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HOMOTOPY NEGLIGIBLE SUBSETS OF BUNDLES

by

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1. Introduction

Recently Eells and Kuiper [2] have shown that for any ANR, locally homotopy negligible closed subsets are homotopy negligible. It is conceivable that such results will be applicable to manifolds, particularly of infinite-dimension. The purpose of this paper is to derive, in the setting of locally trivial bundle ,a condition that a closed subset of the total space will be locally homotopy negligible. A typical result is (for a more precise statement, see Theorem 3) that for a locally trivial bundle $\xi : X \to B$ with base space B and fibre F being manifolds modeled on a closed convex subset of some metric locally convex topological vector space (MLCTVS), a closed subset K of the total space X is locally homotopy negligible if the restriction of K to each fibre $\xi^{-1}(b)$ is locally homotopy negligible in $\xi^{-1}(b)$.

All topological vector spaces (TVS) *considered are metric locally convex topological vector spaces* (LCTVS).

We say that a subset A of a topological space X is: (*weakly*) *homotopy negligible* if the inclusion $X - A \to X$ is a (weak) homotopy equivalence; *locally homotopy negligible* if each point of A has a fundamental system of neighborhoods $\{U\}$ such that the inclusion $U - A \to U$ is a homotopy equivalence; *strongly homotopy negligible* if for any neighborhood U of X, the inclusion $U - A \to U$ is a homotopy equivalence; a *Z-set* (that is, a set with property Z, see Anderson [1]) if for each homotopically trivial non-empty open subset U of X, $U - A$ is non-empty and homotopically trivial.

We say X is a manifold modeled on a space C if X has an open covering by sets homeomorphic to open subsets of C. For our purpose, a *locally trivial bundle* has at least the following structure:

(1) a map $\xi : X \to B$, where X is called the total space, B the base space and $\xi : X \to B$ the projection,

(2) a space F called the fibre,

(3) for each $b \in B$, there is an open set U of b and a homeomorphism φ of $\xi^{-1}(U)$ onto $U \times F$ such that the following diagram commutes

$$\begin{tikzcd} \xi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\xi"'] & & U \times F \arrow[dl, "P_1 \text{ (Projection)}"] \\ & U & \end{tikzcd}$$

$\xi^{-1}(b)$ is called the fibre over b.

A map $f: X \times Y \to X \times Z$ is *X-preserving* if $f(x, y) = (x, z)$ for each $(x, y) \in X \times Y$.

Absolute retract (AR), absolute neighborhood retract (ANR) are metric spaces in the sense of Palais [3].

PROPOSITION 1. *Let X be a Hausdorff paracompact manifold modeled on a closed convex subset of a metrizable locally convex topological vector space* (LCTVS) *E. For a closed subset A of X, we have the following implication.*

$$\begin{tikzcd}[column sep=small] \text{(1) } A \textit{ is strongly homotopy negligible} \arrow[d, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\ \text{(2) } A \textit{ is locally homotopy negligible} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{(5) } A \textit{ is a Z-set} \\ \text{(3) } A \textit{ is homotopy negligible} \arrow[d, Leftrightarrow] & \\ \text{(4) } A \textit{ is weakly homotopy negligible.} & \end{tikzcd}$$

PROOF. $(1) \Rightarrow (2)$, $(1) \Rightarrow (5)$, $(5) \Rightarrow (2)$, $(3) \Rightarrow (4)$, $(1) \Rightarrow (3)$ are trivial. It follows from the hypothesis that X is a metrizable space which is locally an ANR, hence by [3–Thm. 5], X is an ANR. For an ANR, $(4) \Rightarrow (3)$ is a well-known theorem of J. H. C. Whitehead and $(2) \Rightarrow (1)$ is the content of the main lemma of Eells and Kuiper [2].

For application of homotopy negligible subsets to various manifolds and the relation of it with negligible subsets, we refer to [3] and [4].

2. Statement of the Theorems

Throughout this section, by a *manifold* we shall mean a paracompact Hausdorff space modeled on C, where C is a *closed convex* subset of a metric locally convex topological vector space (LCTVS) E. Let R denote the reals. The following examples of C are of interest.

(1) C = a closed convex subset of E

(2) $C = E$

(3) C = a compact convex subset of E

(4) $C = [-1, 1]^n$ for $n = 1, \cdots, \infty$

(5) $C = (-1, 1)^n$ for $n = 1, \cdots, \infty$.

We remark that C is a metrizable AR and X is a metrizable ANR.

The proofs of the following theorems (except for Theorem 3) will be given in section 4.

THEOREM 1. *Let X be a manifold and R' a closed interval in R. Let K be a closed subset of $R' \times X$ such that $K \cap t \times X$ is locally homotopy negligible in each $t \times X$, then K is strongly homotopy negligible in $R' \times X$.*

Furthermore, the same is true when 'locally homotopy negligible' is replaced by 'a Z-set'.

COROLLARY 1. *Let X be a Fréchet manifold and K a closed set in $(0, 1) \times X$ such that $K \cap t \times X$ is a Z-set in each $t \times X$. Then K is a Z-set in $(0, 1) \times X$.*

(See Theorem 4 for a stronger result.)

Thus we settle a question raised in [5–8].

THEOREM 2. *Let X_1, X_2 be manifolds and K a closed subset of $X_1 \times X_2$ such that $K \cap x \times X_2$ is locally homotopy negligible in $x \times X_2$ for each $x \in X_1$. Then K is strongly homotopy negligible in $X_1 \times X_2$.*

COROLLARY 2. *Let X be a manifold and K a closed subset of $s \times X$, where $s = (-1, 1)^\infty$, such that each $x \times X \cap K$ is a Z-set in $x \times X$. Then K is a Z-set in $s \times X$.*

The following theorem is a consequence of Theorem 2.

THEOREM 3. *Let $\xi : X \to B$ be a locally-trivial bundle with fibre F such that X is paracompact Hausdorff and both F and B are manifolds. Then a closed subset K of the total space X is strongly homotopy negligible in X if $K \cap \xi^{-1}(b)$ is locally homotopy negligible in each fibre $\xi^{-1}(b)$ over b.*

PROOF. It suffices to observe that X is locally a product of manifolds $M_1 \times M_2$, where M_1, M_2 are open subsets of B, F respectively, such that $K \cap (x \times M_2)$ is locally homotopy negligible in $x \times M_2$ for each $x \in M_1$. By Theorem 2, $K \cap M_1 \times M_2$ is strongly homotopy negligible in $M_1 \times M_2$. Then we observe that X is an ANR [3–Thm. 5] and $\{M_1 \times M_2\}$ form a neighborhood system of X. Now apply proposition 1.

THEOREM 4. *Let $I^n = [-1, 1]^n$ be the n-cube, $n = 1, 2, \cdots, \infty$, and let $s^n = (-1, 1)^n$ be the interior (or pseudo-interior in case of $n = \infty$) of I^n. Let X be a manifold. Then a closed subset K of $I^n \times X$ is strongly homotopy negligible in $I^n \times X$ if $K \cap (x \times X)$ is locally homotopy negligible in $x \times X$ for each $x \in s^n$.*

Furthermore, the same is true when we replace 'locally homotopy negligible' by 'a Z-set'.

THEOREM 5. *Let C_1, C_2 be spaces such that C_1 is a closed convex subset* *of a metric* LCTVS E_i. *Suppose* K *is a closed subset of* $C_1 \times C_2$ *such that* $K \cap (x \times C_2)$ *is weakly homotopy negligible in* $x \times C_2$ *for each* $x \in C_1$, *then* K *is homotopy negligible in* $C_1 \times C_2$.

COROLLARY 3. *Let* X *be a Fréchet space or the Hilbert cube, then* K *is homotopy negligible in* $R \times X$ *if* $K \cap t \times X$ *is ewakly homotopy negligible in each* $t \times X$.

THEOREM 6. *Let* $I^n = [0, 1]^n$ *and* $s^n = (0, 1)^n$, $n = 1, 2, \cdots, \infty$. *Let* C *be any closed convex subset of a* TVS E. *Suppose* K *is a closed subset of* $I^n \times C$ *such that for each* $x \in s^n$, $K \cap (x \times C)$ *is weakly homotopy negligible in* $x \times C$. *Then* K *is homotopy negligible in* $I^n \times C$.

COROLLARY 4. *Let* $X =$ *a Fréchet space or the Hilbert cube. Then a closed set* K *in* $[0, 1] \times X$ *is homotopy negligible in* $[0, 1] \times X$ *if* $K \cap (t \times X)$ *is weakly homotopy negligible in each* $t \times X$ *for* $0 < t < 1$.

Added in Proof. Since this paper was written, the author in [6–Cor 5] has generalized Theorem 3 to include all ANRs. Precisely, it states that Theorem 3 is true when we replace X by any metric ANR, F by any metric space which is locally an AR (or manifold) and B by any Hausdorff space. This result is based on a Lifting Theorem [6–Thm 8].

3. Lemmas

LEMMA 1. *Let* M_1, M_2 *be metric spaces and* Y *an absolute neighborhood retract* (ANR). *Let* K *be a closed subset of* $M_1 \times M_2$. *Suppose* $f : K \to M_1 \times Y$ *is a* M_1*-preserving map, then* f *can be extended to a* M_1*-preserving map of a neighborhood* U *of* K *into* $M_1 \times Y$.

PROOF. Let $P : M_1 \times Y \to Y$ be the projection. Then $Pf : K \to Y$ extends to a map g of a neighborhood U of K into Y. Define $F : U \to M_1 \times Y$ by $F(x_1, x_2) = (x_1, g(x_1, x_2))$. F is a desired extension.

Let E^n denote the Euclidean n-space $R_1 \times R_2 \times \cdots \times R_n$ where R_i is the real R. Let D^n, S^{n-1} denote respectively the n-ball and $(n-1)$-sphere of E^n. We regard E^n as the subset $E^n \times 0$ in E^{n+1}. Let $a < b < c$ and consider $[a, c]$ as a closed interval in R_{n+2}.

LEMMA 2. *Let* X *be an* ANR *and* K *a closed subset of* $R_{n+2} \times X$. *For* $n \geqq 0$, *and* $i = 1, 2$, *suppose*

$$f_i : I_i \times D^n \to R_{n+2} \times X - K,$$

where $I_1 = [b, c]$, $I_2 = [a, b]$, *are* R_{n+2}*-preserving maps such that* (1) $f_1|_{b \times S^{n-1}} = f_2|_{b \times S^{n-1}}$ *and* (2) $f_1|_{b \times D^n}$ *is homotopic to* $f_2|_{b \times D^n}$ *in* $b \times X - K$ *modulo* $b \times S^{n-1}$, *that is, the entire homotopy agrees with* f_1

on $b \times S^{n-1}$. *Then there is an* R_{n+2}-*preserving map* F *of* $[a, c] \times D^n$ *into* $R_{n+2} \times X - K$ *such that* $F|_{I_1 \times S^{n-1} \cup c \times D^n} = f_1$ *and* $F|_{I_2 \times S^{n-1} \cup a \times D^n} = f_2$.

PROOF. We note that for $n = 0$, $S^{n-1} = \emptyset$. Thus requirements on S^{n-1} become irrelevant and meaningless. With that modification in mind the following proof goes through even for this case.

We may assume $a = -1$, $b = 0$ and $c = 1$. Recall that $I_1 = [0, 1]$, $I_2 = [-1, 0]$ and we regard $[-1, 1]$ as a subset of R_{n+2}. Define for $i = 1, 2$, R_{n+2}-preserving maps

$$\varphi_i : I_i \times D^n \to I_i \times D^{n+1}$$

by

$$\varphi_1(t, x_1, \cdots, x_n) = (t, x_1, \cdots, x_n, (1-t)(1-\sum_{k=1}^{n} x_k^2)^{\frac{1}{2}})$$

and

$$\varphi_2(t, x_1, \cdots, x_n) = (t, x_1, \cdots, x_n, -(t+1)(1-\sum_{k=1}^{n} x_k^2)^{\frac{1}{2}}).$$

It is clear that

$$\begin{aligned}
&\varphi_1(0 \times D^n) = 0 \times S^n_+ , \\
&\varphi_2(0 \times D^n) = 0 \times S^n_- , \\
&\varphi_1|_{I_1 \times S^{n-1} \cup 1 \times D^n} = \text{identity and} \\
&\varphi_2|_{I_2 \times S^{n-1} \cup (-1) \times D^n} = \text{identity.}
\end{aligned}$$

Next define a map h_0 of $0 \times S^n$ into $0 \times X - K$ by

$$h_0(x) = \begin{cases} f_1 \varphi_1^{-1}(x) & x \in 0 \times S^n_+ \\ f_2 \varphi_2^{-1}(x) & x \in 0 \times S^n_- . \end{cases}$$

By (1) of the hypothesis f_1, f_2 agree on $0 \times S^{n-1}$. Hence h_0 is well-defined. By condition (2) of the hypothesis, h_0 can be extended to a map h of D^{n+1} into $0 \times X - K$. Let

$$A = D^{n+1} \cup \varphi_1([0, 1] \times D^n) \cup \varphi_2([-1, 0] \times D^n).$$

Define a R_{n+2}-preserving map $\theta : A \to [-1, 1] \times X - K$ by

$$\theta(x) = \begin{cases} h(x) & x \in D^{n+1} \\ f_1 \varphi_1^{-1}(x) & x \in \varphi_1([0, 1] \times D^n) \\ f_2 \varphi_2^{-1}(x) & x \in \varphi_2([-1, 0] \times D^n). \end{cases}$$

Since $R_{n+2} \times X - K$ is an ANR, by Lemma 1, θ can be extended to a R_{n+2}-preserving map θ_1 of a neighborhood U of A in E^{n+2} into $R_{n+2} \times X - K$.

It is elementary to know that there is a R_{n+2}-preserving map H of E^{n+2} onto itself such that (1) H is the identity outside of U, (2) H is one-to-one on $E^{n+2}-D^{n+1}$, (3) H collapses D^{n+1} onto D^n by $H(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_n, 0)$ and (4) $H|_{[-1,1]\times S^{n-1}\cup\{-1,1\}\times D^n} =$ identity.

We now define a R_{n+2}-preserving map $F: [-1, 1]\times D^n \to R_{n+2}\times X-K$ by

$$F(x) = \begin{cases} \theta_1 H\varphi_1(x) \text{ if } x\in[0, 1]\times D^n \\ \theta_1 H\varphi_2(x) \text{ if } x\in[-1, 0]\times D^n . \end{cases}$$

F is the desired function.

LEMMA 3. *Let E be a metric* TVS *and σ an n-simplex in $R\times E$. Let Y be a metric absolute retract* (AR) *and K a closed subset of $R\times Y$ such that $K\cap t\times Y$ is weakly homotopy negligible in $t\times Y$ for each t. Then any R-preserving map of $Bd(\sigma)$ into $R\times Y-K$ can be extended to an R-preserving map of σ into $R\times Y-K$.*

PROOF. It is known that a metric AR is contractible. It follows that $t\times Y-K$ is homotopically trivial for each t. Let $\dot\sigma =$ boundary σ, $\sigma(t) = \sigma\cap t\times E$ and $\dot\sigma(t) =$ boundary $\sigma(t)$. Let $P: R\times E\to R$ be the projection and let $[a, b] = P(\sigma)$, $a\leqq b$. If $a = b$, the lemma follows immediately from the hypothesis. So suppose $a < b$. By convexity each $\sigma(t)$ is a $(n-1)$-simplex for $a < t < b$.

Let $g: \dot\sigma\to R\times Y-K$ be a R-preserving map. At this point we divide the argument into two cases.

CASE 1. $n > 1$. By hypothesis of K, we have for each t, a map $g_t: \sigma(t)\to t\times Y-K$ such that $g_t|_{\dot\sigma(t)} = g$. For fixed t let $A = \dot\sigma\cup\sigma(t)$ and define a R-preserving map $G_t: A\to t\times Y-K$ by $G_t(x) = g(x)$ when $x\in\dot\sigma$ and $G_t(x) = g_t(x)$ when $x\in\sigma(t)$. Since $R\times Y-K$ is an ANR, by Lemma 1, G_t can be extended to a R-preserving map G'_t of a neighborhood U_t of A into $R\times Y-K$. Since $\sigma(t)$ is compact, there is an open interval (a_t, b_t) containing t such that $G'_t(\sigma(s))\cap K = \emptyset$ for all $a_t\leqq s\leqq b_t$. Since $[a, b]$ is compact, finite number of (a_t, b_t) covers $[a, b]$. It follows that there are a finite number of reals $a = a_0 < a_1 < \cdots < a_{m+1} = b$ and a collection of R-preserving maps $\{f_i\}_{i=0}^m$ such that for all i,

$$f_i: A_i\to R\times Y-K$$

where $A_i = \bigcup\{\sigma(t)|a_i\leqq t\leqq a_{i+1}\}$, $i = 0, 1, \cdots, m$, and

$$f_i|_{\dot\sigma} = g.$$

Since $g(\sigma(a)\cup\sigma(b))\cap K = \emptyset$, we may assume $f_0|_{\sigma(a_1)} = f_1|_{\sigma(a_1)}$ and $f_{m-1}|_{\sigma(a_m)} = f_m|_{\sigma(a_m)}$.

Consider f_1 and f_2. It is clear that A_1, A_2 can be regarded respectively as $[a_1, a_2]\times D^{n-1}$, $[a_2, a_3]\times D^{n-1}$. At the level $a_2\times D^{n-1}$ we have two maps $f_1, f_2 : a_2 \times D^{n-1} \to a_2 \times Y - K$ such that $f_1|_{a_2 \times S^{n-2}} = f_2|_{a_2 \times S^{n-2}} = g$. Since $a_2 \times Y - K$ is homotopically trivial, it follows that $f_1|_{a_2 \times D^{n-1}}$ is homotopic to $f_2|_{a_2 \times D^{n-1}}$ in $a_2 \times Y - K$ modulo $a_2 \times S^{n-2}$. f_1, f_2 satisfy all the conditions of Lemma 2. Hence there is an R-preserving map F_1 of $A_1 \cup A_2$ into $R \times Y - K$ such that $F_1|_{\dot{\sigma}} = g$ and $F_1 = f_1$ on $\sigma(a_1)$. Let $f_0' = f_0, f_1' = F_1|_{A_1}$ and $f_2' = F_1|_{A_2}$. Thus $f_0'|_{\sigma(a_1)} = f_1'|_{\sigma(a_1)}$ and $f_1'|_{\sigma(a_2)} = f_2'|_{\sigma(a_2)}$. Repeat the above process for f_2' and f_3. It follows that there are R-preserving maps f_2'' and f_3' defined respectively on A_2 and A_3 such that both agree with g on $\dot{\sigma}$, $f_2''|_{\sigma(a_2)} = f_1'|_{\sigma(a_2)}$, $f_2''|_{\sigma(a_3)} = f_3'|_{\sigma(a_3)}$ and so on. Define

$$G(x) = \begin{cases} f_0'(x) & x \in A_0 \\ f_1'(x) & x \in A_1 \\ f_2''(x) & x \in A_2. \end{cases}$$

Since $\{A_i\}$ is finite, it is clear that we can extend G to the entire $\sigma = A_0 \cup A_1 \cup \cdots \cup A_m$ with the required properties.

CASE 2. $n = 1$. For each $(t, y) \in R \times Y - K$, there is an open interval L_t containing t such that $L_t \times y \cap K = \emptyset$. Since the inclusion $t \times Y - K \to t \times Y$ is a homotopy equivalence, K cannot contain $t \times Y$. It follows that the set of all L_t covers $[a, b]$. By compactness, there are a finite number $a = a_0 < a_1 < \cdots < a_{m+1} = b$ and a finite subset $\{u_0, u_1, \cdots, u_m\}$ of Y such that for all i, $[a_i, a_{i+1}] \times u_i \cap K = \emptyset$, $(a_0, u_0) = g(\sigma(a))$ and $(a_{m+1}, u_m) = g(\sigma(b))$. Let $A_i = \bigcup \{\sigma(t) | a_i \leqq t \leqq a_{i+1}\}$, $i = 0, \cdots, m$ and define $f_i : A_i \to R \times Y - K$ by $f_i(\sigma(t)) = (t, u_i)$. We may assume $f_0|_{\sigma(a_1)} = f_1|_{\sigma(a_1)}$ and $f_{m-1}|_{\sigma(a_m)} = f_m|_{\sigma(a_m)}$. The rest of the proof is the same as Case 1.

We are now ready to state our main Lemma.

LEMMA 4. *Let E be a metric* TVS *and Y a metric* AR. *Let K be a closed subset of $R \times Y$ such that each $K \cap t \times Y$ is weakly homotopy negligible in $t \times Y$. Let $|T|$ be a finite simplicial complex in $R \times E$ and S a subcomplex of T. Then each R-preserving map of $|S|$ into $R \times Y - K$ can be extended to an R-preserving map of $|T|$ into $R \times Y - K$. Furthermore, the same is true when 'R' is replaced by 'an interval of R'.*

PROOF. Denote the i-skeleton of T by T_i (that is, all simplices of T of dimension $\leqq i$). Let f_0 be an R-preserving of $|S|$ into $R \times Y - K$. Since $t \times Y$ is not contained in K for any t, we may assume f_0 is an R-preserving map of $|S| \cup |T_0|$ into $R \times Y - K$. By Lemma 3, f_0 can be extended to an R-preserving map f_1 of $|S| \cup |T_1|$ into $R \times Y - K$. By Lemma 3 again we can extend f_1 to an R-preserving map f_2 of $|S| \cup T_2$ into $R \times Y - K$. Inductively, f_0 extends to an R-preserving map of $|T|$ into $R \times Y - K$.

4. Proofs

PROOF OF THEOREM 1. We need only to show (see Proposition 1) that K is locally homotopy negligible in $R' \times X$. Let U be a closed neighborhood of $x \in X$ such that there is a homeomorphism φ of U onto a closed neighborhood V of C. Let J be a closed interval of R'. It follows that $J \times U$ is contractible and we want to show $\pi_k(J \times U - K) = 0$ for all $k \geqq 1$. So suppose $f : (D^n, S^{n-1}) \to (J \times U, J \times U - K)$ is a map. Let ψ be the identity map of J onto J and let $\lambda = (\psi, \varphi) : J \times U \to J \times V$. Let $f_0 = \lambda \cdot f$ and $K_0 = \lambda(K \cap J \times U)$. K_0 is closed in $R \times V$ and $f_0 : (D^n, S^{n-1}) \to (J \times V, J \times V - K_0)$. $J \times V$ is a convex subset of $R \times E$. By standard argument it follows that f_0 is homotopic to a simplical map g of (D^n, S^{n-1}) into $(J \times V, J \times V - K_0)$ by maps $\{f_t\}_t$ such that each $f_t : D^n \to J \times V$, $f_1 = g$ and $f_t(S^{n-1}) \cap K_0 = \emptyset$ for all t. By hypothesis of K and Proposition 1, it follows that $K_0 \cap t \times V$ is homotopy negligible in $t \times V$ for each $x \in R$. Since V is an AR, by Lemma 4 there is a map μ of $g(D^n)$ into $J \times V - K_0$ such that $\mu|_{g(S^{n-1})}$. = identity. Let $h = \mu g$. Combining with the homotopy $\{f_t\}$ we therefore have shown that there is a map $F_0 : D^n \to J \times V - K_0$ such that $F_0|_{S^{n-1}} = f_0$. Let $F = \lambda^{-1} F_0$. Thus $F : D^n \to J \times U - K$ and $F|_{S^{n-1}} = f$. This shows $J \times U - K$ is weakly homotopy negligible. By locally convexity of E, $\{J \times U\}$ form a neighborhood system of $R' \times X$. The theorem now follows from Proposition 1.

PROOF OF THEOREM 2. Let U_1, U_2 be closed neighborhood of $x_1 \in X_1$, $x_2 \in X_2$ respectively such that for $i = 1, 2$, there are homeomorphisms φ_i of U_i onto a convex closed neighborhood V_i of C_i, where C_i is a closed convex subset of a TVS E_i. By local convexity of E_1 and E_2, $\{U_1 \times U_2\}$ form a neighborhood system of $X_1 \times X_2$, so by Proposition 1, we need only to show $\pi_k(U_1 \times U_2 - K) = 0$ for all $k \geqq 1$.

Let $f : (D^n, S^{n-1}) \to (U_1 \times U_2, U_1 \times U_2 - K)$ be a map and let $f_0 = \lambda f$, where $\lambda = (\varphi_1, \varphi_2)$. So $f_0 : (D^n, S^{n-1}) \to (V_1 \times V_2, V_1 \times V_2 - K_0)$ where $K_0 = \lambda(K \cap U_1 \times U_2)$. It suffices to show that there is a map $F_0 : D^n \to V_1 \times V_2 - K_0$ such that $F_0|_{S^{n-1}} = f_0$. As in Theorem 1, it is well-known that there is a homotopy $\{f_t\}$ of map of D^n into $V_1 \times V_2$ such that $f_t(S^{n-1}) \cap K_0 = \emptyset$ for all t and $g = f_1$ is simplicial. Let F be the finite-dimensional subspace of $E_1 \times E_2$ generated by $g(D^n)$. Let $F_1 = P_1(F)$ where $P_1 : E_1 \times E_2 \to E_1$ is the projection. K_0 is closed in $E_1 \times V_2$, by hypothesis and Proposition 1, $x \times V_2 \cap K_0$ is locally homotopy negligible in $x \times V_2$ for each $x \in E_1$. By Theorem 1 $K_0 \cap (J_1 \times V_2)$ is strongly homotopy negligible in $J_1 \times V_2$ for any 1-simplex J_1 of E_1. Since F_1 is finite-dimensional, say of dimension n, by induction, $K_0 \cap (J^m \times V_2)$ is strongly homotopy negligible in $J_1 \times V_2$ for any m-cube J^m in F_1. $K_1 = P_1 g(D^n)$ is an m-simplex of F_1, hence homeomorphism to some

m-cube J^m. It follows that $K_0 \cap (K_1 \times V_2)$ is strongly homotopy negligible in $K_1 \times V_2$, which contains $g(S^{n-1})$. Thus $g|_{S^{n-1}}$ can be extended to a g_1 of D^n into $(K_1 \times V_2) - K_0$. Combining with the homotopy $\{f_t\}$, we have shown that there is a map F_0 of D^n into $V_1 \times V_2 - K$ such that, $F_0|_{S^{n-1}} = f_0$. The proof of the theorem is now complete.

Proof of Theorem 4. Let U be a canonical closed neighborhood of I^n that is, U is a product of closed intervals. Let $U_1 = U - s^n$. Let V be a closed neighborhood of X such that there is a homeomorphism λ of V onto a closed convex subset V_1 of a TVS E. For simplicity we assume V_1 is V. It follows from the hypothesis and from Theorem 3 that $K \cap (U_1 \times V)$ is homotopy negligible in $U_1 \times V$. Now let $f: (D^n, S^{n-1}) \to (U \times V, U \times V - K)$ be a map. Since $f(S^{n-1}) \cap K = \emptyset$, let $\varepsilon > 0$ denote the distance between $f(S^{n-1})$ and K (distance in the sense of the usual induced metric on the product space $I^n \times X$). It is evident that there is a homotopy $\{g_t\}$ of maps of $U \cap s^n$ into U such that $g_0 =$ identity, $g_1(U \cap s^n) \subset U_1$ and $d(g_t, g_0) < \varepsilon$ for all t. Now define a homotopy $\{f_t\}$ of maps of $U \times V$ into itself by $f_t(x, y) = (g_t(x), y)$. It follows that $f_0 =$ identity, $f_1(U \times V) \subset U_1 \times V$ and $f_t(S^{n-1}) \cap K = \emptyset$ for all t. Let $h_t = f_t f$. Then $h_1 : (D^n, S^{n-1}) \to (U_1 \times V, U_1 \times V - K)$. Thus $h_1|_{S^{n-1}}$ extends to a map h of D^n into $U_1 \times V - K$. Combining with the homotopy $\{h_t\}$, we have shown that $f|_{S^{n-1}}$ extends to a map of D^n into $U \times V - K$. Thus $U \times V - K$ is homotopically trivial and the proof of the theorem is complete.

Proof of Theorem 5. By the Corollary of [3–Thm. 15] it suffices to show that $C_1 \times C_2 - K$ is homotopically trivial.

We first consider the special case that C_1 is a closed convex subset of the reals R. Let $f: (D^n, S^{n-1}) \to (C_1 \times C_2, C_1 \times C_2 - K)$ be a map. As illustrated in the proofs of the previous theorems (which are rather elementary) we may assume f is simplicial. By Lemma 4 there is a C_1-preserving map g of $f(D^n)$ into $C_1 \times C_2 - K$ such that $g|_{f(S^{n-1})} =$ identity. Hence $C_1 \times C_2 - K$ is homotopically trivial and the proof of the special case is complete.

Now consider the general case. The proof is very similar to that of Theorem 2. We proceed as follows. Let $f: (D^n, S^{n-1}) \to (C_1 \times C_2, C_1 \times C_2 - K)$ be a map. Again we may assume f is simplicial. Let F be the finite dimensional subspace of $E_1 \times E_2$ generated by $f(D^n)$. Let $F_1 = P_1(F)$ where $P_1 : E_1 \times E_2 \to E_1$ is the projection. By the special case proven above, $K \cap (J_1 \times C_2)$ is homotopy negligible in $J_1 \times C_2$ for each 1-simplex J_1 of E_1. Inductively, $K \cap (J^n \times C_2)$ is homotopy negligible in any n-cube of F_1. Since $K_1 = P_1(f(D^n))$ is an m-simplex, which is homeomorphic to an m-cube, it follows that $K \cap (K_1 \times C_2)$ is homotopy negligible in $K_1 \times C_2$, which contains $f(S^{n-1})$. Hence we may make the extension of $f|_{S^{n-1}}$ to D^n in $K_1 \times C_2 - K$ and the proof is complete.

PROOF OF THEOREM 6. The proof makes use of Theorem 5 and is exactly the same as that of Theorem 4.

REFERENCES

R. D. ANDERSON
[1] On topological infinite deficiency, Mich. Math. J. 14 (1967), 365–383.

J. EELLS, Jr. and N. H. KUIPER
[2] Homotopy negligible subsets, Compositio Math. 21 (1969) 155–161.

[3] R. S. PALAIS
Homotopy theory of infinite dimensional manifolds, Topology 5 (1966) 1–16.

R. D. ANDERSON, D. HENDERSON, and J. WEST
[4] Negligible subsets of infinite-dimensional manifolds, Compositio Math. 21 (1969) 143–150.

[5] Problems in Infinite-dimensional spaces and manifolds, Manuscript, LSU, Baton Rouge, La. 1969.

[6] On homeomorphisms of infinite-dimensional bundles, I, to appear.

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