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A homological characterization of local complete intersections


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A HOMOLOGICAL CHARACTERIZATION
OF LOCAL COMPLETE INTERSECTIONS

by

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Let \( R \) denote a local ring with residue field \( k = R/\mathfrak{m} \). Let \( P_R \) be the Poincaré series of \( R \) i.e. the power series

\[
P_R = \sum_{p=0}^{\infty} \dim_k \text{Tor}_p^R(k, k)Z^p.
\]

It is known that \( P_R \) may be written uniquely as a product of the form

\[
P_R = \prod_{i=0}^{\infty} \frac{(1 + Z^{2i+1})^{e_{2i}}}{(1 - Z^{2i+2})^{2i+1}}
\]

where \( e_q(R) = e_q \) \((q = 0, 1, \cdots)\) are non-negative integers only depending on \( R \) (Assmus, Levin). If \( R \) is a local complete intersection (i.e. the \( \mathfrak{m} \)-adic completion of \( R \) is a factor ring of a regular ring \( A \) modulo an \( R \)-sequence) then it is known that \( e_q = 0 \) for all \( q \geq 2 \) (Tate, Zariski). If \( e_2 = 0 \) or \( e_3 = 0 \) then \( R \) is a complete intersection. The case \( e_2 = 0 \) is due to Assmus, the case \( e_3 = 0 \) is due to the author. Cf. [5].

The purpose of this note is to prove the following:

**Theorem.** If \( e_q(R) = 0 \) for all sufficiently large \( q \), then \( R \) is a local complete intersection.

**Notation.** The term ‘\( R \)-algebra’ will be used in the sense of Tate [6] i.e. an associative, graded, differential, strictly skew-commutative algebra \( X \) over \( R \), with unit element 1, such that the homogeneous components \( X_q \) are finitely generated modules over \( R \), \( X_0 = 1 \cdot R \) and \( X_q = 0 \) for \( q < 0 \).

\( Z_+(X) \) (resp. \( H_+(X) \)) will denote the set of homogeneous cycles (resp. homology classes) in \( X \) of positive degree.

If \( X \) is an \( R \)-algebra and \( s \) is a homogeneous cycle in \( X \), then \( X\langle S; dS = s\rangle \) or briefly \( X\langle s\rangle \) denotes the \( R \)-algebra obtained from \( X \) by the adjunction of a variable \( S \) which kills \( s \). Cf. [6].
By the Koszul complex over $R$ generated by elements $t_1, \cdots, t_n$ in $R$ we mean the $R$-algebra obtained from the trivial $R$-algebra $R$ by the adjunction of variables $T_1, \cdots, T_n$ of degree 1 killing $t_1, \cdots, t_n$.

**Lemma 1.** Let $X$ be an $R$-algebra satisfying

(i) $H_0(X) \approx R/\mathfrak{m}$

(ii) $Z_+(X) \subset \mathfrak{m}X$.

Let $n = \dim \mathfrak{m}/\mathfrak{m}^2$.

Then for all $\sigma \in H_+(X)$ we have $\sigma^{n+1} = 0$.

**Proof.** Let $s$ be a cycle representing $\sigma$.

Let $\mathfrak{m}$ be minimally generated by $t_1, \cdots, t_n$. By (ii) there exist $x_1, \cdots, x_n \in X$ such that

$$s = \sum_{i=1}^n t_i x_i.$$ 

By (i) we can choose elements $T_1, \cdots, T_n$ of degree 1 such that $dT_i = t_i$ for $i = 1, \cdots, n$. $s$ is obviously homologous to the cycle

$$s_0 := \sum_{i=1}^n T_i dx_i.$$ 

Since $T_i^2 = 0$ for all $i$ we have $s_0^{n+1} = 0$, hence $\sigma^{n+1} = 0$.

**Definition.** Let $X$ be an $R$-algebra. Define

$$q(X) = \inf \{ r | H_i(X) = 0 \text{ for all } i > r \}$$

$$(\inf \emptyset = \infty).$$

**Lemma 2.** Let $X$ be an $R$-algebra satisfying the assumptions (i) and (ii) of lemma 1. Let $s$ be a homogeneous cycle of positive degree in $X$ and put $Y = X \langle S, dS = s \rangle$. Then

$$q(Y) < \infty \Rightarrow q(X) < \infty$$

**Proof.** Let us assume that $q(Y) < \infty$. We will consider two cases. First assume that $\deg S$ is even. In this case we have an exact sequence of complexes

$$0 \rightarrow X \rightarrow Y \rightarrow 0$$

where $i$ and $j$ are maps of degree 0 and $-\deg S$ respectively. Cf. [6]. Looking at the associated exact homology sequence one sees that $q(X) < \infty$.

Let us now consider that case where $\deg S$ is odd. In this case we have an exact sequence of complexes

$$0 \rightarrow X \rightarrow Y \rightarrow X \rightarrow 0$$

where $i$ and $j$ have degrees 0 and $-\deg S$ respectively and where the con-
necting homomorphism $d_*$ in the associated homology triangle

$$
\begin{array}{c}
H(Y) \\
i_* & \searrow j_* \\
H(X) & \leftarrow & H(X) \\
d_* & & \\
\end{array}
$$

is, up to sign, multiplication by $\sigma$, see the proof of theorem 2 in [6]. Now put $n = \dim \mathcal{M}/\mathcal{M}^2$ and $v = \deg \sigma$. Using (1) we obtain for each $r > q(Y)$ an exact sequence

$$H_r(X) \xrightarrow{d_*} H_{r+v}(X) \to H_{r+v}(Y) = 0$$

Hence $H_{r+v}(Y) = \sigma H_r(X)$ for $r > q(Y)$. It follows that

$$H_{r+(a+1)v}(X) = \sigma^{n+1} H_r(X) \quad \text{for } r > q(Y).$$

By Lemma 1 we have $\sigma^{n+1} = 0$. It follows that $q(X) < \infty$.

**PROOF OF THE THEOREM:** It is enough to prove the theorem for complete local rings, hence we may assume that there exists a regular ring $\mathcal{R}$ and a surjective ring homomorphism $f : \mathcal{R} \to \mathcal{R}$. Put $\mathcal{U} = \ker f$. We may also assume that $\mathcal{U}$ is contained in the square of the maximal ideal $\mathcal{M}$ in $\mathcal{R}$.

Let $a_1, \ldots, a_c$ be a maximal $\mathcal{R}$-sequence in $\mathcal{U}$ and let $\mathcal{U}'$ be the ideal generated by $a_1, \ldots, a_c$. This sequence can be chosen such that it can be extended to a minimal set of generators for $\mathcal{U}$, i.e. the canonical map $\mathcal{U}' \otimes k \to \mathcal{U} \otimes k$ is injective. Put $R' = \mathcal{R}/\mathcal{U}'$ and let $g : R' \to R$ be the homomorphism induced by $f : \mathcal{R} \to R$.

Now assume that $E_q(R) \neq 0$ for all $q$ sufficiently large. We will show that $\ker g = 0$. It suffices to show that $R$ is an $R'$-module of finite projective dimension. Indeed, by construction every element in $\ker g$ is a zero-divisor in the ring $R'$. Hence if $pd_{R'} R < \infty$ it follows from proposition 6.2 in [3] that $\ker g = 0$.

Let $\mathcal{E}$ be the Koszul complex generated over $\mathcal{R}$ by a minimal set of generators for $\mathcal{M}$. Since $\ker f \subset \mathcal{M}^2$, the rings $\mathcal{R}, R'$ and $R$ have the same imbedding dimension. Thus, putting $E' = \mathcal{E} \otimes_{\mathcal{R}} R'$ and $E = \mathcal{E} \otimes_{\mathcal{R}} R = E' \otimes_{R'} R, E'$ and $E$ will be Koszul complexes generated over $R'$ and $R$ by minimal sets of generators for $\mathcal{M}'$ and $\mathcal{M}$ respectively.

Let $s_1, \ldots, s_c$ be cycles representing a basis for the $k$-module $H_1(E')$. Put $F' = E' \langle S_1, \cdots, S_c; dS_i = s_i \rangle$. Then $F'$ is a minimal $R'$-free resolution of $k$. Consider the $R$-algebra $F = F' \otimes_{R'} R$ which contains the $R$-algebra $E$. Since the map $\mathcal{U}' \otimes k \to \mathcal{U} \otimes k$ is injective, the images of $s_1, \cdots, s_c$ in $H_1(E)$ can be extended to a basis for $H_1(E)$. Indeed, we have a commutative diagram
where the left vertical map is induced by the obvious map $E' \rightarrow E$. Therefore, by the theorem in [4], $F$ can be extended to a minimal $R$-algebra resolution $X$ of $k$ of the form $X = F \langle \cdots U_i \cdots \rangle$. Since for $q \geq 2 e_q(R)$ is the number of variables of degree $q + 1$ adjoined to $F$ in order to obtain $X$, and since $e_q(R) = 0$ for all sufficiently large $q$, $X$ has in fact the form

\[ X = F \langle U_1, \cdots, U_r \rangle = F \langle U_1 \rangle \cdots \langle U_r \rangle. \]

for some $r \geq 0$.

Every sub-$R$-algebra of $X$ containing $F$ satisfies the assumptions (i) and (ii) of lemma 1. Since $X$ is acyclic we have $q(X) = 0 < \infty$. Hence, using lemma 2, $r$ times we obtain

\[ q(F) < \infty. \]

But $H(F) = \text{Tor}^R(k, R)$, hence $R$ has finite projective dimension over $R'$.

**Remark.** In [1] André defines homology groups $H_i(A, B, W)$ where $B$ is a commutative algebra over a commutative ring $A$, and $W$ is a $B$-module. For a local ring $R$ with residue field $k$ he defines the simplicial dimension of $R$ as follows

\[ s\text{-dim } R = \inf \{ r | H_i(R, k, k) = 0 \text{ for } i \geq r \} \quad (\inf \emptyset = \infty). \]

In studying the relationship between $\text{Tor}^R(k \cdot k)$ and $H_i(R, k, k)$ we are tempted to conjecture that $e_i(R) = \dim_k H_{i+1}(R, k, k)$ for $i \geq 0$. It would follow from this that the only local rings of finite simplicial dimension are the complete intersection.

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