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THE RADIUS OF UNIVALENCE AND STARLIKENESS OF SOME CLASSES OF REGULAR FUNCTIONS

by

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1.

The writing of this paper has been motivated by recent results of A. E. Livingston [7] and S. D. Bernardi [2].

Let S denote the class of functions $f(z) = z+\sum_2^\infty a_n z^n$ which are regular and univalent in $E\{z : |z| < 1\}$ and which map E onto domains $D(f)$. We denote by S^* and K the subclasses of S where $D(f)$ are, respectively, starlike with respect to the origin, and convex. Let P denote the class of functions $p(z)$ which are regular and satisfy $p(0) = 1$, $\operatorname{Re}(p(z)) > 0$, for z in E. In [1] the following theorem was proven.

THEOREM A. *Let $f(z) = z+\sum_2^\infty a_n z^n$ be a member of S^*. Then*

$$F(z) = (c+1)z^{-c}\int_0^z t^{c-1}f(t)dt = z+\sum_2^\infty \left(\frac{c+1}{c+n}\right) a_n z^n \tag{1.1}$$

is also a member of the same class for $c = 1, 2, 3, \cdots$.

Theorem A represents a generalization of the corresponding theorem by R. J. Libera [6] for the case $c = 1$. Solving the relation (1.1) for the inverse function $f(z)$, we have

$$f(z) = \left(\frac{1}{1+c}\right) z^{1-c}[z^c F(z)]'. \tag{1.2}$$

In [2] S. D. Bernardi proved that if $F(z) \in S^*$, then $f(z)$, defined by (1.2), is univalent and starlike for $|z| < r_0$, where

$$r_0 = [-2+(3+c^2)^{\frac{1}{2}}]/(c-1) \text{ for } c = 2, 3, 4, \cdots.$$

This result is sharp. For $c = 1$, $r_0 = \frac{1}{2}$ and this result is due to A. E. Livingston [7].

In this paper we determine the radius of univalence and starlikeness of functions $f(z) = z+a_2z^2+\cdots$ which are regular in E and satisfy

$$F(z) = \frac{2}{z}\int_0^z \frac{f(t)g(t)}{t}\, dt, \tag{1.3}$$

where $F(z) \in S^*$ and (i) $g(z) \in K$, (ii) $g(z) \in S$ and (iii) $g(z)/z \in P$. We shall employ the same techniques used in [7].

2.

THEOREM 1. *If $f(z)$ is regular in E and satisfies* (1.3), *where $F(z) \in S^*$ and $g(z) \in K$, then $f(z)$ is univalent and starlike for $|z| < 2-\sqrt{3}$. This result is sharp.*

PROOF. Since F is in S^*, $\operatorname{Re}\{zF'(z)/F(z)\} > 0$ for all z in E. Hence there exists w, regular in E, such that $|w(z)| \leqq 1$ for z in E and such that

$$\frac{f(z)g(z) - \int_0^z \frac{f(t)g(t)dt}{t}}{\int_0^z \frac{f(t)g(t)dt}{t}} = \frac{zF'(z)}{F(z)} = \frac{1-zw(z)}{1+zw(z)}.$$

Thus

$$f(z)g(z) = \frac{2}{1+zw(z)} \int_0^z \frac{f(t)g(t)dt}{t},$$

and

$$\frac{zf'(z)}{f(z)} = \frac{2-zw(z)-z^2w'(z)}{1+zw(z)} - \frac{zg'(z)}{g(z)}.$$

Therefore

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq \operatorname{Re}\left\{\frac{2-zw(z)-z^2w'(z)}{1+zw(z)}\right\} - \left|\frac{zg'(z)}{g(z)}\right|. \tag{2.1}$$

Thus $f(z)$ will be univalent and starlike for those values of z for which the right-hand side of (2.1) is positive. Since $g(z)$ is in K,

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{1}{1-|z|} \qquad [4, \text{p. } 13].$$

Therefore the right-hand side of (2.1) will be positive if

$$\operatorname{Re}\left\{\frac{2-zw(z)-z^2w'(z)}{1+zw(z)}\right\} - \frac{1}{1-|z|} > 0. \tag{2.2}$$

Condition (2.2) is equivalent to

$$\operatorname{Re}\{[1+(|z|-2)zw(z)+(|z|-1)z^2w'(z)-2|z|][1+\overline{zw(z)}]\} > 0, \tag{2.3}$$

or

$$\operatorname{Re}\{(1-|z|)z^2w'(z)[1+\overline{zw(z)}]\} < \{1-(1+|z|)\operatorname{Re}(zw(z)) -2|z|+(-2+|z|)|z|^2|w(z)|^2\}. \tag{2.4}$$

Using the well known result

$$|w'(z)| \leqq \frac{1-|w(z)|^2}{1-|z|^2} \qquad [3, \text{p. } 18]$$

and the fact that $\operatorname{Re}(zw(z)) \leqq |z||w(z)|$, we see that (2.4) will be satisfied if

$$(2.5) \qquad (1-|z|)|z|^2 \frac{(1-|w(z)|^2}{1-|z|^2}(1+|z||w(z)|) < \{1-(1+|z|)|w(z)||z|-2|z|+(|z|-2)|z|^2|w(z)|^2\}.$$

Since $|w(z)| \leqq 1$, $(1+|z||w(z)|)/(1+|z|) \leqq 1$ and (2.5) will be satisfied if

$$|z|^2(1-|w(z)|^2) < 1-2|z|-(1+|z|)|z||w(z)|+(|z|-2)|z|^2|w(z)|^2$$

which is equivalent to

$$(2.6) \qquad (|z|^2+2|z|)+(1-|z|)|z|^2|w(z)|^2+(1+|z|)|z||w(z)| < 1.$$

Hence, it suffices to show that (2.6) holds for all functions w, regular in E and satisfying $|w(z)| \leqq 1$, provided $|z| < 2-\sqrt{3}$.

In (2.6) put $a = |z|$, $x = |w(z)|$ and consider the function

$$p(x) = a^2+2a+a(1+a)x+a^2(1-a)x^2.$$

Clearly, $p(x)$ is increasing in $[0, 1]$ and $p(1) = 3a+3a^2-a^3$ is less than one for $0 \leqq a < 2-\sqrt{3}$. Condition (2.2) is thus seen to be satisfied if $|z| < 2-\sqrt{3}$. Hence $f(z)$ is univalent and starlike for $|z| < 2-\sqrt{3}$.

To see that the result is sharp, let $F(z) = z/(1-z)^2$ and $g(z) = z/(1+z)$. Then $F(z)$ is in S^*, $g(z)$ is in K and $f(z) = (z^2+z)/(1-z)^3$. Thus $f'(z) = (z^2+4z+1)/(1-z)^4$ and $f'(-2+\sqrt{3}) = 0$. Hence $f(z)$ is not univalent in $|z| < r$ if $r > 2-\sqrt{3}$.

THEOREM 2. *If $f(z)$ is regular in E and satisfies* (1.3), *where $F(z) \in S^*$ and $g(z) \in S$, then $f(z)$ is univalent and starlike for $|z| < \frac{1}{5}$. This result is sharp.*

The proof of this theorem is similar to that of Theorem 1. The only essential difference is the estimate

$$\left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{1+|z|}{1-|z|} \qquad [4, \text{p. } 5].$$

To see that the result is sharp, let $F(z) = z/(1-z)^2$ and $g(z) = z/(1+z)^2$. Then $f(z)g(z) = z^2/(1-z)^3$ and we have

$$\frac{zf'(z)}{f(z)} = \frac{2+z}{1-z} - \frac{zg'(z)}{g(z)} = \frac{2+z}{1-z} - \frac{1-z}{1+z} = \frac{1+5z}{1-z^2} = 0$$

for $z = -\frac{1}{5}$. Thus $f(z)$ is not starlike in $|z| < r$ if $r > \frac{1}{5}$.

REMARK. The above example shows that we cannot improve on the result of Theorem 2 if instead of $g(z)$ in S we assume $g(z)$ in S^*.

THEOREM 3. *If $f(z)$ is regular in E and satisfies* (1.3), *where $F(z) \in S^*$ and $g(z)/z \in P$, then $f(z)$ is univalent and starlike for $|z| < (5-\sqrt{17})/4$. This result is sharp.*

PROOF. Let $h(z) = g(z)/z$. Then

$$\frac{zf'(z)}{f(z)} = \frac{1-2zw(z)-z^2w'(z)}{1+zw(z)} - \frac{zh'(z)}{h(z)},$$

where $w(z)$ is regular in E and $|w(z)| \leqq 1$. Using the estimate

$$\left|\frac{zh'(z)}{h(z)}\right| \leqq \frac{2|z|}{1-|z|^2} \qquad [5]$$

and the techniques of Theorem 1, the result follows.

To see that the result is sharp, let $F(z) = z/(1-z)^2$ and $g(z) = z(1-z)/(1+z)$. Then $f(z)h(z) = z/(1-z)^3$, $f(z) = (z^2+z)/(1-z)^4$ and $f'(z) = (2z^2+5z+1)/(1-z)^5 = 0$ for $z = (-5+\sqrt{17})/4$. Hence $f(z)$ is not univalent in any disk $|z| < r$ if $r + (5-\sqrt{17})/4$.

REFERENCES

S. D. BERNARDI
[1] Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969) 429–446.

S. D. BERNARDI
[2] The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 24 (1970) 312–318.

C. CARATHEODORY
[3] *Theory of Functions*, Vol. 2, Chelsea, New York, 1954.

W. K. HAYMAN
[4] *Multivalent Functions*, Cambridge Univ. Press, 1958.

T. H. MACGREGOR
[5] The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 14 (1965) 514–520.

R. J. LIBERA
[6] Some classes of regular functions, Proc. Amer. Math. Soc. 16 (1965) 755–758.

A. E. LIVINGSTON
[7] On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966) 352–357.

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