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On the maps of one fibre space into another

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1. Introduction

The purpose of this note is to study, in special cases, the Puppe exact sequence of ex-homotopy theory (for details ¹, see [4]). We begin by recalling the basic notions of the category of ex-spaces and ex-maps, with respect to a fixed base space B. By an ex-space we mean a space X together with a pair of maps

$$B \xrightarrow{\rho} X \xrightarrow{\sigma} B$$

such that $\rho \sigma = 1$. We refer to $\rho$ as the projection, to $\sigma$ as the section, and to $(\rho, \sigma)$ as the ex-structure. Let $X_i (i = 0, 1)$ be an ex-space with ex-structure $(\rho_i, \sigma_i)$. We describe a map $f : X_0 \to X_1$ as an ex-map if

(1.1) $$f \sigma_0 = \sigma_1, \rho_1 f = \rho_0,$$

as shown in the following diagram.

In particular, we refer to $\sigma_1 \rho_0$ as the trivial ex-map. We describe a homotopy $f_t : X_0 \to X_1$ as an ex-homotopy if $f_t$ is an ex-map at every stage. The set of ex-homotopy classes of ex-maps is denoted by $\pi(X_0, X_1)$. Further notions, such as ex-homeomorphism and ex-homotopy equivalence, are defined in the obvious way.

Let $B$ be a pointed space, with basepoint $e \in B$. A functor $\Phi$ can be defined, as follows, from the category of ex-spaces to the category of pointed spaces. If $X$ is an ex-space with ex-structure $(\rho, \sigma)$, then $\Phi X$ is the space $\rho^{-1}e$ with $\sigma e$ as basepoint. If $f : X_0 \to X_1$ is an ex-map, where $X_0, X_1$ are ex-spaces, then $\Phi f : \Phi X_0 \to \Phi X_1$ is the map determined by

¹ Theories of this type have been developed independently by J. C. Becker and J. F. McClendon, amongst others.
restriction of \( f \). We refer to \( \Phi \) as the fibre functor. Note that \( \Phi \) determines a function

\[
\varphi : \pi(X_0, X_1) \to \pi(\Phi X_0, \Phi X_1),
\]

where the codomain means the set of pointed homotopy classes of pointed maps.

In some cases this function \( \varphi \) is both surjective and injective. For example, let \( A \) be a pointed space. Regard the wedge-sum \( A \vee B \) as an ex-space with section the inclusion and projection constant on \( A \). Then \( \Phi(A \vee B) = A \) and we have at once

**Proposition (1.2).** Let \( X \) be an ex-space with fibre \( \Phi X = Y \). Then the function

\[
\varphi : \pi(A \vee B, X) \to \pi(A, Y)
\]

is bijective.

By a fibre ex-space we mean an ex-space with a fibration as projection. When \( X_0 \) and \( X_1 \) are fibre ex-spaces there is a useful necessary condition for an element of \( \pi(\Phi X_0, \Phi X_1) \) to belong to the image of \( \varphi \). This condition involves the brace product, a pairing of homotopy groups derived from the Whitehead product as follows. Let \( X \) be a fibre ex-space with ex-structure \((p, s)\) and fibre \( Y \). Consider the short exact sequence

\[
0 \to \pi_*(Y) \xrightarrow{i_*} \pi_*(X) \xrightarrow{p_*} \pi_*(B) \to 0,
\]

where \( i : Y \subset X \). Given elements \( \beta \in \pi_*(B) \), \( \eta \in \pi_*(Y) \) we form the Whitehead product \([s_\beta, i_\eta] \). This element of \( \pi_*(X) \) lies in the kernel of \( p_* \), since \( p_* i_* = 0 \), and so by exactness there exists a (unique) element \( \{\beta, \eta\} \), say, of \( \pi_*(Y) \) such that

\[
(1.3) \quad i_*[\beta, \eta] = [s_* \beta, i_* \eta].
\]

This operation \( \{,\} \), which we refer to as the brace product, is studied in [5] and [9], where various examples are given. From (1.1), (1.3) and the naturality of the Whitehead product we obtain

**Proposition (1.4).** Let \( X_i \) (\( i = 0, 1 \)) be a fibre ex-space with fibre \( Y_i \). If \( \alpha \in \pi(Y_0, Y_1) \) belongs to the image of \( \varphi \) then

\[
\alpha \circ \{\beta, \eta\} = \{\beta, \alpha \circ \eta\},
\]

for all \( \beta \in \pi_*(B) \), \( \eta \in \pi_*(Y_0) \).

Here the brace product on the left refers to \( X_0 \) while that on the right refers to \( X_1 \). In certain cases, as we shall see, the condition is sufficient as well as necessary.

It appears that sphere-bundles play a special role in ex-homotopy theory just as spheres do in ordinary homotopy theory. Let \( O_q \) (q =
1, 2, \cdots \) denote the group of orthogonal transformations of euclidean $q$-space. For $m \geq q$ we regard the $(m-1)$-sphere $S^{m-1}$ as an $O_q$-space, in the usual way. Given a principal $O_q$-bundle over $B$ let $E_m$ denote the associated $(m-1)$-sphere bundle. When $m > q$ we regard $E_m$ as an ex-space by choosing a cross-section of the bundle. When $m > q+1$ we give $\pi(E_m, X)$ a natural group-structure, as described in §2 below, so that 
\[ \varphi : \pi(E_m, X) \to \pi(S^{m-1}, Y) = \pi_{m-1}(Y) \]
constitutes a homomorphism. We do not give a group-structure to $\pi(E_{q+1}, X)$.

Now consider the case when $B$ is a sphere, say $B = S^n$ ($n > 1$). Let $E_m$ ($m = q+1, q+2, \cdots$) be associated with an $O_q$-bundle over $S^n$, as above, and let $X$ be a fibre ex-space over $S^n$ with fibre $Y$. Let $i_r \in \pi_r(S^n)$ ($r = 1, 2, \cdots$) denote the homotopy class of the identity map and let 
\[ \psi : \pi_r(Y) \to \pi_{r+n-1}(Y) \quad (r \geq q) \]
denote the operator given by
\[ \psi(x) = x \circ \{i_n, i_r\} - \{i_n, x \circ i_r\} \quad (x \in \pi_r(Y)). \]

Here the brace products are to be interpreted as in (1.4). It follows from (1.7) below that $\psi$ is a homomorphism for $r > q$ but this is not true, in general, for $r = q$. Our aim is to set up an exact sequence containing $\psi$, as in (1.5), the fibre function $\varphi$, and a third operator 
\[ \theta : \pi_{m+n}(Y) \to \pi(E_{m+1}, X) \]
which can be defined as follows. Recall (see §3 of [7]) that 
\[ E_{m+1} = (S^m \vee S^n) \cup e^{m+n} \]
as a cell-complex, where $S^m$ is the fibre and $S^n$ is embedded by the cross-section. If $f : E_{m+1} \to X$ is an ex-map such that $\Phi f$ is constant then the separation element $d(f, c) \in \pi_{m+n}(X)$ is defined, with respect to this cell-structure, where $c$ denotes the trivial ex-map. Since $pf = pc$ we have $p_* d(f, c) = 0$ and so $d(f, c) = i_* \beta$, by exactness, where $\beta \in \pi_{m+n}(Y)$. Conversely, given $\beta$, there exists an ex-map $f$, as above, such that $d(f, c) = i_* \beta$. We define $\theta(\beta)$ to be the ex-homotopy class of $f$ in $\pi(E_{m+1}, X)$. It is not difficult to check that $\theta$ constitutes a homomorphism for $m > q$. Having made the necessary definitions we are now ready to state our main result

**Theorem (1.6).** The sequence 
\[ \cdots \to \pi(E_{m+1}, X) \xrightarrow{\varphi} \pi_m(Y) \xrightarrow{\psi} \pi_{m+n-1}(Y) \xrightarrow{\theta} \pi(E_m, X) \]
\[ \to \cdots \to \pi_q(Y) \xrightarrow{\psi} \pi_{q+n-1}(Y), \]
is exact.
It is possible to prove (1.6) by using the methods of Barcus and Barratt [2]. However the proof we give shows, in my opinion, the advantages of exploiting the elementary properties of ex-homotopy theory.

Before we embark on the proof it is convenient to make a few further observations. Suppose that the original q-sphere bundle has classifying element \( \beta \in \pi_{n-1}(O_q) \). By (3.7) of [7] the brace product in the case of \( E_{m+1} \) is given by

\[
\{t_n, t_m\} = S_*^{m-q}J\beta,
\]

where \( S_* \) denotes the suspension functor and \( J\beta \in \pi_{n+q-1}(S^q) \) is defined by the Hopf construction in the usual way. Suppose that \( Y = S^r \) \((r \geq 1)\), regarded as a pointed \( O_r \)-space, and that \( X \) is the \( r \)-sphere bundle with cross-section associated with a principal \( O_r \)-bundle. Then \( \{t_n, t_r\} = J\gamma \)

similarly, where \( \gamma \in \pi_{n-1}(O_r) \) is the classifying element, and hence \( \{t_n, \alpha\} = J\gamma \circ S_*^{n-1}\alpha \), where \( \alpha \in \pi_{m}(S^r) \). Thus

\[
\psi(\alpha) = \alpha \circ S_*^{m-q}J\beta - J\gamma \circ S_*^{n-1}\alpha.
\]

Where the relevant information on the homotopy groups is available we can calculate the kernel and cokernel of \( \psi \), for a range of values of \( m \), and hence calculate \( \pi(E_m, X) \) to within a group extension.

For example, take \( n = 2, q = 2 \). Take \( E_m \) to be the \((m-1)\)-sphere bundle associated with the Hopf bundle over \( S^2 \). Using standard results on the homotopy groups of spheres we find that \( \pi(E_8, E_6) \approx \mathbb{Z}_2 \), in this case. If instead we take \( E_m \) to be the trivial \((m-1)\)-sphere bundle we find that \( \pi(E_8, E_6) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

2. The Puppe sequence

Let \((K, B)\) be a \( CW \)-pair, such that \( B \) is a retract of \( K \), and let \( \rho : K \rightarrow B \) be a cellular retraction. We regard \( K \) as an ex-space with the retraction as projection and the inclusion as section. Let \( \Sigma K \) denote the complex obtained from the union of \( K \times I \) and \( B \) by identifying \((x, t) \in K \times I \) with \( \rho x \in B \) if either \( x \in B \) or \( t = 0, 1 \). A retraction of \( \Sigma K \) on \( B \) is given by \((x, t) \mapsto \rho x \). We give \( \Sigma K \) cell-structure, in the obvious way, so that \( B \) is a subcomplex and the retraction is cellular. We refer to \( \Sigma K \) as the suspension of \( K \), in the ex-category. The \( r \)-fold suspension \((r = 1, 2, \cdots)\) is denoted by \( \Sigma^r K \). Suppose, for simplicity, that \( K \) is locally finite.

Let \( \varphi_X K \), for any ex-space \( X \), denote the function-space of ex-maps \( K \rightarrow X \), with the trivial ex-map as basepoint. By taking adjoints, in the usual way, we identify the homotopy group \( \pi_r(\varphi_X K) \) with \( \pi(\Sigma^r K, X) \). Thus \( \pi(\Sigma^r K, X) \) \((r = 1, 2, \cdots)\) receives a natural group-structure. This group is abelian for \( r \geq 2 \) but not, in general, for \( r = 1 \).
Now suppose that we have a locally finite complex \( K' \) containing \( K \) as a subcomplex and suppose that \( \rho \) can be extended to a cellular retraction \( \rho' : K' \to B \). Let \( K'' \) denote the complex obtained from \( K' \) by identifying points of \( K \) with their images under \( \rho \). Let \( \rho'' : K'' \to B \) denote the retraction induced by \( \rho' \). We regard \( K' \) and \( K'' \) as ex-spaces, with the retractions as projections and the inclusions as sections. Then

\[
K \xrightarrow{i} K' \xrightarrow{j} K''
\]

are ex-maps, where \( i \) is the inclusion and \( j \) is the identification map. Consider the maps

\[
\phi_X K'' \xrightarrow{j*} \phi_X K' \xrightarrow{i*} \phi_X K
\]

given by functional composition. By a straightforward application of the covering homotopy property we obtain

**Theorem (2.1).** If \( X \) is a fibre ex-space then \( i* : \phi_X K' \to \phi_X K \) is a fibration.

Notice that the fibre, over the trivial ex-map, can be identified with \( \phi_X K'' \) by means of \( j* \). Hence the homotopy exact sequence of the fibration can be written in the form

\[
\cdots \to \pi(\Sigma' K'', X) \xrightarrow{j*} \pi(\Sigma' K', X) \xrightarrow{i*} \pi(\Sigma' K, X) \to \cdots
\]

This is an example of the generalization to ex-homotopy theory of the notion of Puppe sequence. An alternative approach (see § 7 of [4]) is to construct a sequence of ex-maps

\[
K \to K' \to K'' \to \Sigma K \to \Sigma K' \to \Sigma K'' \to \cdots
\]

and apply the functor \( \pi(\_ , X) \).

The Puppe sequence gives useful information if one of the three domain ex-spaces is ex-contractible. For example we have

**Corollary (2.2).** Suppose that \( K'' \) is ex-contractible. If \( X \) is a fibre ex-space then

\[
i* : \pi(\Sigma' K', X) \to \pi(\Sigma' K, X)
\]

is bijective, for \( r \geq 1 \), and

\[
i* : \pi(K', X) \to \pi(K, X)
\]

is monic.

Here the term monic is used to mean that the kernel of \( i* \) is trivial; it is not true that \( i* \) is injective, in general.

In § 1 we are given a principal \( O_q \)-bundle over \( B \), and consider the associated \((m-1)\)-sphere bundle \( E_m \) \((m = q, q+1, \cdots)\). We regard \( E_{m+1} \) as the fibre suspension of \( E_m \) in the usual way (see § 7 of [7]). Let \( m > q \).
Then $E_m$ admits a cross-section and so can be regarded as an ex-space. The suspension $\Sigma E_m$ is defined, as above, and the natural ex-map

$$r : E_{m+1} \to \Sigma E_m$$

is an ex-homotopy equivalence, as shown in §6 of [4]. We identify $\pi(E_{m+1}, X)$ with $\pi(\Sigma E_m, X)$ under the bijection induced by $r$, where $X$ is any ex-space. Thus $\pi(E_{m+1}, X)$ receives a group-structure, for $m > q$.

3. The operators in the sequence

Let $D^r$ ($r = 1, 2, \cdots$) denote the $r$-ball bounded by $S^{r-1}$. Choose a map $b_r : D^r \to S^r$ which is constant on $S^{r-1}$ and non-singular on the interior of $D^r$. Given $q$, we regard $D^m$ and $S^{m-1}$ as $O_q$-spaces for $m \geq q$, in the usual way, and choose $b_m$ to be an $O_q$-map.

We recall that an $O_q$-bundle over $S^n$ ($n \geq 2$) corresponds to a map $T : S^{n-1} \to O_q$, in the standard classification. Let $m \geq q$. Write $g(x, y) = T(y) \cdot x$ ($x \in S^{m-1}$, $y \in S^{n-1}$) so that $g : S^{m-1} \times S^{n-1} \to S^{m-1}$. Let $E_m$ denote the space obtained from the union of $S^{m-1} \times D^n$ and $S^{m-1}$ by identifying points of $S^{m-1} \times S^{n-1}$ with their images under $g$, so that the identification map

$$h : (S^{m-1} \times D^n, S^{m-1} \times S^{n-1}) \to (E_m, S^{m-1})$$

is a relative homeomorphism. Write $\pi h(x, y) = b_n y$, where $x \in S^{m-1}$, $y \in D^n$, so that $\pi : E_m \to S^n$. We recall (see §3 of [7]) that $E_m$, with this projection, can be identified with the $(m-1)$-sphere bundle corresponding to $T$, i.e. the $(m-1)$-sphere bundle associated with the given $O_q$-bundle. If we replace $S^{m-1}$ by $D^m$, in this construction, we obtain the associated $m$-ball bundle $E'_m$ with projection $\pi'$, say. We regard $E_m$ as a subspace of $E'_m$, in the obvious way, so that $\pi = \pi'u$, where $u : E_m \subset E'_m$.

Let $F_m, F'_m$ denote the spaces obtained from $E_m, E'_m$ by collapsing $S^{m-1}$ to a point. Then $p'u = vp$, as shown below, where $v$ is the inclusion and $p, p'$ are the collapsing maps.

$$E_m \xrightarrow{u} E'_m$$
$$\downarrow p \quad \downarrow p'$$
$$F_m \xrightarrow{v} F'_m$$

Write $sy = h(e, y)$, where $y \in D^n$, so that $s : D^n \to E_m$. We regard $F_m$ as an ex-space with projection $\rho = \pi p^{-1}$ and section $\sigma = sb_n^{-1}$. Similarly

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\[2\] Here, and elsewhere, it is unnecessary to specify orientation conventions since the validity of (1.6) is independent of the signs of the operators.
we regard $F_m'$ as an ex-space with ex-structure $(\rho', \sigma')$ so that $v$ is an ex-map. Using the method described in §3 of [7] we can endow these spaces with cell-structure so as to satisfy the preliminary conditions of §2. Consider, therefore, the Puppe sequence associated with the pair $(F_m', F_m)$. Recall that $F_m''$, in the notation of §2, is obtained from $F_m'$ by identifying points of $F_m$ with their images under the projection $\rho$. Now $p$ determines a homeomorphism $p'': E_m'' \to F_m''$, where $E_m''$ denotes the space obtained from $E_m'$ by identifying points of $E_m$ with their images under the projection $\pi$. We endow $E_m''$ with ex-structure so as to make $p''$ an ex-homeomorphism. The $0_m$-map $b_m : D^m \to S^m$ is constant on $S^{m-1}$ and so determines an ex-homeomorphism between $E_m''$ and $E_{m+1}$. By composing this with the inverse of $p''$ we obtain an ex-homeomorphism $\beta : F_m'' \to E_{m+1}$. Now $\Phi F_m$ is a point-space, and $\Phi F_m' = S^m = \Phi E_{m+1}$, where $\Phi$ denotes the fibre functor. We use $\Phi \beta$ to identify $\Phi F_m''$ with $S^m$. Let $\omega$ denote the composition

$$
F_m' \xrightarrow{j} F_m'' \xrightarrow{\beta} E_{m+1},
$$

where $j$ is the identification ex-map. Then $\Phi \omega = \Phi$ and hence

$$
(3.1) \quad \varphi \omega^* = \varphi,
$$

as shown in the following diagram where $X$ is a fibre ex-space with fibre $Y$.

$$
\begin{array}{ccc}
\pi(E_{m+1}, X) & \xrightarrow{\omega^*} & \pi(F_m', X) \\
\varphi \downarrow & & \varphi \downarrow \\
\pi(S^m, Y) & & \pi(S^m, Y)
\end{array}
$$

The fibre of $F_m'$ has been identified with $S^m$. We embed the base space $S^n$ in $F_m'$ by means of the section. Let $S^m \vee S^n$ denote the union of these two spheres, with the standard ex-structure. If, in $F_m'$, we identify points of $S^m \vee S^n$ with their images under the projection, we obtain an ex-space which is ex-contractible. Thus (2.2) applies to $\tau^*$, as shown below, where $\tau : S^m \vee S^n \subset F_m'$.

$$
\begin{array}{ccc}
\pi(F_m', X) & \xrightarrow{\tau^*} & \pi(S^m \vee S^n, X) \\
\varphi \downarrow & & \varphi \downarrow \\
\pi(S^m, Y) & & \pi(S^m, Y)
\end{array}
$$

By using (1.2) $\varphi$ is bijective, on the right of the above diagram, and so we obtain
LEMMA (3.2). The fibre function
\[ \varphi : \pi(F_m, X) \to \pi(S^m, Y) \]
is bijective for \( m > q \), monic for \( m = q \).

The next step is to set up a bijection between \( \pi(F_m, X) \) and \( \pi(S^{m+n-1}, Y) \). Certainly \( F_m \) and \( S^{m+n-1} \vee S^n \), as spaces, have the same homotopy type; however in general they do not, as ex-spaces, have the same ex-homotopy type. Situations of this kind can be dealt with as follows. Consider the induced fibration \( \rho^*X \) over \( F_m \). The section of \( X \) over \( S^n \) determines a cross-section of \( \rho^*X \) over \( S^n \). The extensions over \( F_m \) of this partial cross-section correspond to the ex-maps of \( F_m \) into \( X \). Similarly the vertical homotopies of cross-sections, rel \( S^n \), correspond to ex-homotopies. By standard theory (see [1]) such cross-sections are classified by elements of \( \pi_{m+n-1}(Y) \). The corresponding result, in our situation, is that

\[ \xi : \pi(F_m, X) \to \pi_{m+n-1}(Y) \]
is bijective, where \( \xi \) is given as follows. Let \( f : F_m \to X \) be an ex-map of class \( \gamma \in \pi(F_m, X) \), and let \( \iota : F_m \to X \) denote the trivial ex-map. Let \( i : Y \subset X \). Then

\[ i_* \xi(\gamma) = d(f, \iota), \]

where the separation element is defined with respect to the pair \( (F_m, S^n) \).

Let \( l : S^{m+n-1} \to F_m \) be a map \(^3\) of degree 1 such that \( pl : S^{m+n-1} \to S^n \) is null-homotopic. By consideration of the induced fibration \( l^*\rho^*X \) over \( S^{m+n-1} \) we obtain the relation

\[ i_* \xi(\gamma) = \gamma_\lambda(\lambda), \]

where \( \lambda \in \pi_{m+n-1}(F_m) \) denotes the homotopy class of \( l \).

We identify \( S^{m+n-1} \) with the boundary of \( D^m \times D^n \) in the usual way. Let

\[ h' : (D^m \times D^n; D^m \times S^{n-1}, S^{m-1} \times D^n) \to (E'_m; D^m, E_m) \]
denote the identification map used in the construction of the \( m \)-ball bundle \( E'_m \). By restricting \( p'h' \) to the boundary of \( D^m \times D^n \) we obtain a map

\[ H : D^m \times S^{n-1} \cup S^{m-1} \times D^n \to F_m'. \]

Notice that \( H \) is constant on \( S^{m-1} \times S^{n-1} \). Also \( H \) maps \( D^m \times S^{n-1} \) into \( S^m \) and \( S^{m-1} \times D^n \) into \( F_m \). Let \( \kappa \in \pi_{m+n-1}(S^m) \) denote the class of the map

\(^3\) We use the phrase which follows to mean that \( l \) determines a homeomorphism when \( S^n \subset F_m \) is collapsed to a point.
which agrees with $H$ on $D^m \times S^{n-1}$ and is constant on $S^{m-1} \times D^n$. Also let $\lambda \in \pi_{m+n-1}(F_m)$ denote the class of the map

$$I : D^m \times S^{n-1} \cup S^{m-1} \times D^n \to F_m$$

which agrees with $H$ on $S^{m-1} \times D^n$ and is constant on $D^m \times S^{n-1}$. By (3.9) of [10] the class of $H$ in $\pi_{m+n-1}(F_m)$ is equal (with suitable conventions) to

$$j_*([\kappa]) + v_*([\lambda]) - \tau_*[i_n, i_m],$$

where $\tau, j, v$ are the inclusion maps and where $i_n, i_m \in \pi_*(S^m \vee S^n)$ are the classes of the inclusion maps of $S^m, S^n$ respectively. But $H$ is null-homotopic, since $h'$ extends over $D^m \times D^n$, and so we conclude that

$$(3.5) \quad v_*([\lambda]) = \tau_*[i_n, i_m] - j_*([\kappa]).$$

It is easy to check that $I$, as above, is a map of degree 1, and that

$$\rho l : D^m \times S^{n-1} \cup S^{m-1} \times D^n \to S^n$$

is null-homotopic. Moreover it follows from the basic theory of the Hopf construction (see [10]) that

$$\pm \kappa = S^{m-1}_* \eta x = \{i_n, i_m\},$$

where $\alpha \in \pi_{n-1}(O_q)$ denotes the homotopy class of $T$ and the brace product is defined with respect to $E_m$. We arrange our orientation conventions so that $\kappa = \{i_n, i_m\}$.

The next step in the proof of our main theorem is to establish that

$$(3.6) \quad \psi \phi = \xi v_*,$$

as shown in the following diagram, where $\psi$ is defined by means of the brace product, as in (1.5).

Let $\gamma' \in \pi(F'_m, X)$. Write $\phi \gamma' = \eta \in \pi_m(Y)$, $v^*(\gamma') = \gamma \in \pi(F_m, X)$. By naturality $\gamma_* \tau_*[i_n, i_m] = [s_* i_n, i_* \eta] = i_* \{i_n, \eta\}$, by (1.3), where $s : S^n \to X$ denotes the section. Also $\gamma_* j_*([\kappa]) = i_* \eta_*(\kappa) = i_* \eta_* \{i_n, i_m\}$, as we have just seen. Now compose both sides of (3.5) with $\gamma'_*$ and we obtain the relation

$$i_* \psi \phi(\gamma') = \gamma_* v_*(\lambda) = \gamma_*(\lambda) = i_* \xi(\gamma),$$

by (3.4). Since $i_*$ is injective this proves (3.6).
It follows at once from (3.3) by naturality that

\[(3.7) \quad \theta \xi = p^*,\]

as shown in the following diagram, where \(\theta\) is defined by means of the separation element as in § 1.

\[
\begin{array}{ccc}
\pi(F_m, X) & \xrightarrow{p^*} & \pi(E_m, X) \\
\downarrow \zeta & & \downarrow \theta \\
\pi_{m+n-1}(Y) & & \\
\end{array}
\]

Now we are ready to complete the proof of (1.6). As we have seen, the Puppe sequence associated with the pair \((F'_q, F_q)\) can be written in the form

\[
\cdots \to \pi(E_{m+1}, X) \xrightarrow{\alpha^*} \pi(F'_m, X) \xrightarrow{\xi^*} \pi(F_m, X) \xrightarrow{p^*} \pi(E_m, X) \to \cdots.
\]

We use \(\phi\) and \(\xi\) to replace the terms in the middle by homotopy groups of \(Y\). We recall that \(\phi\) is monic for \(m = q\), bijective for \(m > q\); also \(\xi\) is bijective for \(m \geq q\). Hence and from (3.1), (3.6) and (3.7) we obtain the exact sequence of § 1.

4. Proper cross-sections

The above theory can be used to give an alternative proof of the main result of [3], which concerns the following problem. Let \(B\) be a pointed space. Let \(X_i (i = 0, 1)\) be an ex-space of \(B\) with fibre \(X_i = Y_i\), say. Suppose we have an ex-map \(p : X_1 \to X_0\) which, as an ordinary map, constitutes a fibration with fibre \(Z\), say, over the basepoint of \(Y_0 \subset X_0\). Write \(\Phi p = q\). Then \(q : Y_1 \to Y_0\) can be regarded as a fibration with fibre \(Z\) so that we have the situation indicated in the following diagram, where \(u, v\) and \(u_i\) are the inclusions.

\[
\begin{array}{ccc}
Z & \xrightarrow{1} & Z \\
\downarrow v & & \downarrow u \\
Y_1 & \xrightarrow{u_1} & X_1 & \xrightarrow{\rho_1} & B \\
\downarrow q & & \downarrow p & & \downarrow 1 \\
Y_0 & \xrightarrow{u_0} & X_0 & \xrightarrow{\rho_0} & B
\end{array}
\]

Under what conditions does the fibration \(p\) admit a cross-section? As in [3] we describe such a cross-section \(f : X_0 \to X_1\) as proper if
Note that proper cross-sections are ex-maps, since \( p \) is an ex-map. If \( f \) is proper then \( g : Y_0 \to Y_1 \) constitutes a cross-section of \( q \), where \( g = \Phi f \). As in [3] we describe \( g \), or its vertical homotopy class, as the type of \( f \). We approach our problem by asking, for each cross-section \( g \) of \( q \), whether \( p \) admits a proper cross-section of type \( g \).

Now let \( B \) be a (pointed) CW-complex. Suppose that \( (X_i, Y_i) \) is a CW-pair, and that the section \( \sigma_i \) embeds \( B \) as a subcomplex. Then \( (X_i, B \vee Y_i) \) forms a CW-pair, and the answer to our question is independent of the choice of \( g \) in its class, by the homotopy lifting property. Consider the homomorphism

\[
\theta : \pi_r(B) \oplus \pi_r(Y_0) \to \pi_r(X_0)
\]

given by \( \sigma_0* \) on the first summand and \( \tau_0* \) on the second. Certainly \( \theta \) is an isomorphism for \( r \geq 2 \), since \( \sigma_0 \) is a right inverse of \( \rho_0 \). Suppose that \( \theta \) is also an isomorphism for \( r = 1 \). This is the case, for example, if \( \pi_1(X_0) \) is abelian, or if \( \pi_1(B) \) is trivial. Under this hypothesis we prove

**Lemma (4.2).** Let \( f : X_0 \to X_1 \) be an ex-map such that \( \Phi f : Y_0 \to Y_1 \) is a cross-section of \( q \). Then there exists a proper cross-section \( f' \) of \( p \) such that \( \Phi f' = \Phi f \).

Since \( pf : X_0 \to X_0 \) maps \( B \vee Y_0 \) identically we have \( (pf)_* \theta = \theta \), where \( \theta \) is as above and

\[
(pf)_* : \pi_*(X_0) \to \pi_*(X_0).
\]

Hence \( (pf)_* = 1 \), since \( \theta \) is an isomorphism, and so \( pf \) is a homotopy equivalence, by Theorem 1 of [11]. Hence \( pf \) determines a homotopy equivalence of the pair \( (X_0, B \vee Y_0) \) with itself, by (3.1) of [6], since \( pf \) maps \( B \vee Y_0 \) identically. Therefore there exists an inverse homotopy equivalence \( k : X_0 \to X_0 \), say, which also maps \( B \vee Y_0 \) identically. Write \( f'' = fk : X_0 \to X_1 \). Then \( f'' \) agrees with \( f \) on \( B \vee Y_0 \). Let \( l_0 : X_0 \to X_0 \) be a homotopy of \( pf'' \) into the identity such that \( l_0(B \vee Y_0) \subset B \vee Y_0 \).

By composing with \( h = f|B \vee Y_0 \) we lift \( l_0|B \vee Y_0 \) to a homotopy \( l_0' : B \vee Y_0 \to X_1 \) such that \( l_0' = h = l_1' \). We extend \( l_0' \) over \( X_0 \) so as to cover \( l_0 \), using the homotopy lifting property, and thus deform \( f'' \) into \( f' \), say. Then \( f' \) is a proper cross-section of \( p \) and \( \Phi f' = \Phi f'' = \Phi f \), as required.

To apply (4.2) we return to the situation considered in § 1, where \( E_{m+1} \) is an \( m \)-sphere bundle over \( S^n \). The fundamental group is abelian, since \( n > 1 \). We apply (4.2) with \( (X_0, Y_0) = (E_{m+1}, S^m) \) and write \( (X_1, Y_1) = (X, Y) \), so that \( p : X \to E_{m+1} \) and \( q : Y \to S^m \). Hence and from the exactness of our sequence we obtain
COROLLARY (4.3). Let $\gamma \in \pi_m(Y)$ be the class of a cross-section of $q$. Then $p$ admits a proper cross-section of type $\gamma$ if and only if

$$\gamma \circ \{t_n, t_m\} = \{t_n, \gamma\}.$$ 

In particular, suppose that $Y$ is a pointed $O_m$-space, and that $q$ is a pointed $O_m$-map. Suppose that $X$ and $E_{m+1}$ share the same principal $O_m$-bundle and that $p$ is the map associated with $q$. To obtain the main result of [3] from (4.3) it is only necessary to convert the brace products into the kind of product used in [3], as shown in [5].

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