I. M. JAMES

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PRODUCTS BETWEEN HOMOTOPY GROUPS

by

I. M. James

This note is mainly concerned with the mixed product introduced by McCarty [13] and its relation to various other constructions such as the Whitehead product and the intrinsic join of [7]. Basically the mixed product is part of the theory of the Hopf construction, studied by G. W. Whitehead in [17]. However, the special case treated by McCarty, which we call the relative Samelson product, has desirable formal properties, and to some extent appears to be susceptible to calculation. We describe this in § 1 and then, using fibre bundle theory, we show in § 2 how the relative Samelson product can be characterized in terms of the Whitehead product. This enables us, in § 3, to give a description of the Whitehead product structure for the total space of a fibre bundle with cross-section. It also enables us, in § 4, to deduce relations satisfied by the relative Samelson product, such as the Jacobi identity, from the corresponding results for the Whitehead product. McCarty [14] has described a different way of doing this. The results established here are needed for a study [9] of the Bott suspension and for some work [8], [11] on certain higher obstructions to the existence of cross-sections. Examples and applications of a different type are given in the last two sections of the present paper.

We work throughout in the category of pointed spaces. The basepoint of any space is denoted by e. In the case of a topological group this is the neutral element. Unless otherwise stated all maps and homotopies are basepoint-preserving.

1. The mixed product

Let $u : X \to Y$ be a map, where X, Y are spaces. Let A be a space and let $\mu : A \times X \to Y$ be a map such that

$$(1.1) \qquad \begin{cases} \mu(a, e) = e & (a \in A), \\ \mu(e, x) = ux & (x \in X). \end{cases}$$

Then a product of $\pi_p(A)$ with $\pi_q(X)$ to $\pi_{p+q}(Y)$ is defined as follows, for $p, q \geqq 1$. Let $f : S^p \to A$, $g : S^q \to X$ represent $\alpha \in \pi_p(A)$, $\beta \in \pi_q(X)$, respectively. Then $\mu(f\times g) : S^p \times S^q \to Y$ satisfies (1.1), with u replaced by ug. Hence $\mu(f\times g)$ agrees with $u\rho$ on the subspace

$$S^p \vee S^q = S^p \times e \cup e \times S^q \subset S^p \times S^q,$$

where $\rho : S^p \times S^q \to S^q$ denotes the right projection. Thus the separation element

$$d(\mu(f\times g), u\rho) \in \pi_{p+q}(Y)$$

is defined, in the usual way. We denote this element by $\langle \alpha, \beta \rangle$ and refer to it as the *mixed product* [1] associated with the given map μ. It is also convenient, for formal reasons, to define

$$\text{(1.2)} \qquad \langle \beta, \alpha \rangle = (-1)^{pq+1} \langle \alpha, \beta \rangle.$$

The basic properties of this type of construction have been established by G. W. Whitehead in [17]. We recall some of these for use later. Each element of A determines, through μ, a map of X into Y. If we suspend this map and reverse the process we obtain a map $\nu : A \times SX \to SY$, satisfying (1.1) with u replaced by $Su : SX \to SY$. It is shown [2] in (3.10) of [17] that

$$\text{(1.3)} \qquad S_* \langle \alpha, \beta \rangle = -\langle \alpha, S_* \beta \rangle,$$

where S_* denotes the Freudenthal suspension. In other words the following diagram is anti-commutative, where the upper row is given by the mixed product associated with μ and the lower row by the mixed product associated with ν.

$$\begin{array}{ccc} \pi_p(A) \times \pi_q(X) & \longrightarrow & \pi_{p+q}(Y) \\ \downarrow{\scriptstyle 1 \times S_*} & & \downarrow{\scriptstyle S_*} \\ \pi_p(A) \times \pi_{q+1}(SX) & \longrightarrow & \pi_{p+q+1}(SY) \end{array}$$

There is also a simple relationship between the mixed product and various binary functors in the category of pointed spaces. For example, consider the join functor, which we denote by $*$ in the usual way. Let Z be a space. Each element of A determines, through μ, a map of X into Y. If we take the join of this map with the identity on Z then, reversing the process, we obtain a map

$$\nu : A \times (X * Z) \to (Y * Z),$$

[1] The terminology we adopt is slightly different from McCarty's, but this is unlikely to cause any confusion.

[2] All we require, in the way of sign conventions, is a system where this relation is valid. The situation is discussed in [19].

which satisfies (1.1) with u replaced by $u * 1 : X * Z \to Y * Z$. The mixed products associated with these two maps satisfy the relation

$$(1.4) \qquad \langle \alpha, \beta \rangle * \gamma = (-1)^{r+1} \langle \alpha, \beta * \gamma \rangle,$$

where $\gamma \in \pi_r(Z)$. To see this we first observe that it is sufficient, by naturality, to establish the relation in case $Z = S^r$, with γ the class of the identity. Then we recall that the join operation in this case can be identified with the $(r+1)$-fold suspension, and hence (1.4) follows from (1.3) by induction.

Most examples of the mixed product have $X = Y$ and u the identity. This is the case, in particular, when $A = X = Y = H$, where H is a topological group and where $\mu : H \times H \to H$ is defined by

$$\mu(x, y) = x^{-1}yx \qquad (x, y \in H).$$

It is easy to check that the mixed product thus defined is equivalent to the Samelson (commutator) product in $\pi_*(H)$.

However the main example, for what follows, arises when H is a subgroup of a topological group G. We take $A = H$ and take $X = Y = G/H$, the factor space of right cosets. We assume that the natural projection of G onto G/H admits a local cross-section, and hence constitutes a fibration. The action of H on G by left translation determines a map

$$\mu : H \times G/H \to G/H$$

which satisfies (1.1) with u the identity. This action of H on G/H we shall refer to as the *standard action*. The associated mixed product of $\pi_p(H)$ with $\pi_q(G/H)$ to $\pi_{p+q}(G/H)$ we shall refer to as the *relative Samelson product*. Notice that an element $\alpha \in \pi_p(H)$ operates, by a Samelson product, on each term of the exact sequence

$$\cdots \to \pi_*(H) \to \pi_*(G) \to \pi_*(G/H) \to \cdots$$

associated with the natural fibration of G over G/H. On the first term shown the operation is given by the ordinary Samelson product with the given element α. On the second term the operation is given by the ordinary Samelson product with $\sigma_* \alpha \in \pi_p(G)$, where $\sigma : H \subset G$. On the third term the operation is given by the relative Samelson product with α, as defined above. It is a straightforward exercise to show that these operations, apart from sign, commute with the operators of the exact sequence.

The calculation of relative Samelson products provides many interesting problems. We have, for a start, the results on the J-homomorphism, which can be regarded as a special case, with G/H a sphere. Other results can be obtained, as in § 7 of [9], by exploiting such relations as the

Jacobi identity (see § 4 below). Again, if an ordinary Samelson product can be calculated then, through the exact sequence mentioned above, it may be possible to deduce a result on the relative Samelson product.

More general results can be deduced from special cases such as these through the following useful relation between mixed products and the intrinsic join construction of [7]. To state this, consider the group O_m ($m = 1, 2, \cdots$) of orthogonal transformations in euclidean m-space. We make the usual embeddings

$$O_1 \subset O_2 \subset \cdots \subset O_m \subset \cdots.$$

We recall that the factor space $V_{m,k} = O_m/O_{m-k}$ can be identified with the Stiefel manifold of orthonormal k-frames in m-space. Through this identification the standard action of O_{m-k} on $V_{m,k}$ is given by

$$\mu(a, (u_1, \cdots, u_k)) = (a \cdot u_1, \cdots, a \cdot u_k),$$

where $a \in O_{m-k}$ and where $(u_1, \cdots, u_k)$ is an orthonormal k-frame.

Next we recall that the intrinsic map

$$h : V_{m,k} * V_{n,k} \to V_{m+n,k}$$

can be defined as follows. Let

$$u = (u_1, \cdots, u_k), \; v = (v_1, \cdots, v_k)$$

be orthonormal k-frames in m-space, n-space respectively. Then

$$h(u, t, v) = (w_1, \cdots, w_k),$$

the orthonormal k-frame in $(m+n)$-space given by

$$w_r = \left(u_r \cos \frac{\pi}{2} t, v_r \sin \frac{\pi}{2} t\right) \qquad (1 \leqq r \leqq k).$$

Consider the standard action μ of O_{m-k} on $V_{m,k}$, and the standard action λ of O_{m-k} (through O_{m+n-k}) on $V_{m+n,k}$. If we apply to μ the construction described above in our general discussion of the join operation, we obtain an action ν which makes the following diagram commutative.

$$\begin{array}{ccc} O_{m-k} \times (V_{m,k} * V_{n,k}) & \xrightarrow{\nu} & V_{m,k} * V_{n,k} \\ \Big\downarrow{\scriptstyle 1 \times h} & & \Big\downarrow{\scriptstyle h} \\ O_{m-k} \times (V_{m+n,k}) & \xrightarrow[\lambda]{} & V_{m+n,k} \end{array}$$

Now the ordinary join operation defines a pairing of $\pi_*(V_{m,k})$ with $\pi_*(V_{n,k})$ to $\pi_*(V_{m,k} * V_{n,k})$. By composing this with the homomorphism

h_*, induced by the intrinsic map, we obtain a pairing of the same groups with values in $\pi_*(V_{m+n,k})$. Since the above diagram is commutative it follows immediately from (1.4) by naturality that

$$(1.5) \qquad \langle \alpha, \beta \rangle * \gamma = (-1)^{r+1} \langle \alpha, \beta * \gamma \rangle,$$

where $\alpha \in \pi_p(0_{m-k})$, $\beta \in \pi_q(V_{m,k})$, $\gamma \in \pi_r(V_{n,k})$, and where $*$ denotes the intrinsic join on both sides. A similar relation holds if we replace the orthogonal group by the unitary or symplectic group, and replace the real Stiefel manifold by the complex or quaternionic Stiefel manifold, as appropriate.

2. A relation with the Whitehead product

As before, let G be a topological group with subgroup H and factor space $G/H = Y$, say. Consider the standard action of H on Y. This is a pointed action (see § 1 of [10]), by (1.1), and so every H-bundle with fibre Y obtains a canonical cross-section, as shown in § 9 of [15]. Classes of H-bundles over S^n, we recall, correspond to elements of $\pi_{n-1}(H)$. Consider the bundle E with fibre Y and base S^n which corresponds to an element $\alpha \in \pi_{n-1}(H)$. Let

$$i_* : \pi_*(Y) \to \pi_*(E)$$

be induced by the inclusion $i : Y \to E$. Let $\xi \in \pi_n(E)$ denote the class of the canonical cross-section. I assert that, for any element $\beta \in \pi_q(Y)$, we have

$$(2.1) \qquad i_* \langle \alpha, \beta \rangle = [\xi, i_* \beta]$$

where the brackets on the left denote the relative Samelson product associated with the standard action of H on G/H and those on the right denote the Whitehead product in $\pi_*(E)$. Note that (2.1) characterizes the relative Samelson product since the existence of a cross-section implies that i_* is injective. Various consequences of this basic relation will be obtained in the next section, after we have given the proof.

We begin by recalling how the Hurewicz isomorphism is defined. Given any space Z let $\Omega^q_* Z$ denote the function-space of *free* maps of S^q into Z. Consider the fibration

$$r : \Omega^q_* Z \to Z$$

defined by evaluation at the basepoint of S^q. The fibre over the basepoint of Z is the space $\Omega^q Z$ of pointed maps of S^q into Z. We choose one pointed map $u : S^q \to Z$ to be the basepoint in the function-space. Now the adjoint of a (basepoint-preserving) map $f' : S^p \to \Omega^q Z$ is a map $f: S^p \times S^q \to Z$, satisfying (1.1). The Hurewicz isomorphism

$$\psi : \pi_p(\Omega^q Z) \to \pi_{p+q}(Z)$$

is given in terms of the separation element by

$$\psi\{f'\} = d(f, u\rho), \tag{2.2}$$

where ρ denotes the right projection.

Next we construct a model for E by the method described in § 3 of [12]. Choose a map $v : S^{n-1} \to H$, representing α, and let f denote the composition

$$S^{n-1} \times Y \xrightarrow{v \times 1} H \times Y \xrightarrow{\mu} Y.$$

Choose a relative homeomorphism

$$k : (D^n, S^{n-1}) \to (S^n, e),$$

where D^n is an n-ball with boundary S^{n-1}. The induced bundle over D^n is trivial, since the ball is contractible, and moreover the trivialization can be chosen so that a relative homeomorphism

$$h : (D^n \times Y, S^{n-1} \times Y) \to (E, Y)$$

is obtained which agrees with f on $S^{n-1} \times Y$.

We take a representative of the given element $\beta \in \pi_q(Y)$ as basepoint in $\Omega^q Y$ and consider the following diagram, where i_* denotes the injection and δ the boundary operator in the homotopy sequence of the pair.

$$\begin{tikzcd}
\pi_n(\Omega^q_* E, \Omega^q Y) \arrow[r, "\delta"] \arrow[d, "i_*"] & \pi_{n-1}(\Omega^q Y) \arrow[r, "\psi"] \arrow[d, "i_*"] & \pi_{n+q-1}(Y) \arrow[d, "i_*"] \\
\pi_n(\Omega^q_* E, \Omega^q E) \arrow[r, "\delta"'] & \pi_{n-1}(\Omega^q E) \arrow[r, "\psi"'] & \pi_{n+q-1}(E)
\end{tikzcd}$$

Since $\Omega^q E$ is the fibre of the fibration $r : \Omega^q_* E \to E$ we have an isomorphism

$$r_* : \pi_n(\Omega^q_* E, \Omega^q E) \approx \pi_n(E).$$

We recall from (3.2) of [17] that

$$[r_* \eta, i_* \beta] = \psi\delta(\eta), \tag{2.3}$$

for any $\eta \in \pi_n(\Omega^q_* E, \Omega^q E)$.

Since h is induced by k it follows that $rh' = sk$, as shown below, where h' denotes the adjoint of h, and where r is given by the evaluation map and s by the canonical cross-section.

$$\begin{array}{ccc} (D^n, S^{n-1}) & \xrightarrow{h'} & (\Omega_*^q E, \Omega^q Y) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle r} \\ (S^n, e) & \xrightarrow[s]{} & (E, e) \end{array}$$

Hence $r_* i_* \zeta = \xi$, the class of the canonical cross-section, where $\zeta \in \pi_n(\Omega_*^q E, \Omega^q Y)$ denotes the class of h'. Take $\eta = i_* \zeta$, in (2.3), and we obtain

$$[r_* i_* \zeta, i_* \beta] = \psi \delta i_*(\zeta) = i_* \psi \delta(\zeta),$$

by commutativity of the earlier diagram. Thus we conclude that

$$(2.4) \qquad [\xi, i_* \beta] = i_* \psi \delta(\zeta).$$

Since $h|S^{n-1} \times Y$ is given by f, as described above, it follows at once that $h'|S^{n-1}$ is given by

$$f^1 : S^{n-1} \to \Omega^q Y,$$

the adjoint of f. Thus $\delta\zeta$ is represented by f' and so $\psi\delta(\zeta) = d(f, u\rho)$, by (2.2). But $f = \mu(v \times 1)$, by definition, where $v : S^{n-1} \to H$ represents α and μ is the standard action. Hence $d(f, u\rho) = \langle \alpha, \beta \rangle$, by definition of the mixed product, and so $\psi\delta(\zeta) = \langle \alpha, \beta \rangle$. Combining this with (2.4) we obtain (2.1) as asserted.

3. Some consequences of (2.1)

Let (G, H) be as before. We use the standard notation for classifying spaces. We regard B_H as an H-bundle over B_G with fibre $Y = G/H$ and projection $p : B_H \to B_G$. If B is a space and $h : B \to B_H$ is a map let $h^\# B_H$ denote the H-bundle over B with fibre Y induced by $ph : B \to B_G$. The canonical cross-section of $h^\# B_H$ is determined by the pull-back of ph to h. The class of $h^\# B_H$, as a bundle with cross-section, depends only on the homotopy class θ, say, of h, and will therefore be denoted by $\theta^\# B_H$. When B is a finite complex any H-bundle over B with fibre Y is equivalent to a bundle of the form $h^\# B_H$.

Write $E = h^\# B_H$ and consider the homomorphisms

$$\pi_*(Y) \underset{i_*}{\to} \pi_*(E) \overset{s_*}{\underset{p_*}{\leftrightarrows}} \pi_*(B),$$

where i is the inclusion, p is the projection and s is the canonical cross-section. The transgression homomorphism in the homotopy sequence of the principal bundle associated with E is given by the composition

$$\pi_n(B) \underset{h_*}{\to} \pi_n(B_H) \underset{\Delta}{\to} \pi_{n-1}(H),$$

where Δ denotes the transgression isomorphism in the homotopy sequence of the universal H-bundle. Given an element $\xi \in \pi_n(B)$ we can apply (2.1) to the induced bundle ξ^*E over S^n. Hence if $\beta \in \pi_q(Y)$ we obtain the relation

$$i_*\langle \Delta h_* \xi, \beta\rangle = [s_* \xi, i_* \beta], \tag{3.1}$$

by naturality. Using (1.2) this transforms into the relation

$$i_*\langle \beta, \Delta h_* \xi\rangle = (-1)^{q+1}[i_* \beta, s_* \xi], \tag{3.2}$$

which it is convenient to have explicitly stated.

To clarify the significance of these relations consider the direct-sum decomposition

$$\pi_r(Y) \oplus \pi_r(B) \approx \pi_r(E) \qquad (r \geqq 2),$$

given by i_* on the first summand and s_* on the second. Notice that i_* and s_*, being induced homomorphisms, respect the Whitehead product. Thus the Whitehead products in $\pi_*(E)$ can be expressed in terms of the Whitehead products in $\pi_*(Y)$ and $\pi_*(B)$ together with the relative Samelson product as above. Of course the situation in the fundamental group is another matter.

Let ΣE denote the bundle over B with fibre SY formed by taking the double mapping cylinder of the projection $p : E \to B$. By combining (1.3) with (3.1) we obtain relations between the Whitehead products in $\pi_*(E)$ and $\pi_*(\Sigma E)$.

Let V be a euclidean bundle and let W denote the Whitney sum of V and a trivial bundle of any dimension. Consider the associated bundles V_k, W_k of orthonormal k-frames. If V_k admits a cross-section then, through (1.5) and (3.1), we obtain relations between the Whitehead products in $\pi_*(V_k)$ and $\pi_*(W_k)$. Such products are of special interest since (see [11]) they appear as higher order obstructions to the extension and classification of mappings between fibre spaces.

Various properties of the mixed product, such as bilinearity, can be established from the definition without difficulty. However it is often convenient to exploit the known properties of the Whitehead product, with the help of (3.1), especially where the deeper properties are concerned. The composition law is one example. Let $\alpha \in \pi_p(H)$, $\beta \in \pi_q(Y)$, $\gamma \in \pi_r(S^q)$. When γ is a suspension element we have $\langle \alpha, \beta \circ \gamma\rangle = \langle \alpha, \beta\rangle \circ S^p_* \gamma$. In the general case the law of composition is given by an expansion of the form

$$\begin{aligned}\langle \alpha, \beta \circ \gamma\rangle = \langle \alpha, \beta\rangle \circ S^p_* \gamma + [\langle \alpha, \beta,\rangle, \beta] \circ S^p_* \gamma' \\ + [[\langle \alpha, \beta\rangle, \beta], \beta] \circ S^p_* \gamma'' + \cdots,\end{aligned} \tag{3.3}$$

where $\gamma', \gamma'', \cdots$ are generalized Hopf invariants of γ. This relation follows immediately, using (3.1), from the composition law for Whitehead products, given in (7.4) of [2]. In case Y is a sphere the quadruple Whitehead products vanish, as shown in [5], and so the expansion terminates after the first three terms. Applications of this formula are given in § 5 below, after we have established the Jacobi identity for the relative Samelson product.

4. The Jacobi identity

One proof of the Jacobi identity for the relative Samelson product is given by McCarty in [14]. Here we give another proof, using the results of § 3 above. Both proofs are based on the Jacobi identity for the Whitehead product.

The relative form of the Jacobi identity for the Samelson product can be stated as follows. Let $\beta \in \pi_q(Y)$, where $q \geqq 2$ and $Y = G/H$ as before [3]. Consider the operator

$$M : \pi_p(H) \to \pi_{p+q}(Y) \qquad (p = 1, 2, \cdots)$$

which is given by taking the relative Samelson product on the right with β. The purpose of this section is to prove that, for $\alpha_r \in \pi_{p_r}(H)$ $(r = 1, 2)$ we have

$$M\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_1, M\alpha_2 \rangle + (-1)^{p_2 q} \langle M\alpha_1, \alpha_2 \rangle, \tag{4.1}$$

where the brackets on the left refer to the ordinary Samelson product and those on the right to the relative Samelson product. If we replace M by the operator M' defined by taking the product with β on the left then the corresponding relation reads

$$M'\langle \alpha_1, \alpha_2 \rangle = \langle M'\alpha_1, \alpha_2 \rangle + (-1)^{p_1 q} \langle \alpha_1, M'\alpha_2 \rangle. \tag{4.2}$$

Let Δ, as before, denote the transgression isomorphism in the homotopy sequence of the universal H-bundle. Write $n_r = p_r + 1$ and write

$$\theta_r = \Delta^{-1}\alpha_r \in \pi_{n_r}(B_H).$$

Recall that $\theta_r^\# B_H$, as defined in § 3, is a class of H-bundle over S^{n_r} with fibre Y. Choose representatives E_r of $\theta_r^\# B_H$ so that $E_1 \cap E_2 = Y$. Then $E_1 \cup E_2$, as a bundle over $S^{n_1} \vee S^{n_2}$, represents $(\theta_1 \vee \theta_2)^\# B_H$. Consider the following commutative diagram where all the maps are natural inclusions except s_r and s which are canonical cross-sections.

[3] Where we require the existence of a local cross-section McCarty requires the first countability axiom. Such restrictions can be eliminated by using a different type of proof.

$$\begin{array}{ccccc} Y & \xrightarrow{i_r} & E_r & \xleftarrow{s_r} & S^{n_r} \\ \downarrow 1 & & \downarrow j_r & & \downarrow k_r \\ Y & \xrightarrow[i]{} & E_1 \cup E_2 & \xleftarrow[s]{} & S^{n_1} \vee S^{n_2} \end{array}$$

In $\pi_*(E_1 \cup E_2)$ let P denote the operation of taking the Whitehead product, on the right, with the element $i_* \beta \in \pi_q(E_1 \cup E_2)$. By the Jacobi identity for the Whitehead product (see [17], for example) we have

$$(4.3) \qquad -P[\eta_1, \eta_2] = (-1)^{n_1}[\eta_1, P\eta_2] + (-1)^{n_2 q}[P\eta_1, \eta_2],$$

where $\eta_r \in \pi_{n_r}(E_1 \cup E_2)$. We apply this with $\eta_r = j_{r*}\xi_r$, where $\xi_r \in \pi_{n_r}(E_r)$ denotes the class of s_r. Let us take the three terms of (4.3) in turn, beginning with the left-hand side.

Since $sk_r = j_r s_r$, in the diagram above, we have $\eta_r = s_* \kappa_r$, where $\kappa_r \in \pi_{n_r}(S^{n_1} \vee S^{n_2})$ denotes the class of k_r. Hence $[\eta_1, \eta_2] = s_* [\kappa_1, \kappa_2]$, in $\pi_*(E_1 \cup E_2)$, and so from (3.1) we obtain

$$P[\eta_1, \eta_2] = i_* M\Delta(\theta_1 \vee \theta_2)_*[\kappa_1, \kappa_2].$$

By naturality

$$\Delta(\theta_1 \vee \theta_2)_*[\kappa_1, \kappa_2] = \Delta[\theta_1, \theta_2] = (-1)^{p_1}\langle \alpha_1, \alpha_2 \rangle,$$

since $\Delta\theta_r = \alpha_r$. Thus we obtain the relation

$$P[\eta_1, \eta_2] = (-1)^{p_1}\langle \alpha_1, \alpha_2 \rangle.$$

By (2.1) we have at once that

$$-[\xi_r, i_{r*}\beta] = i_{r*} M\alpha_r,$$

in $\pi_*(E_r)$. This implies that

$$P\eta_r = j_{r*}[\xi_r, i_{r*}\beta] = i_* M\alpha_r$$

in $\pi_*(E_1 \cup E_2)$, since $j_r i_r = i$. Hence

$$\begin{aligned} [\eta_1, P\eta_2] &= [\eta_1, i_* M\alpha_2] \\ &= j_{1*}[\xi_1, i_{1*} M\alpha_2]. \end{aligned}$$

However $[\xi_1, i_{1*} M\alpha_2] = i_{1*}\langle \alpha_1, M\alpha_2 \rangle$, by (2.1) with β replaced by $M\alpha_2$, and so we obtain

$$[\eta_1, P\eta_2] = i_*\langle \alpha_1, M\alpha_2 \rangle.$$

This remains true if we interchange the suffixes and so, using (1.2), we obtain

$$(-1)^{n_1+q}[P\eta_1, \eta_2] = i_*\langle M\alpha_1, \alpha_2 \rangle.$$

We substitute in (4.3) from these relations, and obtain

$$i_* M\langle \alpha_1, \alpha_2 \rangle = i_*\langle \alpha_1, M\alpha_2\rangle + (-1)^{p_2 q} i_*\langle M\alpha_1, \alpha_2\rangle.$$

Since i_* is injective this proves (4.1).

5. The iterated Samelson product

Let G be a topological group. With each element $\lambda \in \pi_1(G)$ we associate the operator

$$\lambda_\# : \pi_r(G) \to \pi_{r+1}(G) \qquad (r = 1, 2, \cdots)$$

defined by taking the Samelson product with λ. From the Jacobi identity, each of these operators constitutes a derivation, with respect to the Samelson product in $\pi_*(G)$. Suppose that $\pi_2(G) = 0$, as is the case when G is a Lie group. Then these derivations form an anticommuting set of operators, and in particular

$$2\lambda_\#^2 = 0. \tag{5.1}$$

Note that $\lambda_\# = 0$ if λ can be represented by a loop within the centre of G.

Let R_t denote the group of rotations of euclidean t-space, where $t = 2, 3, \cdots$. Let

$$D : \pi_r(R_t) \to \pi_{r+1}(R_t)$$

be defined by taking the Samelson product with the generator $\lambda \in \pi_1(R_t)$. Of course D is trivial when $t = 2$. We prove

THEOREM (5.2). *Let $t > 2$ and $t \equiv 2 \bmod 4$. Then*

$$D^2 : \pi_r(R_t) \to \pi_{r+2}(R_t)$$

is trivial.

Consider the factor group PR_t of R_t by its centre $\{1, -1\}$. Let $\rho : R_t \to PR_t$ denote the natural projection. The induced homomorphism

$$\rho_* : \pi_*(R_t) \to \pi_*(PR_t)$$

respects the Samelson product since ρ is a homomorphism of topological groups. Since $t \equiv 2 \bmod 4$ we have $\pi_1(PR_t) \approx Z_4$, and so $\rho_*\lambda = 2\mu$, where $\mu \in \pi_1(PR_t)$. Hence $\rho_*\lambda_\# = 2\mu_\#$, by linearity, and so $\rho_*\lambda_\#^2 = 4\mu_\#^2 = 0$, by (5.1). Since ρ_* is an isomorphism for the higher homotopy groups this proves (5.2).

In the course of this section we prove a similar result which holds for all values of t, namely

THEOREM (5.3). *The operator*

$$D^6 : \pi_r(R_t) \to \pi_{r+6}(R_t)$$

is trivial.

An example will be given where D^4 is non-trivial. I do not know of any example where D^5 is non-trivial. We shall also show that D^4 is trivial when $t = 3$ or 4.

We identify R_{q+1}/R_q with S^q in the usual way, where $q \geqq 2$. Consider the operator

$$D : \pi_r(S^q) \to \pi_{r+1}(S^q)$$

defined by taking the relative Samelson product with the generator $\lambda \in \pi_1(R_q)$. Write $\eta_q = D\iota_q$, where ι_q generates $\pi_q(S^q)$. Then $\eta_q = J\lambda$, which generates $\pi_{q+1}(S^q)$. The triple Whitehead product is of odd order in the case of S^q, as shown in [5]. Also $[\eta_2, \iota_2] = 0$, and $2\eta_q = 0$ for $q > 2$. Hence $[[\eta_q, \iota_q], \iota_q] = 0$ for $q \geqq 2$ and the composition law (3.3) reduces to

$$D\gamma = \eta_q \circ S_* \gamma \pm [\eta_q, \iota_q] \circ S_* H\gamma, \tag{5.4}$$

where $\gamma \in \pi_r(S^q)$ and H denotes the generalized Hopf invariant. Now $H[\eta_q, \iota_q] = 0$, since $[\eta_q, \iota_q]$ is a suspension element, and $S_*[\eta_q, \iota_q] = 0$. Hence by iterating (5.4) we find that $D^2\gamma = \eta_q \circ \eta_{q+1} \circ S_*^2 \gamma$ and then, after two more stages, that $D^4\gamma = 0$, since $\eta_q \circ \eta_{q+1} \circ \eta_{q+2} \circ \eta_{q+3} = 0$. This proves

LEMMA (5.5). *If $q \geqq 2$ then*

$$D^4 : \pi_r(S^q) \to \pi_{r+4}(S^q)$$

is trivial.

We have shown, incidentally, that $D^3\iota_q = \eta_q \circ \eta_{q+1} \circ \eta_{q+2} \neq 0$. Hence it follows, with $q = 3$, that $D^3 : \pi_3(R_3) \to \pi_6(R_3)$ is non-trivial.

We use (5.5) to prove (5.3) for odd values of t. Suppose that $t \equiv 3 \bmod 4$. Regard D as operating on the homotopy exact sequence

$$\cdots \to \pi_r(R_{t-1}) \xrightarrow{u_*} \pi_r(R_t) \xrightarrow{p_*} \pi_r(S^{t-1}) \to \cdots,$$

as described in § 1. If $\theta \in \pi_r(R_t)$ then $p_* D^4 \theta = \pm D^4 p_* \theta = 0$, by (5.5) with $q = t-1$. Hence $D^4\theta = u_*\varphi$, by exactness, where $\varphi \in \pi_r(R_{t-1})$. But $D^2\varphi = 0$, by (5.2) with $t-1$ in place of t, and so

$$D^6\theta = D^2 u_* \varphi = \pm u_* D^2 \varphi = 0.$$

This proves (5.3) when $t \equiv 3 \bmod 4$. When $t \equiv 1 \bmod 4$ we use the homotopy exact sequence of the fibration $R_{t+1} \to S^t$ rather than that of the fibration $R_t \to S^{t-1}$. The details are very similar and will be left to the reader. When $t \equiv 0 \bmod 4$ we can also use (5.2) and (5.5) to show that $D^8 : \pi_r(R_t) \to \pi_{r+8}(R_t)$ is trivial, but to show that D^6 is trivial we need to refine the method as follows.

Consider the Stiefel manifold $V_{2n,2}$, where $n \geqq 2$. Let

$$D : \pi_r(V_{2n,2}) \to \pi_{r+1}(V_{2n,2})$$

be defined by taking the relative Samelson product with the generator $\lambda \in \pi_1(R_{2n-2})$. Let

$$S^{2n-2} = V_{2n-1,1} \xrightarrow{u} V_{2n,2} \xleftarrow{v} W_{n,1} = S^{2n-1}$$

be the standard inclusions. Write

$$u_* \iota_{2n-2} = \alpha_n \in \pi_{2n-2}(V_{2n,2}), \; v_* \iota_{2n-1} = \beta_n \in \pi_{2n-1}(V_{2n,2}).$$

I assert that

$$(5.6) \qquad D\alpha_n = \alpha_n \circ \eta_{2n-2}, \; D\beta_n = \beta_n \circ \eta_{2n-1}.$$

To prove the first part of (5.6), we have

$$\langle \alpha_n, \lambda \rangle = u_* \langle \iota_{2n-2}, \lambda \rangle = u_* \eta_{2n-2} = \alpha_n \circ \eta_{2n-2}.$$

To prove the second part we choose a generator $\mu \in \pi_1(U_{n-1})$ and then have

$$\langle \beta_n, \lambda \rangle = v_* \langle \iota_{2n-1}, \mu \rangle = v_* \eta_{2n-1} = \beta_n \circ \eta_{2n-1}.$$

Thus (5.6) is established. Furthermore, every element of $\pi_r(V_{2n,2})$ can be expressed uniquely in the form $\alpha_n \circ \theta + \beta_n \circ \varphi$, where $\theta \in \pi_r(S^{2n-2})$, $\varphi \in \pi_r(S^{2n-1})$, and hence the value of D can be calculated, in the usual terms, by using (5.6) and the composition law (3.3). After a short calculation, similar to one in § 5 of [9], we obtain

LEMMA (5.7). *If $n \geqq 2$ then*

$$D^4 : \pi_r(V_{2n,2}) \to \pi_{r+4}(V_{2n,2})$$

is trivial.

To complete the proof of (5.3) we take $t \equiv 0 \bmod 4$ and write $t = 2n$. We use the same type of argument as in the case when t is odd, but use (5.7) in place of (5.5). The details are left to the reader.

6. Relative version of a result of Bott's

The Samelson product has been studied by Bott [3], in respect of the classical groups. The purpose of this section is to extend this work to the relative case. Consider the symmetric space R_{2t}/U_t ($t = 2, 3, \cdots$) which we shall denote by U'_t. By the periodicity theorem, $\pi_{2t-1}(U_{s+t})$ is cyclic infinite for $s > 0$ and, when t is even, $\pi_{2t-2}(U'_{s+t})$ is cyclic infinite for $s \geqq 0$. Recall that U'_t fibres over S^{2t-2} with fibre U'_{t-1}. It is shown in [4] that the image of the transgression homomorphism

$$\Delta' : \pi_{2t-2}(S^{2t-2}) \to \pi_{2t-3}(U'_{t-1})$$

is of infinite order for odd values of t, and is of order $(t-2)!\,a_t$ for even values of t, where $a_t = \frac{1}{2}$ or 1 according as $t \equiv 0$ or $2 \bmod 4$. With this notation we shall prove

THEOREM (6.1). *Let $m, n \geqq 1$, with n even. Consider the relative Samelson product*

$$\langle \varphi_m, \varphi'_n \rangle \in \pi_{2m+2n-3}(U'_{m+n-1}),$$

where $\varphi_m \in \pi_{2m-1}(U_{m+n-1})$ and $\varphi'_n \in \pi_{2n-2}(U'_{m+n-1})$ are generators. If m is odd the product is of infinite order. If m is even the product is of order

$$\frac{(m+n-2)!\; a_{m+n}}{(m-1)!\,(n-2)!\; a_n}$$

By taking various values of m and n we can obtain, for $t = 2$ and for all $t \geqq 4$, examples of non-trivial relative Samelson products in the case of U'_t. Hence we deduce

COROLLARY (6.2). *If $t = 2$ or $t \geqq 4$ then U_t is not homotopy-normal in R_{2t} (in the sense of McCarty* [13]*).*

To prove (6.1) we follow closely the method used by Bott in [3]. An alternative proof of Bott's result has been given by Arkowitz [4] [1], and it seems likely that this method could also be relativized.

Much of the homotopy theory of Stiefel manifolds can be relativized, on the following lines. For this purpose it is convenient to write $(V_{2t,2k}, W_{t,k}) = W'_{t,k}$. The standard exact sequence

$$\cdots \to \pi_{r+1}(W_{t,k}) \xrightarrow{\delta} \pi_r(U_{t-k}) \xrightarrow{\sigma_*} \pi_r(U_t) \xrightarrow{\rho_*} \pi_r(W_{t,k}) \to \cdots$$

can be derived from one of the homotopy exact sequences of the triad $(U_t; U_{t-k}, U_t)$ by making appropriate identifications. If we change the triad to $(R_{2t}; R_{2t-2k}, U_t)$ the result is an exact sequence of the form

$$\cdots \to \pi_{r+1}(W'_{t,k}) \xrightarrow{\delta'} \pi_r(U'_{t-k}) \xrightarrow{\sigma'_*} \pi_r(U'_t) \xrightarrow{\rho'_*} \pi_r(W'_{t,k}) \to \cdots.$$

Of course U'_t can be fibred with fibre U'_{t-1} but not, in general, with fibre U'_{t-k}.

Now consider the relative join

$$W_{m,k} * W'_{n,k} = (W_{m,k} * V_{2n,2k}, W_{m,k} * W_{n,k}),$$

where $m, n \geqq k$. The intrinsic map of [7] determines a map of $W_{m,k} * W'_{n,k}$ into $W'_{m+n,k}$, and so enables us to construct a relative intrinsic join operation, pairing $\pi_i(W_{m,k})$ with $\pi_j(W'_{n,k})$ to $\pi_{i+j+1}(W'_{m+n,k})$.

[4] I am most grateful to Professor Arkowitz for sending me a preprint of this work.

It is a straightforward exercise (cf. § 6 of [8]) to prove

LEMMA (6.3). *If* ψ_m *generates* $\pi_{2m-1}(W_{m,1})$ *and* ψ'_n *generates* $\pi_{2n-2}(W'_{n,1})$ *then the relative intrinsic join* $\psi_m * \psi'_n$ *generates* $\pi_{2m+2n-2}(W'_{m+n,1})$.

The formal properties of the relative intrinsic join are similar to those which hold in the ordinary case. In particular, we have that

$$P(\sigma_* \times \sigma'_*) = \pm\Gamma(\rho_* \times \rho'_*), \tag{6.4}$$

as shown in the following diagram, where P denotes the relative Samelson product and where Γ denotes the composition of the relative intrinsic join and the boundary operator δ' in the exact sequence mentioned above.

$$\begin{array}{ccc}
\pi_i(U_m)\times\pi_j(U'_n) & \xrightarrow{\sigma_*\times\sigma'_*} & \pi_i(U_{m+n-k})\times\pi_j(U'_{m+n-k}) \\
{\scriptstyle \rho_*\times\rho'_*}\big\downarrow & & \big\downarrow{\scriptstyle P} \\
\pi_i(W_{m,k})\times\pi_j(W'_{n,k}) & \xrightarrow[\Gamma]{} & \pi_{i+j}(U'_{m+n-k})
\end{array}$$

The proof of (6.4) will be omitted since it is very similar to that of the corresponding result in the ordinary case, as given originally by Bott [3] and perfected by Husseini [6].

To obtain (6.1) we apply this result with $k = 1$. Let n be even. Recall that $\psi_m = (m-1)!\,\theta_m$, where θ_m generates $\pi_{2m-1}(U_m)$, and that $\psi'_n = (n-2)!\,a_n\theta'_n$, where θ'_n generates $\pi_{2n-2}(U'_n)$. Hence (6.4) shows that

$$\langle\sigma_*\theta_m, \sigma'_*\theta'_n\rangle = \pm(m-1)!\,(n-2)!\,a_n\delta'(\psi_m * \psi'_n).$$

However, the image of

$$\delta' : \pi_{2m+2n-2}(W'_{m+n,1}) \to \pi_{2m+2n-3}(U'_{m+n-1})$$

is of order infinity or $(m+n-2)!\,a_{m+n}$, according as $m+n$ is odd or even. Hence and from (6.3) it follows that $\langle\sigma_*\theta_m, \sigma'_*\theta'_n\rangle$ is of order infinity or $(m+n-2)!\,a_{m+n}/(m-1)!(n-2)!\,a_n$, according as $m+n$ is odd or even. This proves (6.1), since $\sigma_*\theta_m$ generates $\pi_{2m-1}(U_{m+n-1})$ and $\sigma'_*\theta'_n$ generates $\pi_{2n-2}(U'_{m+n-1})$.

The same argument, with slightly simpler details, applies in the case of the symmetric space $Sp'_t = U_{2t}/Sp_t$. The result obtained is

THEOREM (6.5). *Let* $m, n \geqq 1$. *Set* $b_{m,n} = 2$ *if* m *is odd and* n *is even, otherwise* $b_{m,n} = \frac{1}{2}$. *Let* $\varphi_m \in \pi_{4m-1}(Sp_{m+n-1})$, $\varphi'_n \in \pi_{4n-3}(Sp'_{m+n-1})$ *be generators. Then the relative Samelson product*

$$\langle\varphi_m, \varphi'_n\rangle \in \pi_{4m+4n-4}(Sp'_{m+n-1})$$

is of order $(2m+2n-2)!\,b_{m,n}/(2m-1)!(2n-2)!$

Corollary (6.6). *If $t \geqq 2$ then Sp_t is not homotopy-normal in U_{2t} (in the sense of McCarty* [13]).

The special unitary group SU_q is homotopy-normal in U_q $(q = 2, 3, \cdots)$, as can easily be shown. In particular $Sp_1 = SU_2$ is homotopy-normal in U_2. I do not know (with reference to (6.2)) whether or not U_3 is homotopy-normal in R_6.

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Oxford University
Mathematical Institute
24–29 St. Giles
OXFORD, England