Compositio Mathematica

T. H. Gulliksen

A homological characterization of local complete intersections

Compositio Mathematica, tome 23, nº 3 (1971), p. 251-255
<http://www.numdam.org/item?id=CM_1971__23_3_251_0>

COMPOSITIO MATHEMATICA, Vol. 23, Fasc. 3, 1971, pag. 251–255
Wolters-Noordhoff Publishing
Printed in the Netherlands

A HOMOLOGICAL CHARACTERIZATION OF LOCAL COMPLETE INTERSECTIONS

by

T. H. Gulliksen

5th Nordic Summer-School in Mathematics,
Oslo, August 5–25, 1970

Let R denote a local ring with residue field $k = R/\mathfrak{M}$. Let P_R be the Poincaré series of R i.e. the power series

$$P_R = \sum_{p=0}^{\infty} \dim_k \operatorname{Tor}_p^R(k, k) Z^p.$$

It is known that P_R may be written uniquely as a product of the form

$$P_R = \prod_{i=0}^{\infty} \frac{(1+Z^{2i+1})^{\varepsilon_{2i}}}{(1-Z^{2i+2})^{\varepsilon_{2i+1}}}$$

where $\varepsilon_q(R) = \varepsilon_q$ $(q = 0, 1, \cdots)$ are non-negative integers only depending on R (Assmus, Levin). If R is a local complete intersection (i.e. the $\mathfrak{M}$-adic completion of R is a factor ring of a regular ring $\tilde{R}$ modulo an $\tilde{R}$-sequence) then it is known that $\varepsilon_q = 0$ for all $q \geqq 2$ (Tate, Zariski). If $\varepsilon_2 = 0$ or $\varepsilon_3 = 0$ then R is a complete intersection. The case $\varepsilon_2 = 0$ is due to Assmus, the case $\varepsilon_3 = 0$ is due to the author. Cf. [5].

The purpose of this note is to prove the following:

THEOREM. *If $\varepsilon_q(R) = 0$ for all sufficiently large q, then R is a local complete intersection.*

NOTATION. The term 'R-algebra' will be used in the sense of Tate [6] i.e. an associative, graded, differential, stricktly skew-commutative algebra X over R, with unit element 1, such that the homogeneous components X_q are finitely generated modules over R, $X_0 = 1 \cdot R$ and $X_q = 0$ for $q < 0$.

$Z_+(X)$ (resp. $H_+(X)$) will denote the set of homogeneous cycles (resp. homologyclasses) in X of positive degree.

If X is an R-algebra and s is a homogeneous cycle in X, then $X\langle S; dS = s\rangle$ or briefly $X\langle S\rangle$ denotes the R-algebra obtained from X by the adjunction of a variable S which kills s. Cf. [6].

By the Koszul complex over R generated by elements $t_1, \cdots, t_n$ in R we mean the R-algebra obtained from the trivial R-algebra R by the adjunction of variables $T_1, \cdots, T_n$ of degree 1 killing $t_1, \cdots, t_n$.

Lemma 1. Let X be an R-algebra satisfying

(i) $H_0(X) \approx R/\mathfrak{M}$

(ii) $Z_+(X) \subset \mathfrak{M}X$.

Let $n = \dim \mathfrak{M}/\mathfrak{M}^2$.

Then for all $\sigma \in H_+(X)$ we have $\sigma^{n+1} = 0$.

Proof. Let s be a cycle representing σ.

Let $\mathfrak{M}$ be minimally generated by $t_1, \cdots, t_n$. By (ii) there exist $x_1, \cdots, x_n \in X$ such that

$$s = \sum_{i=1}^{n} t_i x_i.$$

By (i) we can choose elements $T_1, \cdots, T_n$ of degree 1 such that $dT_i = t_i$ for $i = 1, \cdots, n$. s is obviously homologous to the cycle

$$s_0 := \sum_{i=1}^{n} T_i \, dx_i.$$

Since $T_i^2 = 0$ for all i we have $s_0^{n+1} = 0$, hence $\sigma^{n+1} = 0$.

Definition. Let X be an R-algebra. Define

$$\mathrm{q}(X) = \inf\{r | H_i(X) = 0 \text{ for all } i > r\}$$
$$(\inf \emptyset = \infty).$$

Lemma 2. *Let X be an R-algebra satisfying the assumptions* (i) *and* (ii) *of lemma* 1. *Let s be a homogeneous cycle of positive degree in X and put $Y = X\langle S; dS = s\rangle$. Then*

$$q(Y) < \infty \Rightarrow q(X) < \infty$$

Proof. Let us assume that $q(Y) < \infty$. We will consider two cases. First assume that $\deg S$ is even. In this case we have an exact sequence of complexes

$$0 \to X \xrightarrow{i} Y \xrightarrow{j} Y \to 0$$

where i and j are maps of degree 0 and $-\deg S$ respectively. Cf. [6]. Looking at the associated exact homology sequence one sees that $q(X) < \infty$.

Let us now consider that case where $\deg S$ is odd. In this case we have an exact sequence of complexes

$$0 \to X \xrightarrow{i} Y \xrightarrow{j} X \to 0$$

where i and j have degrees 0 and $-\deg S$ respectively and where the connecting homomorphism d_* in the associated homology triangle

$$\begin{tikzcd} & H(Y) \arrow[dr, "j_*"] & \\ H(X) \arrow[ur, "i_*"] & & H(X) \arrow[ll, "d_*"'] \end{tikzcd}$$

is, up to sign, multiplication by σ, see the proof of theorem 2 in [6]. Now put $n = \dim \mathfrak{M}/\mathfrak{M}^2$ and $v = \deg \sigma$. Using (1) we obtain for each $r > q(Y)$ an exact sequence

$$H_r(X) \overset{d_{*r}}{\to} H_{r+v}(X) \to H_{r+v}(Y) = 0$$

Hence $H_{r+v}(Y) = \sigma H_r(X)$ for $r > q(Y)$. It follows that

$$H_{r+(n+1)v}(X) = \sigma^{n+1} H_r(X) \qquad \text{for} \quad r > q(Y).$$

By Lemma 1 we have $\sigma^{n+1} = 0$. It follows that $q(X) < \infty$.

Proof of the theorem: It is enough to prove the theorem for complete local rings, hence we may assume that there exists a regular ring $\tilde{R}$ and a surjective ringhomomorphism $f: \tilde{R} \to R$. Put $\mathfrak{A} = \ker f$. We may also assume that $\mathfrak{A}$ is contained in the square of the maximal ideal $\tilde{\mathfrak{M}}$ in $\tilde{R}$.

Let $a_1, \cdots, a_c$ be a maximal $\tilde{R}$-sequence in $\mathfrak{A}$ and let $\mathfrak{A}'$ be the ideal generated by $a_1, \cdots, a_c$. This sequence can be chosen such that it can be extended to a minimal set of generators for $\mathfrak{A}$, i.e. the canonical map $\mathfrak{A}' \otimes k \to \mathfrak{A} \otimes k$ is injective. Put $R' = \tilde{R}/\mathfrak{A}'$ and let $g : R' \to R$ be the homomorphism induced by $f: \tilde{R} \to R$.

Now assume that $\varepsilon_q(R) = 0$ for all q sufficiently large. We will show that $\ker g = 0$. It suffices to show that R is an R'-module of finite projective dimension. Indeed, by construction every element in $\ker g$ is a zero-divisor in the ring R'. Hence if $pd_{R'} R < \infty$ it follows from proposotion 6.2 in [3] that $\ker g = 0$.

Let $\tilde{E}$ be the Koszul complex generated over $\tilde{R}$ by a minimal set of generators for $\tilde{\mathfrak{M}}$. Since $\ker f \subset \tilde{\mathfrak{M}}^2$, the rings $\tilde{R}$, R' and R have the same imbedding dimension. Thus, putting $E' := \tilde{E} \otimes_{\tilde{R}} R'$ and $E := \tilde{E} \otimes_{\tilde{R}} R = E' \otimes_{R'} R$, E' and E will be Koszul complexes generated over R' and R by minimal sets of generators for $\mathfrak{M}'$ and $\mathfrak{M}$ respectively.

Let $s_1, \cdots, s_c$ be cycles representing a basis for the k-module $H_1(E')$. Put $F' = E' \langle S_1, \cdots, S_c; dS_i = s_i \rangle$. Then F' is a minimal R'-free resolution of k. Consider the R-algebra $F := F' \otimes_{R'} R$ which contains the R-algebra E. Since the map $\mathfrak{A}' \otimes k \to \mathfrak{A} \otimes k$ is injective, the images of $s_1, \cdots, s_c$ in $H_1(E)$ can be extended to a basis for $H_1(E)$. Indeed, we have a commutative diagram

$$\begin{array}{ccc} H_1(E') & \xrightarrow[\approx]{} & \mathfrak{A}' \otimes k \\ \downarrow & & \downarrow \cap \\ H_1(E) & \xrightarrow[\approx]{} & \mathfrak{A} \otimes k \end{array}$$

where the left vertical map is induced by the obvious map $E' \to E$. Therefore, by the theorem in [4], F can be extended to a minimal R-algebra resolution X of k of the form $X = F\langle \cdots U_i \cdots \rangle$. Since for $q \geqq 2$ $\varepsilon_q(R)$ is the number of variables of degree $q+1$ adjoined to F in order to obtain X, and since $\varepsilon_q(R) = 0$ for all sufficiently large q, X has in fact the form

$$X = F\langle U_1, \cdots, U_r\rangle = F\langle U_1\rangle \cdots \langle U_r\rangle.$$

for some $r \geqq 0$.

Every sub-R-algebra of X containing F satisfies the assumptions (i) and (ii) of lemma 1. Since X is acyclic we have $q(X) = 0 < \infty$. Hence, using lemma 2, r times we obtain

$$q(F) < \infty.$$

But $H(F) = \mathrm{Tor}^{R'}(k, R)$, hence R has finite projective dimension over R'.

Remark. In [1] André defines homology groups $H.(A, B, W)$ where B is a commutative algebra over a commutative ring A, and W is a B-module. For a local ring R with residue field k he defines the simplicial dimension of R as follows

$$s\text{-dim } R = \inf\{r \mid H_i(R, k, k) = 0 \text{ for } i \geqq r\} \ (\inf \emptyset = \infty).$$

In studying the relationship between $\mathrm{Tor}^R_{\cdot}(k \cdot k)$ and $H.(R, k, k)$ we are tempted to conjecture that $\varepsilon_i(R) = \dim_k H_{i+1}(R, k, k)$ for $i \geqq 0$. It would follow from this that the only local rings of finite simplicial dimension are the complete intersection.

REFERENCES

M. André

[1] Méthode Simpliciale en Algèbre Homologique et Algèbre Commutative. Springer lecture Notes 32 (1967).

E. F. Assmus

[2] On the homology of local rings, Illinois J. of Math. 3 (1959) 187–199.

M. Auslander, D. A. Buchsbaum

[3] Codimension and multiplicity. Annals of Math. 68 (1958) 625–657.

T. H. Gulliksen

[4] A proof of the existence of minimal R-algebra resolutions. Acta Math. 120 (1968).

T. H. GULLIKSEN, G. LEVIN
[5] Homology of local rings. Queen's papers in pure and appl. Math. No. 20 (1969).
J. TATE
[6] Homology of Noetherian rings and local rings. Illinois J. of Math. 1 (1957) 14–25.

(Oblatum 5–X–1970)

T. H. Gulliksen
Universitetet i Oslo
Matematisk Institutt
BLINDERN, Oslo 3
Norway

This paper also appears in 'Algebraic Geometry, Oslo 1970', Proceedings of the 5th Nordic Summer-School in Mathematics, Wolters-Noordhoff Publishing 1971.