k(s)$, and n an integer. If we have $\dim_{k(\eta_s)}(\mathscr{M} \otimes k(\eta_s)) \leqq n$, there exists a neighbourhood U of η_s and a surjective morphism $0_U^n \to \mathscr{M}|U$. The image V of U is open in S. Of course, if $r > n$, we have $F_r \cap V = \phi$. Hence, to prove that F_r is represented by a subscheme of S, we can first replace S by a suitable open subscheme, in such a way that for any point s of S, we have $\dim_{k(\eta_s)}(\mathscr{M} \otimes k(\eta_s)) \leqq r$. We shall show that in this case, F_r is a closed subscheme of S. Such an assertion is local on S. Let s be a point of S. We can find an open neighbourhood U of η_s and a surjective morphism $u : 0_U^r \to \mathscr{M}|U$.

Then, after a restriction to suitable open subschemes of S (resp. U), we are reduced to the case $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $\mathscr{M} = \tilde{M}$, and we may assume that there exists a surjective morphism $u : B^r \to M$. But lemma 1 of th. 1, ch.I implies that M is locally free of rank r at the generic point of a fibre of X over S if and only if u is bijective. Hence F_r is represented by a closed subscheme of S (ch. II, prop. 4).

Chapter 4

Flattening by blowing up

1.

Let S be a noetherian scheme, X an S-scheme of finite type, $\mathscr{M}$ a coherent sheaf on X. Consider a blowing up $S' \to S$ of an ideal of 0_S, and let Z be the closed subscheme of S defined by this ideal (i.e. Z is the center of the blowing up). Set

$$X' = X \times_S S', \mathscr{M}' = \mathscr{M} \times_S S', Z' = Z \times_S S', Y = X \times_S Z,$$
$$Y' = Y \times_S S' = Z' \times_{S'} X'.$$

Then Z' is a divisor of S' (i.e. Z' is locally equal to $V(f')$, where f' is not a zero divisor of $0_{S'}$).

We now introduce the coherent subsheaf $\mathscr{N}'$ of $\mathscr{M}'$ defined as follows: for any affine open subscheme U' of X', $\Gamma(U', \mathscr{N}')$ is the submodule of $\Gamma(U', \mathscr{M}')$ of sections with support in $Y' \cap U'$.

DEFINITION 1: The pure transform $\mathscr{M}^{\Delta}$ of $\mathscr{M}$, by the blowing up $S' \to S$, is the coherent sheaf $\mathscr{M}'/\mathscr{N}'$.

So the pure transform $\mathscr{M}^{\Delta}$ is characterized by the following properties:

a) $\mathscr{M}^{\Delta}$ is a coherent quotient of the usual inverse image $\mathscr{M}'$.

b) The canonical morphism $\mathscr{M}' \to \mathscr{M}^{\Delta}$ is an isomorphism on $X'-Y' \simeq X-Y$.

c) $\mathrm{Ass}(\mathscr{M}^{\Delta}) \subset X'-Y'$ (EGA IV, 3.1.8).

Now, if $\mathscr{M}$ is S-flat, then $\mathscr{M}'$ is S'-flat. But, since Z' is a divisor of S', we have $\mathrm{Ass}(S') \subset S'-Z'$, and so $\mathrm{Ass}(\mathscr{M}') \subset X'-Y'$ (EGA IV, 3.3.1); hence $\mathscr{M}' = \mathscr{M}^{\Delta}$, and the pure transform of $\mathscr{M}$ coincides with the ordinary inverse image.

We shall prove the following result:

THEOREM 1. *Let $(S, X, \mathscr{M})$ be as before, and suppose that U is an open subscheme of S such that $\mathscr{M}|X\times_S U$ is U-flat. Then we can find a blowing up $S' \to S$, with center in $S-U$, such that the pure transform $\mathscr{M}^{\Delta}$ of $\mathscr{M}$ becomes S'-flat.*

2. Proof of the theorem in the projective case

Suppose further that X is projective over S. Then we shall see that we can find a canonical, projective morphism $S' \to S$, which is an isomorphism over U, in such a way that the pure transform $\mathscr{M}^{\Delta}$ of $\mathscr{M}$ by the morphism $S' \to S$ becomes S'-flat. The morphism $S' \to S$ is not necessarily isomorphic to any blowing up with center in $S-U$, but we can find a blowing up $S'' \to S'$, such that the composite morphism $S'' \to S$ is a blowing up with center in $S-U$; hence we get theorem 1 in that case.

For any S-scheme T, set $X_T = X\times_S T$, $\mathscr{M}_T = \mathscr{M}\times_S T$, and consider the set $Q(T)$ of isomorphism classes of coherent quotients $\mathscr{N}$ of $\mathscr{M}_T$ which are T-flat. We get, in a natural way, a contravariant functor

$$Q : (\mathrm{Sch}/S)^0 \to Ens$$
$$T \to Q(T).$$

Grothendieck has proved that the functor Q is represented by an S-scheme, which is a disjoint sum of projective S-schemes Q_i ([2]).

By hypothesis, $\mathscr{M}|X\times_S U$ is U-flat, hence defines a canonical point of $Q(U)$, i.e. an S-morphism $s : U \to Q$. Let S' be the schematic closure of $s(U)$ in Q. Then the projection $S' \to S$ is a projective morphism which induces an isomorphism over U. Let $X' = X\times_S S'$, $\mathscr{M}' = \mathscr{M}\times_S S'$. The S-morphism $S' \to Q$ corresponds to a point of $Q(S')$, hence to a coherent quotient $\overline{\mathscr{M}}'$ of $\mathscr{M}'$ which is S'-flat. Of course, the canonical morphism $\mathcal{M}' \to \bar{\mathcal{M}}'$ is an isomorphism over U. Moreover, since S' is the schematic closure of $s(U)$, we have $\mathrm{Ass}(S') \subset U$, and the flatness of $\bar{\mathcal{M}}'$ implies $\mathrm{Ass}(\bar{\mathcal{M}}') \subset X' \times_{S'} U$. Therefore, $\bar{\mathcal{M}}'$ is the pure transfoım of $\mathcal{M}$, and we are through.

3. Some indications on the proof of theorem 1

The proof proceeds by induction on $\dim(\mathcal{M}/S)$.

Definition 2. Let $(S, X, \mathcal{M})$ be as before. Let n be an integer, and F the closed set of points $x \in X$ such that $\mathcal{M}$ is not S-flat at x. We say that $\mathcal{M}$ is S-flat in dimension $\geqq n$ if $\dim(F/S) < n$. In fact, we shall prove the following refinement of theorem 1:

Theorem 1 bis. *Let $(S, X, \mathcal{M})$ be as before, U an open set of S, n an integer. Suppose that $\mathcal{M}|X \times_S U$ is U-flat in dimension $\geqq n$. Then we can find a blowing up $S' \to S$, with center in $S - U$, such that the pure transform $\mathcal{M}^{\Delta}$ of $\mathcal{M}$ becomes S'-flat in dimension $\geqq n$.*

Preliminary remarks: 1) Let S be a noetherian scheme, U an open set of S, $f: S' \to S$ a blowing up with center in $S - U$, $g: S'' \to S'$ a blowing up with center in $S' - f^{-1}(U)$. Then $fg : S'' \to S$ is a blowing up with center in $S - U$. Hence, to prove theorem 1 bis, we may proceed in several steps.

2) Let S be a noetherian scheme, U and V two open sets of S, $V' \to V$ a blowing up with center in $V - U \cap V$. Then there exists a blowing up $S' \to S$, with center in $S - U$, which extends $V' \to V$ (cf. EGA I, 9.4).

3) Let I and J be two ideals of a noetherian scheme S. Let $S' \to S$ be the blowing up of I, and $S'' \to S'$ the blowing up of $J0_{S'}$. Then $S'' \to S$ is the blowing up of the ideal IJ.

From these remarks we easily deduce that theorem 1 bis is of local nature on S and on X; hence we may assume S and X to be affine.

Then, after some technical reductions, and a suitable use of theorem 1, ch. I, we come to the most important step of the proof:

Proposition 1. *Let S be a noetherian, affine scheme, $X \to S$ a smooth morphism with geometrically irreducible fibres, $\mathcal{M}$ a coherent sheaf on X, U an open subscheme of S. Suppose that $\mathcal{M}$ is S-flat at the generic point of each fibre of X over U. Then, there exists a blowing up $S' \to S$, with center in $S - U$, such that the pure transform $\mathcal{M}^{\Delta}$ of $\mathcal{M}$ becomes S'-flat at the generic point of each fibre of the morphism $X' = X \times_S S' \to S'$.*

Proof: Let s be a point of S and η_s the generic point of $X \otimes_S k(s)$. Then we know (th.1, ch.I) that $\mathcal{M}$ is S-flat at the point η_s, if and only if the 0_X-module $\mathcal{M}$ is free at the point η_s. Hence, the hypothesis implies that there exists an open set V of X, which covers U, such that $\mathcal{M}|V$ is locally free. But the rank of the stalks of a locally free module, is locally constant, and the smooth morphism $V \to U$ is open; therefore we get a canonical splitting of $U = \coprod U_i$, $i \in N$ such that $\mathscr{M}$ is free of fixed rank i on $V \times_U U_i$.

Then, it is not difficult to see that we can find a blowing up $h : S' \to S$, with center in $S - U$, such that $S' = \coprod \overline{U}'_i$, where $\overline{U}'_i$ is the schematic closure of $h^{-1}(U_i)$ in S'. Hence, we are reduced to the case where $\mathscr{M}$ is of fixed rank r on V.

To conclude the proof, we shall use some elementary facts about Fitting ideals.

Fitting ideals of a module.

Let A be a noetherian ring, M and A-module of finite type, r an integer. Consider a presentation of M:

$$A^m \xrightarrow{u} A^n \to M \to 0$$

and the corresponding morphism of exterior powers

$$\bigwedge^{n-r}(u) : \bigwedge^{n-r}(A^m) \to \bigwedge^{n-r}(A^n)$$

Definition 3. The r-th Fitting ideal, $F_r(M)$, of M, is the ideal of A generated by the coordinates of the image of $\bigwedge^{n-r}(u)$ (i.e. if the vectors $(a_i) = (a_{i,j})$ $j = 1, \cdots, n$, $i = 1, \cdots, m$ are the images, by u, of the canonical basis of A^m, then $F_r(M)$ is generated by the minors of order $n-r$ of the matrix $(a_{i,j})$).

In fact, the definition of $F_r(M)$ does not depend on the presentation of M, and consequently extends to the case of a coherent sheaf $\mathscr{M}$ on a noetherian scheme S. It is clear that the formation of the Fitting ideal, commutes with a base change $S' \to S$. Furthermore, we have

$$\text{Support}\,(F_r(\mathscr{M})) = \{s \in S | \dim_{k(s)}(\mathscr{M} \otimes k(s)) \geqq r+1\}$$

Lemma 1. *Let M be an A-module of finite type, r an integer. Suppose that $F_r(M)$ is generated by an element, a, which is not a zero divisor in A, and suppose also that M is locally free of rank r on Spec$(A) - V(a)$. Let N be the submodule of M annihilated by a. Then M/N is locally free of rank r.*

Proof. Choose a presentation of M, as in definition 3. Then the minors of order $n-r$ of the matrix $(a_{i,j})$, generate the Fitting ideal (a). Hence, locally for the Zariski topology on Spec(A), and after a suitable permutation, we may assume that there exists a unit, h, of A, such that $\det(a_{ij}) = ah$ $(r+1 \leqq i, j \leqq n)$. Moreover, the other minors of order $n-r$ are multiples of a. Let $\{e_i\}_{i=1,\ldots,n}$ be the image in M of the canonical basis of A^n. Then, applying Cramer's rule, we get

$$ah\, e_i = a \sum_{j=1}^{r} b_{ij} e_j, \qquad i = r+1, \cdots, n.$$

Hence, locally, M/N is generated by r elements, and we can find an exact sequence

$$0 \to K \to A^r \to M/N \to 0$$

Since M is locally free of rank r on $\operatorname{Spec}(A) - V(a)$, K is killed by some power of a; as a is not a zero divisor, this implies $K = 0$.

We now return to the proof of proposition 1.

Let s be a point of S. We can find an open affine neighbourhood $U = \operatorname{Spec}(A)$ of s, and an affine open subscheme $W = \operatorname{Spec}(B)$ of X, which covers U, and such that B is a free A-module (ch. II, th. 1, cor. 1). We have noted that the proof of theorem 1 bis is of local nature on S; the same holds for proposition 1. Hence we may replace S by U, and X by W.

So assume that B is a free A-module, and choose a basis $\{e_i\}_{i\in I}$ for B over A. Consider the r-th Fitting ideal F of M; let $a_\lambda = \sum_i a_{i\lambda} e_i$, $\lambda \in \Lambda$, be a family of generators of F, and K the ideal of A generated by the family $\{a_{i\lambda}\}_{\substack{i\in I\\ \lambda\in\Lambda}}$. We shall see that we can take for S' the blowing up of K in $\operatorname{Spec}(A)$.

a) By assumption, M is locally free of rank r at the generic point of each fibre over U; thus $V(F)$ does not contain any fibre over U, and so $V(K)$ is contained in $S - U$. Hence $S' \to S$ is a blowing up with center in $S - U$.

b) Set $X' = X \times_S S'$, $M' = M \times_S S'$, $F' = F\mathcal{O}_{X'}$; $K' = K\mathcal{O}_{S'}$. Then K' is an invertible ideal. More precisely, let $S'_{i\lambda}$ be the greatest open subscheme of S' where the inverse image $a'_{i\lambda}$ of $a_{i\lambda}$ generates K'. Then $S'_{i\lambda}$ is affine, and the open sets $S'_{i\lambda}$ cover S'. Further, on $S'_{i\lambda}$ we have $a'_{j\mu} = a'_{i\lambda}\alpha_{j\mu}$, and the $\alpha_{j\mu}$ generate the unit ideal. Let a'_λ (resp. e'_j) be the inverse image of a_λ (resp. e_j) om X'. Then, over $S'_{i\lambda}$, we have $a'_\mu = a'_{i\lambda}(\sum \alpha_{j\mu} e'_j) = a'_{i\lambda} h'_\mu$.

Hence, over $S'_{i\lambda}$, the r-th Fitting ideal of M', F', is generated by the family $a'_{i\lambda} h'_\mu$. But, by construction, we have $\alpha_{i\lambda} = 1$; therefore, h'_λ cannot be identically zero on any fibre over $S'_{i\lambda}$, and, consequently, h'_λ is invertible on an open set V' of X' which covers $S'_{i\lambda}$. Thus, on V', F' is generated by $a'_{i\lambda}$. Applying lemma 1, we conclude that M^Δ is locally free, of rank r, on V'.

4. Applications

Let S be a noetherian scheme, and $X \to S$ a morphism of finite type.

Proposition 2. *Let r be an integer, and U an open set of S, such that* $dim(X\times_S U/U) \leqq r$. *Then there exists a blowing up* $S' \to S$, *with center in* $S-U$, *such that*

$$dim(X^{\Delta}/S') \leqq r.$$

Proof: Apply theorem 1 bis, with $\mathscr{M} = 0_X$ and $n = r+1$.

Proposition 3. *Suppose that* $X \to S$ *is separated and is an open immersion over an open subscheme U of S. Then there is a blowing up* $S' \to S$ *with center in* $S-U$, *such that the pure transform* X^{Δ} *of X is an open subscheme of* S'.

Proof: We first apply proposition 2 to reduce the case dim $(X/S) = 0$. Moreover, X is separated over S, and Ass $(X) \subset U$. We then apply the Main Theorem of Zariski to prove that X is an open subscheme of S.

Proposition 4. *Suppose that* $X \to S$ *is proper and is an isomorphism over U. Then we can find a commutative diagram*

$$\begin{array}{ccc} X' & \longrightarrow & X \\ & {}_{h}\searrow \quad \swarrow_{f} & \\ & S & \end{array}$$

where u (resp. h) is a blowing up with center in $X - f^{-1}(U)$ *(resp.* $S-U$*).*

Proposition 5 (**Chow's Lemma**). *Suppose that* $X \to S$ *is separated, and let U be an open subscheme of X which is quasi-projective over S. Then we can find a blowing up* $X' \to X$; *with center in* $X-U$, *such that* X' *is quasi-projective over S.*

Proof: By assumption, U is an open subscheme of a projective S-scheme Z. Let Γ be the schematic schematic closure in $X\times_S Z$ of the graph of the open immersion $U \to Z$. We get a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{p} & \Gamma \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q} \\ S & \xleftarrow[h]{} & Z \end{array}$$

where p is a projective morphism which is an isomorphism over U, and q is separated and is an isomorphism over U. We now apply proposition 3 to the morphism q: we can find a blowing up $Z' \to Z$ with center in $Z-U$, such that the pure transform $Y = \Gamma^{\Delta}$ is an open subscheme of Z', and so is quasi-projective over S. Then the composite morphism $Y \to \Gamma \to X$ is projective and is an isomorphism over U. We then apply proposition 4 to get a blowing up of X with center in $X-U$.

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