COMPOSITIO MATHEMATICA

TAMMO TOM DIECK

Partitions of unity in homotopy theory

Compositio Mathematica, tome 23, n° 2 (1971), p. 159-167

<http://www.numdam.org/item?id=CM_1971__23_2_159_0>

COMPOSITIO MATHEMATICA, Vol. 23, Fasc. 2, 1971, pag. 159-167
Wolters-Noordhoff Publishing
Printed in the Netherlands

PARTITIONS OF UNITY IN HOMOTOPY THEORY

by

Tammo tom Dieck (Saarbrücken)

We prove, roughly, that a map f is a homotopy equivalence if f is locally a homotopy equivalence. We also prove that $p : E \to B$ is a fibration if the restrictions of p to the sets E_α of a suitable covering (E_α) of E are fibrations.

The paper was inspired by talks of Dold (see [5]) and might well be considered a second part to Dold [4]. The essential difference to the work of Dold is that we have to consider numerable coverings of a space X which are closed under finite intersections. We use the fundamental observation of G. Segal ([11], Prop. 4.1) that the „classifying space" of such a covering is homotopy equivalent to X. It seems that this theorem of Segal and the section extension theorem of Dold ([4], 2.7) are the two foundation stones of the theory.

1. The main results

A covering $(E_\alpha | \alpha \in A)$ of a space E is called numerable if there exists a locally finite partition of unity $(t_\alpha | \alpha \in A)$ such that the closure of $t_\alpha^{-1}]0, 1]$ is contained in E_α. If $\sigma \subset A$ we put

$$A_\sigma = \bigcap_{\alpha \in \sigma} A_\alpha .$$

(From now on we use only non-empty σ in this context!) If B is a fixed topological space we have the category Top/B of spaces over B and we have a notion of homotopy and homotopy equivalence over B (see Dold [4], 1).

THEOREM 1. *Let $p : X \to B$ and $q : Y \to B$ be spaces over B and $f : X \to Y$ a map over B (i.e. $qf = p$). Let $U = (X_\alpha | \alpha \in A)$ resp. $V = (Y_\alpha | \alpha \in A)$ be a numerable covering of X resp. Y. Assume $f(X_\alpha) \subset Y_\alpha$ and that for every finite $\sigma \subset A$ the map $f_\sigma : X_\sigma \to Y_\sigma$ induced by f is a homotopy equivalence over B. Then f is a homotopy equivalence over B.*

We call $p : E \to B$ a fibration if it has the covering homotopy property for all spaces (Hurewicz fibration). We call $p : E \to B$ an h-fibration if p is homotopy equivalent over B to a fibration. (Then p has the weak covering homotopy property (WCHP) in the sense of Dold [4], 5. See also

[3] for details.) We call $p : E \to B$ shrinkable if p is homotopy equivalent over B to $\mathrm{id} : B \to B$.

THEOREM 2. *Let $p : E \to X$ be a continuous map. Let $U = (E_\alpha | \alpha \in A)$ be a family of subsets of E and let $V = (X_\alpha | \alpha \in A)$ be a numerable covering of X. Assume $p(E_\alpha) \subset X_\alpha$ and that for finite $\sigma \subset A$ the map $p_\sigma : E_\sigma \to X_\sigma$ induced by p is shrinkable. Then p has a section.*

The following theorem answers questions of Dold and D. Puppe (see [5]).

THEOREM 3. *Let $p : E \to B$ be a continuous map. Let $U = (E_\alpha | \alpha \in A)$ be a numerable covering such that for every finite $\sigma \subset A$ the restriction $p_\sigma : E_\sigma \to B$ of p to E_σ is a fibration (an h- fibration, shrinkable). Then p is a fibration (an h-fibration, shrinkable).*

The above theorems and their proofs have many corollaries and applications. We mention some of them.

THEOREM 4. *Let $U = (X_\alpha | \in A)$ be a numerable covering of a space X. If all the X_σ have the homotopy type of a CW-complex then X has the homotopy type of a CW-complex.*

The hypothesis of Theorem 4 is, for instance, satisfied if all the X_σ are either empty or contractible. This in turn is true for spaces which are equi-locally convex (Milnor [9]). Another application of Theorem 4 is the following: If $p : E \to B$ is an h-fibration, if B has the homotopy type of a CW-complex and if every fibre $p^{-1}(b)$, $b \in B$, has the homotopy type of a CW-complex, then E has the homotopy type of a CW-complex.

THEOREM 5. *Let $U = (X_\alpha | \alpha \in A)$ be an open covering of X and $V = (Y_\alpha | \alpha \in A)$ an open covering of Y. Let $f : X \to Y$ be a continuous map with $f(X_\alpha) \subset Y_\alpha$.*
(a) *If the $f_\sigma : X_\sigma \to Y_\sigma$ are homotopy equivalences then f induces for every paracompact space Z a bijection*

$$f_* : [Z, X] \to [Z, Y]$$

of homotopy sets.

(b) *If the f_σ are weak homotopy equivalences then f is a weak homotopy equivalence.*

THEOREM 5(b) is a variant of a result of McCord [8, Theorem 6]. Compare also the special case discussed by Eells and Kuiper [6].

2. Homotopy equivalences

In this section we prove Theorems 1, 4 and 5. We begin with the proof of Theorem 1. For simplicity we omit the phrase 'over B'. In the following lemmas, for instance, we use cofibrations 'over B' and homotopies 'over B'.

The covering U of X leads to the classifying space BX_U introduced by G. Segal ([11], p. 108). We recall the basic properties of this space. The map f induces $F : BX_U \to BY_V$, because the construction of BX_U is functorial. We have a commutative diagram

$$\begin{array}{ccc} BX_U & \xrightarrow{F} & BY_V \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle pr} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the vertical maps are homotopy equivalences (Prop. 4.1 of Segal [11]). Note that compactly generated spaces do not enter that proposition. Note also that BX_U is a space 'over B' and that pr is a homotopy equivalence 'over B'. It is useful to observe that pr is in fact shrinkable – as the proof of Segal shows – and hence in particular an h-fibration. The space BX_U, being the geometric realisation of a semi-simplicial space, has a functorial filtration by skeletons $BX_U^{(n)}$, $n = 0, 1, 2, \cdots$. We need the following lemma in order to prove that F induces homotopy equivalences

$$F^{(n)} : BX_U^{(n)} \to BY_V^{(n)}.$$

LEMMA 1. *Given a commutative diagram*

$$\begin{array}{ccccc} A_1 & \xleftarrow{f_1} & A_0 & \xrightarrow{f_2} & A_2 \\ {\scriptstyle h_1}\downarrow & & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_2}\downarrow \\ B_1 & \xleftarrow[g_1]{} & B_0 & \xrightarrow[g_2]{} & B_2 \end{array}$$

where f_1, g_1 are cofibrations and h_0, h_1, h_2 are homotopy equivalences. Then h_0, h_1, h_2 induce a homotopy equivalence $h : A \to B$ where A is the push-out of (f_1, f_2) and B the push-out of (g_1, g_2).

PROOF. The lemma is of course well known, see R. Brown [1], 7.5.7. We sketch a proof because we need the basic ingredient also for other purposes. Using the homotopy theorem for cofibrations (compare [3], 7.42) we can assume without loss of generality that f_2 and g_2 are cofibrations, too. But then it is clear that Lemma 1 follows from Lemma 2 below. (Compare the detailed proof of a dual lemma in R. Brown and P. R. Heath [2]).

LEMMA 2. *Given a commutative diagram*

$$\begin{array}{ccc} A_0 & \xrightarrow{f} & A_1 \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\ B_0 & \xrightarrow[g]{} & B_1 \end{array}$$

where f and g are cofibrations and h_0 and h_1 homotopy equivalences. Given a homotopy equivalence $H_0 : B_0 \to A_0$ and a homotopy $\varphi : A_0 \times I \to A_0$ with $\varphi(a, 0) = H_0 h_0(a)$, $\varphi(a, 1) = a$ for $a \in A_0$. Then we can find a homotopy equivalence $H_1 : B_1 \to A_1$ with $fH_0 = H_1 g$ and a homotopy $\psi : A_1 \times I \to A_1$ with $\psi(a, 0) = H_1 h_1(a)$, $\psi(a, 1) = a$ for $a \in A_1$ and

$$\psi(fa, t) = \begin{cases} f\varphi(a, 2t) & a \varepsilon A_0;\ t \leqq \frac{1}{2} \\ f(a) & a \varepsilon A_0;\ t \geqq \frac{1}{2}. \end{cases}$$

Proof. [3], 2.5.

We can now prove by induction over n

LEMMA 3. *The map $F^{(n)} : BX_U^{(n)} \to BY_V^{(n)}$ is a homotopy equivalence.*

PROOF. The space $BX_U^{(0)}$ is the topological sum of the X_σ, $\sigma \subset A$ finite. Hence $F^{(0)}$ is obviously a homotopy equivalence. We can construct $BX_U^{(n)}$ from $BX_U^{(n-1)}$ via the following push-out diagram

$$\begin{array}{ccc} \coprod_{\tau \in A_n} (X_{q(\tau)} \times \partial \Delta^n) & \xrightarrow{k_n} & BX_U^{(n-1)} \\ \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle J_n} \\ \coprod_{\tau \in A_n} (X_{q(\tau)} \times \Delta^n) & \xrightarrow[K_n]{} & BX_U^{(n)} \end{array}$$

Explanation: Δ^n is the standard n-simplex with boundary $\partial\Delta^n$ and j_n is induced by the inclusion $\partial\Delta^n \subset \Delta^n$. Note that j_n is a cofibration (over B!). The topological sum is over $\tau \in A_n$, where

$$A_n = \{(\sigma_0, \cdots, \sigma_n) | \sigma_0 \subsetneqq \cdots \subsetneqq \sigma_n,\ \sigma_n \subset A \text{ finite}\},$$

and $q(\sigma_0, \cdots, \sigma_n) = \sigma_n$. The map k_n is the attaching map for the n-simplices. Lemma 1 gives the inductive step.

As a corollary to the preceding proof we have

LEMMA 4. *The map $J_n : BX_U^{(n-1)} \to BX_U^{(n)}$ is a cofibration.*

We also need

LEMMA 5. *The space BX_U is the topological direct limit of the $BX_U^{(n)}$.*

Proof. Geometric realisation commutes with direct limits.

In view of Lemma 3 to 5 the following lemma will finish the proof of Theorem 1. Consider a commutative diagram

$$\begin{array}{ccccccc} X_0 & \xrightarrow{i_1} & X_1 & \xrightarrow{i_2} & X_2 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ Y_0 & \xrightarrow[I_1]{} & Y_1 & \xrightarrow[I_2]{} & Y_2 & \longrightarrow & \cdots \end{array}$$

where $i_1, i_2, \cdots$ and $I_1, I_2, \cdots$ are cofibrations and $f_0, f_1, \cdots$ are homotopy equivalences. Let X be the topological direct limit of the i_k, Y the limit of the I_k and $f : X \to Y$ the map induced by the f_k.

Lemma 6. *The map $f : X \to Y$ is a homotopy equivalence.*

Proof. (Compare [3], § 10.) Using Lemma 2 we construct inductively homotopy equivalences $F_n : Y_n \to X_n$ with $i_n F_{n-1} = F_n I_n$ and homotopies $\varphi_n : X_n \to X_n$ from $F_n f_n$ to $\mathrm{id}(X_n)$ such that φ_n is constant for $t \geqq 1 - 2^{-(n+1)}$ and such that $(i_n \times \mathrm{id})\varphi_n = \varphi_{n-1}$. The F_n and φ_n induce $F : Y \to X$ and $\varphi : X \times I \to X$ such that $\varphi(x, 0) = F_n(x)$ and $\varphi(x, 1) = x$ for $x \in X$. Hence f has a homotopy left inverse.

Remark 1. Lemma 6 shows in particular that $X = \lim X_k$ is the homotopy direct limit of the X_k in the sense of Milnor [(10], p. 149), i.e. the projection of the telescope of the i_n onto X is a homotopy equivalence.

Remark 2. The numerability of the covering U is only used to establish the homotopy equivalence $BX_U \simeq X$. The map $F : BX_U \to BY_V$ is always a homotopy equivalence, if the f_σ are homotopy equivalences. There are other cases in which $\mathrm{pr} : BX_U \to X$ is a homotopy equivalence, e.g. if U is closed, finite-dimensional and the inclusions $X_\sigma \subset X_\tau$ are cofibrations.

Proof of Theorem 4. We show that BX_U has the homotopy type of a CW-complex. The procedure is the same as in the proof of Theorem 1. If in the diagram

$$A_1 \xleftarrow[f]{} A_0 \xrightarrow[g]{} A_2$$

all spaces have the homotopy type of a CW-complex and if f is a cofibration, then the push-out has the homotopy type of a CW-complex. This shows inductively that the $BX_U^{(n)}$ have the homotopy type of a CW-complex. One finishes the proof using Lemma 5, Lemma 6 and Remark 1.

Proof of Theorem 5. Let $U = (X_\alpha | \alpha \in A)$ be any covering of X. Consider $\mathrm{pr} : BX_U \to X$. We claim that for every $\alpha \in A$ the map $\mathrm{pr}_\alpha : \mathrm{pr}^{-1} X_\alpha \to X_\alpha$

is shrinkable. If $U(\alpha)$ is the covering $(X_\alpha \bigcap X_\beta | \beta \in A)$ of X_α we show that its classifying space, B_α say, is canonically homeomorphic to $\mathrm{pr}^{-1}X_\alpha$. The result then follows since $U(\alpha)$ is clearly a numerable covering of X_α because it contains X_α. The homeomorphism $B_\alpha \cong \mathrm{pr}^{-1}X_\alpha$ follows along the lines of Gabriel-Zisman [7], Ch. III, 3.2.

Let now U be an open covering of X. We show that for a paracompact Z the map $\mathrm{pr}_* : [Z, BX_U] \to [Z, X]$ is bijective. We consider a pull-back diagram

$$\begin{array}{ccc} E & \xrightarrow{q} & BX_U \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle pr} \\ Z & \xrightarrow{f} & X \end{array}$$

for given f. By Corollary 3.2 of Dold [4] we see that q is shrinkable. Let $s : Z \to E$ be a section of q. Then gs satisfies $\mathrm{pr} \circ gs = f$ and hence pr_* is surjective. Injectivity follows similarly; one has to use Prop. 3.1 of Dold [4]. Theorem 5(a) follows.

To prove Theorem 5(b) we show that $F : BX_U \to BY_V$ is a weak homotopy equivalence if the f_σ are weak homotopy equivalences. We prove analogues of Lemmas 3 to 6. But this is standard homotopy theory.

3. Sections

We prove Theorem 2. We use the notations of the previous section.

We construct a map s such that the following diagram is commutative

$$\begin{array}{ccc} & & E \\ & {\scriptstyle s}\nearrow & \downarrow{\scriptstyle p} \\ BX_U & \xrightarrow{pr} & B. \end{array}$$

More precisely we construct inductively maps $s^{(n)} : BX_U^{(n)} \to E$ with $ps^{(n)} = \mathrm{pr}|BX_U^{(n)}$, $J_n s^{(n)} = s^{(n-1)}$, and an additional property to be mentioned soon.

The map

$$s^{(0)} : \coprod_{\sigma \in A_0} X_\sigma \to E$$

is given as follows: $s^{(0)}|X_\sigma \to E$ is a section $X_\sigma \to E_\sigma$ composed with the inclusion $E_\sigma \subset E$. The section exists because $E_\sigma \to B_\sigma$ is shrinkable. The equality $ps^{(0)} = \mathrm{pr}|BX_U^{(0)}$ clearly holds. Suppose $s^{(n-1)}$ is given. We want to extend

$$s^{(n-1)}k_n : \coprod (X_{q(\tau)} \times \partial \Delta^n) \to E$$

over $\coprod (X_{q(\tau)} \times \Delta^n)$. If $\tau = (\sigma_0, \cdots, \sigma_n)$, we impose the additional induction hypothesis that the image of $X_{q(\tau)} \times \partial \Delta^n$ under $s^{(n-1)}k_n$ is contained in E_{σ_0}. The construction of $s^{(0)}$ agrees with this requirement. With our new hypothesis we have the commutative diagram

$$\begin{array}{ccc} X_{q(\tau)} \times \partial \Delta^n & \xrightarrow{s^{(n-1)}k_n} & E_{\sigma_0} \\ \downarrow{\scriptstyle pr} & & \downarrow \\ X_{q(\tau)} & \xrightarrow[\subset]{} & X_{\sigma_0} \end{array}$$

From Dold [4], Prop. 3.1(b), we see that $s^{(n-1)}k_n$ can be extended over $\coprod X_{q(\tau)} \times \Delta_n$ and hence we can construct $s^{(n)}$ via the push-out diagram entering the proof of Lemma 3. The properties $ps^{(n)} = \mathrm{pr}|BX_U^{(n)}$ and $J_n s^{(n)} = s^{(n-1)}$ are obvious from the construction. We show that $s^{(n)}$ satisfies the additional induction hypothesis. Given $\tau = (\sigma_0, \cdots, \sigma_{n+1})$ we describe

$$k_{n+1} : X_{q(\tau)} \times \partial \Delta^{n+1} \to BX_U^{(n)}.$$

Let $d_i : \Delta^n \to \Delta_i^{n+1}$ be the standard map onto the i-th face of Δ^{n+1} and let e_i be the inverse homeomorphism. Let

$$\partial_i : X_{q(\tau)} \to X_{q(\varepsilon_i \tau)}$$

be the inclusion, where

$$\varepsilon_i \tau = (\sigma_0, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_{n+1}).$$

The restriction of k_{n+1} to $X_{q(\tau)} \times \Delta_i^{n+1}$ is $K_n(\partial_i \times e_i)$. By construction of $s^{(n)}$ the image of $s^{(n)}K_n(\partial_i \times e_i)$ is contained in E_{σ_0} (for $i > 0$) or E_{σ_1} (for $i = 0$). But $E_{\sigma_1} \subset E_{\sigma_0}$, hence $s^{(n)}$ has the desired property. Because of Lemma 5 the maps $s^{(n)}$ combine to give $s : BX_U \to E$.

If (X_α) is numerable then $\mathrm{pr} : BX_U \to X$ has a section t and $st : B \to E$ will then be a section of p. This proves Theorem 2.

4. Fibrations

If $p : E \to B$ is a map we denote by W_p the subspace

$$W_p = \{(w, e) | w(1) = pe\} \subset B^I \times E,$$

where B^I is the path space with compact open topology. The map

$$\pi_p : E^I \to W_p,$$

defined by $\pi_p(v) = (\mathrm{pv}, \mathrm{v}(1))$, is shrinkable if p is a fibration. Conversely, if π_p has a section then p is a fibration.

In general we have a commutative diagram

$$\begin{array}{ccc} W_p & \xrightarrow{k_p} & E \\ & {\scriptstyle j_p}\searrow \quad \swarrow{\scriptstyle p} & \\ & B & \end{array}$$

$k_p(w, e) = e, j_p(w, e) = pe$. The map k_p is a homotopy equivalence and j_p is a fibration. From our definition of h-fibrations and Theorem 6.1 of Dold [4] it follows immediately that p is an h-fibration if an only if k_p is a homotopy equivalence over B.

We have recalled these characterisations of fibrations and h-fibrations because we want to use them in the following proof of Theorem 3.

Proof of Theorem 3. To begin with let us assume that the p_σ are h-fibrations. The W_{p_σ} form a numerable covering of W_p and we have $k_p(W_{p_\sigma}) \subset E_\sigma$. Moreover we know that $W_{p_\sigma} \to E_\sigma$ is a homotopy equivalence over B because p_σ is an h-fibration. We are now in a position to apply Theorem 1, which tells us that k_p is a homotopy equivalence over B. Hence p is an h-fibration.

Now assume that the p_σ are fibrations. We want to show that π_p has a section. We use Theorem 2. We have the numerable covering $(W_{p_\alpha} | \alpha \in A)$ of W_p and we have the family of subsets $(E_\alpha^I | \alpha \in A)$. Moreover $E_\sigma^I \to W_p$ is shrinkable because p_σ is a fibration. (Note that this is also true if E_σ is empty.) Theorem 2 gives the desired section of π_p.

Finally assume that the p_σ are shrinkable, i.e. homotopy equivalences over B. Theorem 1 shows that p is a homotopy equivalences over B. The proof of Theorem 3 is now finished.

REFERENCES

R. Brown

[1] Elements of Modern Topology. McGraw–Hill, London (1968).

R. Brown and P. R. Heath

[2] Coglueing Homotopy Equivalences. Math. Z. 113, 313–325 (1970).

T. tom Dieck, K. H. Kamps and D. Puppe

[3] Homotopietheorie. Springer Lecture Notes Vol. 157 (1970).

A. Dold

[4] Partitions of unity in the theory of fibrations. Ann. of Math. 78, 223–255 (1963).

A. Dold

[5] Local extention properties in topology. Proc. Adv. Study Inst. Alg. Top., Aarhus (1970).

J. E. Eells, and Kuiper, N. H.
[6] Homotopy negligible subsets. Compositio math. 21, 155–161 (1965).

P. Gabiel and M. Zisman
[7] Calculus of fractions and homotopy theory. Springer-Verlag, Berlin-Heidelberg-New York (1967).

M. McCord
[8] Singular homology groups and homotopy groups of finite topological spaces. Duke Math. J. 33, 465–474 (1966).

J. Milnor
[9] On spaces having the homotopy type of a CW-complex. Trans. Amer. Math. Soc. 90, 272–280 (1959).

J. Milnor
[10] Morse theory. Princeton Univ. Press, Princeton (1963).

G. Segal
[11] Classifying spaces and spectral sequences. Publ. math. I.H.E.S. 34, 105–112 (1968).

(Oblatum: 17–XI–(70)

Mathematisches Institut der
Universität des Saarlandes
66 Saarbrücken
Germany