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TOPOLOGICAL STABILITY FOR INFINITE-DIMENSIONAL MANIFOLDS [1]

by

R. Schori

1. Introduction

R. D. Anderson and R. Schori prove in [2] and [3] the following theorem that will be referred to as Theorem I. If M is a separable metric manifold modeled on s, the countable infinite product of lines, then $M \times s$ is homeomorphic ($\cong$) to M. The proof of this theorem uses the product structure of s very crucially. However, since s is homeomorphic to each infinite-dimensional separable Fréchet space [1], we may replace s in the statement of Theorem I by any such Fréchet space, for example, separable infinite-dimensional Hilbert space, l_2. A *Fréchet space* is a locally convex, complete, metric topological vector space and a *Fréchet manifold* is a metrizable manifold modeled on an infinite dimensional Fréchet space. In [8] and [9], David W. Henderson proves that if M is any separable Fréchet manifold, then $M \times l_2$ can be embedded as an open subset of l_2. Thus, using the result that $M \times l_2 \cong M$ Henderson has the open embedding theorem for separable Fréchet manifolds.

In this paper we give a substantial generalization of Theorem I. The following is a main theorem of this paper and is stated as Corollary 5.5. If M is a paracompact manifold modeled on a metrizable topological vector space F such that $F \cong F^{\omega}$ (the countable infinite product of F), then $M \times F \cong M$. (For more general results see Theorems 5.4 and 5.10) The proof of this theorem is shorter than but follows in broad outline the proof of Theorem I. However, it should be noted that all references to separability, completeness, and local convexity that are implicit in the hypothzsis of Theorem I have been deleted. The condition $F \cong F^{\omega}$ is obviously satisfied by all separable infinite-dimensional Fréchet spaces as they are all homeomorphic to s. Furthermore, it is known that many of the non-separable spaces (see section 4) also satisfy the condition; for example, the space of all bounded sequences, l_{∞}, and all infinite-dimensional Hilbert and reflexive Banach spaces. There are no known in-

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finite dimensional Fréchet spaces for which the condition is not satisfied. In order to utilize the product structure on F that is imposed by this condition, as the product structure on s is used, it is necessary to identify some special conditions that are formalized in Definition 4.1 of this paper.

Many of the techniques in the proof of Theorem 1 were strongly dependent upon separable metric properties and a large portion of this paper is devoted to developing alternative procedures. For example, the separable metric property of quaranteeing the existence of certain star finite open covers was used extensively in [3]. A theorem of E. A. Michael [13, Theorem 3.6] bridges one of these gaps and is used to obtain some rather general conditions for spaces Y and Z such that $Y \times Z \cong Z$. Another important tool in generalizing from the separable to the non-separable case is Lemma 5.2 of this paper.

Henderson has strongly generalized and supplemented his previously mentioned results. The author and Henderson, in a joint paper [12], combine the results of this paper and those in [10] and [11] to prove the following theorems: If M is a connected paracompact manifold modeled on a metric locally convex *TVS*, $F \cong F^{\omega}$, then M can be embedded as an open subset of F. If M and N are connected paracompact manifolds modeled on a normed *TVS*, $F \cong F^{\omega}$, then M and N are homeomorphic if and only if they have the same homotopy type.

The author thanks R. D. Anderson for introducing the problem to him, James Eells for suggesting the hypothesis $F \cong F^{\omega}$, and David W. Henderson and the members of his seminar at Cornell for many helpful comments.

2. Stability for open subsets of F

This section is included to give the reader an introduction to the language and methods of the rest of the paper and, in addition, yields some useful corollaries.

By a *space* we will mean a topological space. By TVS we will mean a Hausdorff topological vector space. Let F be a TVS, let $F_0^{\omega} = F \times F^{\omega}$, and let $\pi : F \times F^{\omega} \to F^{\omega}$ be the natural projection onto F^{ω}. For $n > 0$, let $\pi_n : F^{\omega} \to F^{\omega}$ be defined by $\pi_n(z) = (z_1, \cdots, z_n, 0, 0, \cdots)$ where $z = (z_1, z_2, \cdots) \in F^{\omega}$. Also, for Y a space and $g : F^{\omega} \to Y$, define $g^* : F \times F^{\omega} \to Y$ by $g^* = g\pi$. Let I be the closed unit interval.

By a *map* we will mean a continuous function. For spaces X, Y, and Z, a map $h : X \times Y \to Z$ is an *invertible isotopy* if the map $h_2 : X \times Y \to Z \times Y$ defined by $h_2(x, y) = (h(x, y), y)$ is a homeomorphism.

2.1 LEMMA. *There exists a function* $h : F_0^{\omega} \times [1, \infty) \to F^{\omega}$ *such that*

a) *h is an invertible isotopy, and*

b) *if* $u \in [1, \infty)$ *is fixed and* $n \leqq u$, *then* $H : F_0^\omega \to F^\omega$ *defined by* $H(x) = h(x, u)$ *has the property that* $\pi_n^* = \pi_n H$.

PROOF. Let $x = (x_0, x_1, x_2, \cdots) \in F_0^\omega$ and for $n > 0$, let

$$h(x, n) = (x_1, \cdots, x_n, -x_0, -x_{n+1}, -x_{n+2}, \cdots).$$

Extend h linearly, that is, for $n > 1$, $n-1 < u < n$, and $s = u-(n-1)$, let

$$h(x, u) = (x_1, \cdots, x_{n-1}, -(1-s)x_0+sx_n, -sx_0-(1-s)x_n, -x_{n+1}, \cdots.$$

The corresponding function h_2, from the definition of invertible isotopy, is continuous and is seen to be a homeomorphism by showing that it has a continuous inverse function. To do this it is sufficient, by Cramer's Rule, to verify that the determinant of the following matrix is non-zero for all $s \in I$.

$$\begin{pmatrix} -(1-s) & s \\ -s & -(1-s) \end{pmatrix}.$$

The determinant is equal to $(1-s)^2+s^2$ which is positive for all $s \in I$. Thus, condition a) is satisfied and condition b) is observed to be true by noticing in the definition of $h(x, n)$ that the $x_1, \cdots, x_n$ coordinates of x remained fixed.

Before proving stability for open subsets of F, we need a few concepts. Let $X = \pi_{i>0} X_i$ be the product of spaces $X_1, X_2, \cdots$. By an *n-basic* subset of X we will mean the product of a subset in $\pi_{i=1}^n X_i$ with $\pi_{i>n} X_i$. Note that if B is n-basic and $m \geqq n$, then B is also m-basic. A *basic* set is an n-basic set for some n. If B is a subset of a space X, then by $Cl_X B$, $\text{Int}_X B$, and $Bd_X B$ we mean the closure, interior, and boundary, respectively, of B in X.

A space X is *perfectly normal* if it is normal and each open set is an F_σ. It is known [6; Theorem 30A.6] that a perfectly normal space is hereditarily perfectly normal, that is, every subspace is also perfectly normal. Furthermore, it is well known that every metric space is perfectly normal.

2.2 THEOREM. *If* M *is an open subset of a perfectly normal TVS,* $F \cong F^\omega$, *then* $F \times M \cong M$.

PROOF. Assume $M \subset F^\omega$ and cover M with a collection B of basic open sets contained in M. By Lemma 5.2 of this paper there exists a collection $\{K_n\}_{n>0}$ of closed sets in F^ω such that $M = \bigcup_{n>0} K_n$, $K_n \subset \text{Int } K_{n+1}$ for $n > 0$, and each K_n is n-basic and a subset of $\bigcup\{b \in B : b \text{ is } n\text{-basic}\}$.

Define $g : M \to [1, \infty)$ inductively as follows. Let $g(K_1) = 2$, let $n > 1$, and assume g has been defined on K_{n-1}. Note that $BdK_n =$

$\pi_n^{-1}\pi_n(BdK_n)$. Let $g(BdK_n) = n+1$ and extend g to $\pi_n(K_n)$ with the Tietze extension theorem where the range of g is restricted to the interval $[n, n+1]$. Then extend g to the rest of K_n *by the product structure*. That is, for $x \in K_n$ where $g(x)$ has not been previously defined, let $g(x) = g(\pi_n(x))$. Thus, by induction we have defined g on $M = \bigcup_{n>0} K_n$ where g has the *local product property* with respect to the $\{K_n\}$, that is, if $x \in M$ and $m = \min\{i : x \in K_i\}$, then $g(x) = g(\pi_m(x)) \geqq m$.

Now define $H : F \times M \to F^\omega$ by $H(x) = h(x, g^*(x))$ where h is the function of 2.1 and $x \in F \times M$. The map H takes $F \times M$ into M since g has the local product property and since property b) of 2.1 holds. We now show that H maps $F \times M$ onto M. If $y \in M$, then $H_y : F_0^\omega \to F^\omega$ defined by $H_y(x) = h(x, g(y))$, for $x \in F_0^\omega$, is a homeomorphism by a) of 2.1. Thus, there exists $z \in F_0^\omega$ such that $H_y(z) = y$. By b) of 2.1 and since g has the local product property we have $g^*(z) = gH_y(z) = g(y)$ and hence $y = h(z, g^*(z)) = H(z)$. Clearly $z \in F \times M$. We now verify that H is a homeomorphism. Observe the following diagram

$$F \times M \xrightarrow{(i,\, g^*)} (F \times F^\omega) \times [1, \infty) \xrightarrow{h_2|A} F^\omega \times [1, \infty) \xrightarrow{P|B} F^\omega$$

where i is the injection of $F \times M$ into $F \times F^\omega$, $A =$ image (i, g^*), $B =$ image $(h_2|A)$, and p is the projection map. Note that H is the composition of these three maps. Each of (i, g^*) and $h_2|A$ is automatically a homeomorphism onto its image and we will show that $p|B$ is a homeomorphism onto M since it has a continuous inverse. The map $p|B$ takes $(H(x), g^*(x))$ to $H(x)$ where $x \in F \times M$ and thus the map taking $H(x)$ to $(H(x), g(H(x)) = (h(x), g^*(x))$ is a continuous inverse. Hence, $H : F \times M \to M$ is a homeomorphism.

2.3 Corollary. *Let M be an open subset of a perfectly normal TVS, $F \cong F^\omega$, and let G be an open cover of M. There exists a homeomorphism $H : F \times M \to M$ such that for each $x \in F \times M$, there exists $V \in G$ containing each of $\pi(x)$ and $H(x)$.*

Proof. In the proof of 2.2 pick the collection B of basic open sets so it refines G. If $z \in b \in B$ and $y \in F$, then $H(y, z) \in b$ since g has the local product property and b) of 2.1 holds. Since B refines G, the result follows.

2.4 Corollary. *The same hypothesis as* 2.3 *with the additional assumption that F is locally convex. Then the homeomorphism $H : F \times M \to M$ can be chosen to have the addition property of being homotopic to the projection π.*

Proof. We may assume that the members of the collection G of 2.3 were convex sets and define the homotopy ψ from H to π by $\psi(x, t) = (1-t)H(x) + t\pi(x)$ for $x \in F \times M$ and $t \in I$.

3. A product theorem criterion

In this section we give some rather general conditions for spaces Y and Z such that $Y \times Z \cong Z$. A function is an *embedding* if it is a homeomorphism onto its image. If $f: Y \times I \to Y$ and $t \in I$, then $f_t : Y \to Y$ is defined by $f_t(y) = f(y, t)$, for $y \in Y$.

3.1 Definition. A space Y is *strongly contractible* if there exists $y_0 \in Y$ and a map $f: Y \times I \to Y$ such that

1) $f_0(Y) = \{y_0\}$
2) $f_1 = id_Y$ and
3) $f_t(y_0) = y_0$ for all $t \in I$.

We will sometimes say that the triple (Y, y_0, f) is strongly contractible.

3.2 Definition. Let Y and Z be spaces where (Y, y_0, f) is strongly contractible and let $r : Z \to I$ be a map. Then the *variable product* of Y by Z with respect to r

$$Y \times_r Z = \{(f(y, r(z)), z) \in Y \times Z : (y, z) \in Y \times Z\}.$$

If $U \subset Z$, then by $Y \times_r U$ we mean the variable product of Y by U with respect to $r|U$ and call it the *restricted variable product above U*, sometimes abbreviated by *above U*. Note that if $r = 1$, then $Y \times_r Z = Y \times Z$.

3.3 Definition. Let (Y, y_0, f) be a strongly contractible space. A space Z has *Property P_Y* if each open subset U of Z is normal and satisfies the following condition: If $Y \times_{r_0} U$ is a variable product of Y by U and $K \subset W \subset U$ where K is closed in U and W is open, then there exists a homeomorphism H of $Y \times_{r_0} U$ onto a variable product $Y \times_r U$ such that $r(K) = 0$, $r \leqq r_0$, $r = r_0$ on $U - W$, and $H = id$ above $U - W$.

A space has Property P_Y *locally* if each of its points has a neighborhood with Property P_Y. A locally finite collection of sets in a space is *discrete* if the closures of its elements are pairwise disjoint.

The following too often neglected theorem of E. A. Michael will be used to prove the next theorem.

3.4 Theorem (*Michael* [13, *Theorem* 3.6]). *Let Z be a paracompact space and let P be a property of topological spaces such that*

a) *if X is a subspace of Z and has Property P, then every open subset of X has Property P,*

b) *if X is the union of two open subsets of Z both of which have Property P, then X has Property P, and*

c) *if X is the union of a discrete collection of open subsets of Z all of which have Property P, then X has Property P.*

Then, Z having Property P locally implies that Z has Property P.

We are now ready for the main theorem of this section.

3.5 THEOREM. *If Y is strongly contractible and Z is a paracompact space having Property P_Y locally, then $Y \times Z \cong Z$.*

PROOF. By 3.4 it suffices to prove that if $X = U_1 \cup U_2$ where each of U_1 and U_2 is an open subset of Z and has Property $P = P_Y$, then X has Property P. It then will follow that Z has Property P and the theorem follows by letting $r_0 = 1$ and $K = Z$ in the condition of Property P.

Let U be an open subset of X, let $Y \times_{r_0} U$ be a variable product of Y by U and let $K \subset W \subset U$ where K is closed in U and W is open. Without loss of generality we may assume $U = U_1 \cup U_2$. Let V_1 and V_2 be disjoint open sets in U containing the disjoint closed sets $U - U_2$ and $U - U_1$, respectively. Let $K_1 = K - V_2$ and let W_1 be an open set such that $K_1 \subset W_1 \subset \mathrm{Cl}\, W_1 \subset W \cap U_1$. Since U_1 has Property P there exists a homeomorphism H_1 of $Y \times_{r_0} U_1$ onto a variable product $Y \times_{s_0} U_1$ where $s_0(K_1) = 0$, $s_0 \leqq r_0$, $s_0 = r_0$ on $U_1 - W_1$, and $H_1 = id$ above $U_1 - W_1$. Let $Y \times_{r_1} U_2$ be the variable product where $r_1 = r_0$ on $U_2 - U_1$ and $r_1 = s_0$ on $U_1 \cap U_2$. Let $K_2 = K - V_1$ and let W_2 be an open set such that $K_2 \subset W_2 \subset \mathrm{Cl}\, W_2 \subset W \cap U_2$. Since U_2 has Property P there exists a homeomorphism H_2 of $Y \times_{r_1} U_2$ onto a variable product $Y \times_{s_1} U_2$ where $s_1(K_2) = 0$, $s_1 \leqq r_1$, $s_1 = r_1$ on $U_2 - W_2$, and $H_2 = id$ above $U_2 - W_2$. Let $Y \times_r U$ be the variable product where $r = s_0$ on $U_1 - U_2$ and $r = s_1$ on U_2. Then H defined on $Y \times_{r_0} U$ by $H = H_2 \circ H_1$ above $U_1 \cap U_2$, $H = H_1$ above $U_1 - U_2$, and $H = H_2$ above $U_2 - U_1$ is the desired homeomorphism onto $Y \times_r U$.

4. Topological vector spaces

Every TVS, F, is strongly contractible to the origin by the map $f : F \times I \to F$ defined by $f(y, t) = ty$. In the proof of the main theorem of this paper we will need a stronger version of strongly contractible.

4.1 DEFINITION. A strongly contractible space (Y, y_0, f) is an *S-space* if the following three conditions are satisfied.

1) The map $f_2 : Y \times I \to Y \times I$ defined by $f_2(y, t) = (f(y, t), t)$ is an embedding when restricted to $Y \times (0, 1]$.

2) For each neighborhood U of y_0 there exists $t \in (0, 1]$ such that if $0 < s < t$, then $f_s(Y) \subset U$, and

3) $f(f(y, t), u) = f(y, tu)$ for all $y \in F$ and $t, u \in I$.

4.2 THEOREM. *Every metrizable TVS (denoted MTVS) is an S-space.*

PROOF. Eidelheit and Mazur [7] prove that each MTVS, F, admits a strictly monotonic metric d in the sense that if $x \in F$ and $x \neq 0$, then $0 < s < t$ iff $d(sx, 0) < d(tx, 0)$. Furthermore, we may assume d is bounded by 1 since the equivalent metric $d_1 = d(1+d)^{-1}$ is also strictly monotonic.

We will define $f : F \times I \to F$ such that it pulls F linearly, with respect to the metric, to the zero vector. Let $A = \{(x, y) \in F \times F : (x, y) = (0, 0)$ or y belongs to the ray from 0 through $x \neq 0\}$ and define $\varphi : A \to I$ by $\varphi(x, y) = d(y, 0)$. For $x \in F$, let $A_x = A \cap (\{x\} \times F)$ and define $\varphi_x : A_x \to I$ by $\varphi_x(y) = \varphi(x, y)$. Then φ_x is an embedding since d is strictly monotonic. Since $\varphi_2 : A \to F \times I$ defined by $\varphi_2(x, y) = (x, \varphi(x, y))$ is an embedding it is easy to see that $f : F \times I \to F$ defined by $f(x, t) = \varphi_x^{-1}(t\varphi_x(x))$ satisfies all the conditions for an S-space.

It was observed in section 2 that every metric space is perfectly normal. However, not every TVS is perfectly normal; by [15, Theorem 4] is it known that any uncountable product of real lines is not normal.

In this paper we are also concerned with TVS's $F \cong F^{\omega}$. We have the following theorem that is the combined work of several people.

4.3 THEOREM. *If F is one of the following spaces, then $F \cong F^{\omega}$.*

a) *Any infinite-dimensional separable Fréchet space.*
b) *Any infinite-dimensional Hilbert or reflexive Banach space.*
c) *The space of bounded sequences, l_{∞}.*

PROOF. Part a) follows since each infinite-dimensional separable Fréchet space is homeomorphic to s by [1]. For part b) and more general theorems see [4]. Part c) follows from b) and [5, Corollary 5, p. 760] which says that l_{∞} is homeomorphic to a Hilbert space.

5. Stability for non-separable manifolds

5.2 LEMMA. *Let $X = \pi_{i>0} X_i$ be a product of spaces where each finite product $\pi_{i=1}^{n} X_i$ is perfectly normal. If W is an open subset of X and B is a cover of W with basic open sets contained in W, then there exists a collection of closed sets $\{K_n\}_{n>0}$ such that $W = \bigcup_{n>0} K_n$, $K_n \subset \operatorname{Int} K_{n+1}$ for $n > 0$, and each K_n is n-basic and contained in $\bigcup\{b \in B : b$ is n-basic$\}$.*

PROOF. For each $n > 0$, let $W_n = \bigcup \{b \in B : b$ is n-basic$\}$. Then $W_1 \subset W_2 \subset \cdots$, $W = \bigcup_{n>0} W_n$, and each W_n is open and n-basic. Thus $W_n = E_n \times \pi_{i>n} X_i$ where E_n is open in $\pi_{i=1}^{n} X_i$. Since $\pi_{i=1}^{n} X_i$ is perfectly normal we have $E_n = \bigcup_{i>0} C_i^n$ where each C_i^n is closed in $\pi_{i=1}^{n} X_i$. Using normality, let U_1^n be an open set in $\pi_{i=1}^{n} X_i$ such that $C_1^n \subset U_1^n \subset \operatorname{Cl} U_1^n \subset E_n$. Let $k \geqq 1$, assume U_k^n has been defined, and let U_{k+1}^n be an open set such that $\mathrm{Cl}\, U_k^n \cup C_{k+1}^n \subset U_{k+1}^n \subset \mathrm{Cl}\, U_{k+1}^n \subset E_n$. For $V_m^n = \mathrm{Cl}\, U_m^n \times \pi_{i \geqq n} X_i$ we have $W_n = \bigcup_{m>0} V_m^n$ and for each $m > 0$, $V_m^n \subset \mathrm{Int}\, V_{m+1}^n$. If $K_n = \bigcup_{m \leqq n} V_n^m$, then $\{K_n\}_{n>0}$ satisfies the conclusion of the lemma.

The function h of the following lemma is essentially the h that is geometrically described in [3] by Anderson and Schori.

5.3 LEMMA. *Let F be a TVS and let $f : F \times I \to F$ be a map such that $(F, 0, f)$ is an S-space. There exists a function $h : F_0^\omega \times I \times [1, \infty) \to F_0^\omega$ such that*

a) *the function h_2 from $F_0^\omega \times I \times [1, \infty)$ to itself defined by $h_2(x, t, u) = (h(x, t, u), t, u)$ is an embedding,*

b) *if $t \in I$ and $u \in [1, \infty)$ are fixed, the map $H : F_0^\omega \to F_0^\omega$ defined by $H(x) = h(x, t, u)$ is a homeomorphism of F_0^ω onto $f(F, 1-t) \times F^\omega$ where $H = id$ if $t = 0$, and*

c) *if $n \leqq u$, then H defined above has the property that $\pi_n^* = \pi_n^* H$.*

PROOF. We will first define h' and then modify it to obtain h. Define $h' : F_0^\omega \times I \times \{n\} \to F_0^\omega$ for a fixed $n > 0$ as follows: If $x = (x_0, x_1, x_2, \cdots) \in F_0^\omega$, let $h'(x, 0, n) = x$. If $i > 0$ let

$$(*) \qquad h'(x, 1-2^{-i}, n) = (x_{n+i}, x_1, \cdots, x_n, -x_0, -x_{n+1}, \cdots, -x_{n+i-1}, x_{n+i+1}, \cdots)$$

and let

$$h'(x, 1, n) = (0, x_1, \cdots, x_n, -x_0, -x_{n+1}, -x_{n+2}, \cdots).$$

Extend h' linearly as follows. For $n > 0$, $i \geqq 0$, $1-2^{-i} < t < 1-2^{-(i+1)}$, and $r = r(t)$ a linear functional in t where $r(1-2^{-i}) = 0$ and $r(1-2^{-(i+1)}) = 1$, let

$h'(x, t, n) = (1-r)h'(x, 1-2^{-i}, n) + rh'(x, 1-2^{-(i+1)}, n)$. We give the coordinate presentation of this for the case $i = 0$, that is $0 < t < \frac{1}{2}$, as follows:

$$h'(x, t, n) = ((1-r)x_0 + rx_{n+1}, x_1, \cdots, x_n, -rx_0 + (1-r)x_{n+1}, x_{n+2}, \cdots).$$

We will now extend h' to all of $F_0^\omega \times I \times [1, \infty)$. For $t \in I$, $n > 1$, $n-1 < u < n$, and $s = u-(n-1)$, let

$$h'(x, t, u) = (1-s)h'(x, t, n-1) + sh'(x, t, n).$$

Note that $h' | F_0^\omega \times \{1\} \times [1, \infty)$ is actually the function h of Lemma 2.1 and hence is a homeomorphism onto $\{0\} \times F^\omega \times \{1\} \times [1, \infty)$. We claim that the function $h' | F_0^\omega \times [0, 1) \times [1, \infty)$ is an invertible isotopy. There are three cases: 1) $0 \leqq t \leqq \frac{1}{2}$, 2) $\frac{1}{2} \leqq t \leqq \frac{3}{4}$, and 3) $1-2^{-i} \leqq t \leqq 1-2^{-(i+1)}$ for $i > 1$. We will give the coordinate-wise presentation of h' for case 1).

If $0 \leqq t \leqq \frac{1}{2}, u \in [1, \infty)$ where $n-1 < u < n, r = 2t$, and $s = u-(n-1)$, then

$$h'(x, t, u) = ((1-r)r_0 + r(1-s)x_n + rsx_{n+1}, x_1, \cdots, x_{n-1}, -r(1-s)x_0 + [(1-r)(1-s)+s]x_n, -rsx_0 + [(1-s)+(1-r)s]x_{n+1}, x_{n+2}, \cdots).$$

To guarantee a continuous inverse function for the corresponding function h_2' in this case, it is sufficient by Cramer's Rule, to verify that the determinant of the following matrix is non-zero for all values of $r, s \in I$.

$$\begin{pmatrix} (1-r) & r(1-s) & rs \\ -r(1-s) & 1-r+rs & 0 \\ -rs & 0 & 1-rs \end{pmatrix}$$

The determinant is equal to $(1-r)(1-r+rs)(1-rs)+r^2(1-s)^2(1-rs)+r^2s^2(1-r+rs)$ which is positive for all values of $r, s \in I$.

CASE 2) is handled similarly where the corresponding matrix is 4×4. Likewise for case 3) where we have independent 2×2 and 3×3 matrices. Thus $h_2'|F_0^\omega \times [0, 1) \times [1, \infty)$ has a continuous inverse and hence is a homeomorphism. However, h' is not continuous for $t = 1$. To remedy this we define $\mu : F_0^\omega \times I \to F_0^\omega$ by $\mu(x, t) = (f(x_0, 1-t), x_1, x_2, \cdots)$ and define $h : F_0^\omega \times I \times [1, \infty) \to F_0^\omega$ by $h(x, t, u) = \mu(h'(x, t, u), t)$. Since f satisfies condition 2) of 4.1 we have that h is continuous. We now must show that the corresponding h_2 is an embedding. Since f satisfies condition 1) of 4.1 we have that $h_2|F_0^\omega \times [0, 1) \times [1, \infty)$ is an embedding and since $h_2 = h'$ on $F_0^\omega \times \{1\} \times [1, \infty)$ we have that h_2 restricted to this set is an embedding. Hence h_2 is both continuous and one-to-one. Thus, all that remains is to show that its inverse function is continuous. Since $\{0\} \times F^\omega \times \{1\} \times [1, \infty)$ is closed it is sufficient to show that if $\{(x^\alpha, t^\alpha, u^\alpha)\}$ is a generalized sequence in $F_0^\omega \times [0, 1) \times [1, \infty)$ and $(x, 1, u) \in F_0^\omega \times \{1\} \times [1, \infty)$ where $\{h_2(x^\alpha, t^\alpha, u^\alpha)\} \to h_2(x, 1, u)$, then $\{(x^\alpha, t^\alpha, u^\alpha)\} \to (x, 1, u)$. We automatically have $t^\alpha \to 1$ and $u^\alpha \to u$. Since $t^\alpha \to 1$ implies that for each $i > 0$, t^α is eventually greater than $1-2^{-i}$, we can see from (*) that for each $k > 0$ (where the 0th coordinate is the F coordinate) that the kth coordinate of the inverse of $h_2(x^\alpha, t^\alpha, u^\alpha)$ converges to the kth coordinate of the inverse of $h_2(x, 1, u)$, which says that $\{(x^\alpha, t^\alpha, u^\alpha)\} \to (x, 1, u)$ and hence the inverse of h_2 is continuous. Thus h_2 is an embedding.

Part b) of 5.3 follows since $h_2'|F_0^\omega \times [0, 1) \times [1, \infty)$ is a homeomorphism and f satisfies 2) and 4) of 3.1. Finally, part c) is true since as seen in (*) the $x_1, \cdots, x_n$ coordinates are left fixed.

If B is a collection of sets, let B^* be the setwise union of the elements of B.

5.4 THEOREM. *If M is a paracompact manifold modeled on a perfectly normal TVS, F, where F is an S-space and $F \cong F^{\omega}$, then $F \times M \cong M$.*

PROOF. By 3.5 it suffices to show that if U is an open subset of F^{ω}, then U satisfies the condition of Property P_F. Let f be a map so that $(F, 0, f)$ is an S-space, let $F \times_{r_0} U$ be a variable product of F by U, and let $K \subset W \subset U$ where K is closed in U and W is open. Let V be open such that $K \subset V \subset \mathrm{Cl}\, V \subset W$, let $A = r_0^{-1}(0)$, and let $X' = X - A$ for $X \subset U$. Using normality, construct collections B_1 and B_2 of basic open sets in F^{ω} such that (closure will mean closure in W')$W' = B_1^* \bigcup B_2^*$, $K' \subset B_1^* \subset \mathrm{Cl}\, B_1^* \subset V$, and $K' \cap \mathrm{Cl}\, B_2^* = \emptyset$.

By 5.2 take a collection of closed sets $\{K_n\}_{n>0}$ such that $W' = \bigcup_{n>0} K_n$, $K_n \subset \mathrm{Int}\, K_{n+1}$ for $n > 0$, and each K_n is n-basic and contained in $\bigcup\{b \in B_1 \cup B_2 : b \text{ is } n\text{-basic}\}$.

Define $g : W' \to [1, \infty)$ exactly as done in 2.2. Thus, if $x \in W'$ and $m = \min\{i : x \in K_i\}$, then $g(x) = g(\pi_m(x)) \geqq m$. Furthermore, g is *unbounded near* A, that is, if $x \in A \cap \mathrm{Cl}_U W'$ and $n > 0$, there exists a neighborhood N of x such that $g(N \cap W') > n$.

We now need a function φ such that i) $\varphi : W' \to I$ is continuous; ii) $\varphi(K') = 1$; iii) φ satisfies the *local product* property with respect to the $\{K_n\}$, that is, if $x \in W'$ and $m = \min\{i : x \in K_i\}$, then $\varphi(x) = \varphi(\pi_m(x))$; and iv) φ *goes to zero* near $\mathrm{Bd}_{U'} W'$, that is, if $x \in \mathrm{Bd}_{U'} W'$ and $\varepsilon > 0$ there exists a neighborhood N of x such that $\varphi(N \cap W') < \varepsilon$.

We define φ as follows. For $n > 0$, let $D_n = \mathrm{Cl}\,(\bigcup\{b \in B_2 : b \text{ is } n\text{-basic}\})$ and thus $K' \cap D_n = \emptyset$. Note that each of D_n, K_n, and $\mathrm{Bd}\, K_n$ is n-basic. Let $\varphi(K' \cup K_1) = 1$ and for $n > 1$, let $\varphi(\mathrm{Bd}\, K_n \cap D_n) = 1/n$. (If $K_1 = \emptyset$, let $m = \min\{i : K_i \neq \emptyset\}$ and go directly to the mth stage of the construction.) We will first extend φ to $K' \cup K_1 \cup (\bigcup_{i>0} D_i)$ inductively as follows. We have $\varphi(\pi_2(K_1 \cap D_1)) = 1$ and $\varphi(\pi_2(\mathrm{Bd}\, K_2 \cup D_2)) = \frac{1}{2}$. Extend φ to the rest of $\pi_2(K_2 \cap D_2)$ with the Tietze extension theorem where the range is restricted to $[\frac{1}{2}, 1]$ and then extend φ to all of $K_2 \cup D_2$ by the product structure. Let $n > 2$ and assume φ has been extended to $K_{n-1} \cap D_{n-1}$. We have $\varphi(\pi_i(\mathrm{Bd}\, K_i \cap D_i)) = 1/i$ for $i = n-1$, n. Extend φ to $\pi_n[(K_n - \mathrm{Int}\, K_{n-1}) \cap D_n]$ with the Tietze theorem where the range of φ is restricted to $[1/n, 1/n-1]$ and then extend to $(K_n - \mathrm{Int}\, K_{n-1}) \cap D_n$ by the product structure. Thus φ has been defined on $K' \cup K_1 \cup (\bigcup_{i>0} D_i)$.

We now extend φ to the rest of W'. Since $K' \cup D_n = \emptyset$ and D_n is n-basic we have for all n that $\pi_n(K') \cap \pi_n(D_n) = \emptyset$. Henceforth, when extending φ, the range of φ will be restricted to I. Since φ has been defined on K_1, the first step of the induction has been done. Let $n > 1$ and assume φ has been extended to $K' \cup (\bigcup_{i=1}^{n-1} K_i) \cup (\bigcup_{i>0} D_i)$. Let

$\varphi(\pi_n(K' \bigcap K_n)) = 1$ and extend φ to the rest of $\pi_n(K_n)$ with the Tietze theorem and then extend φ to the rest of K_n by the product structure. Thus, by induction φ has been defined on W'. Conditions i), ii), and iii) are automatically satisfied from the definition of φ. To see condition iv) note that a) we can assume $N \subset U' - \text{Cl}\, V$, b) $W' - \text{Cl}\, V \subset B_2^* \subset \bigcup_{i>0} D_i$ which implies that if $y \in K_n \cap (W' - \text{Cl}\, V)$, then $y \in K_n \cap D_n$, and c) if $y \in (K_n - \text{Int}\, K_{n-1}) \cap D_n$, then $1/n \leqq \varphi(y) \leqq 1/n-1$.

The map $k : F \times_{r_0} W' \to F \times W'$ defined by $k(y, z) = (f_{r_0(z)}^{-1}(y), z)$ is a homeomorphism. Let h be the map of 5.3 and define $H_1 : F \times W' \to F \times W'$ by $H_1(x) = h(x, \varphi^*(x), g^*(x))$ for $x \in F \times W'$. By essentially duplicating the argument given in the proof of 2.2 we can prove that H_1 is a homeomorphism of $F \times W'$ onto $F \times_{1-\varphi} W'$. Now, define the map H_2 on $F_0 \times_{r_0} W'$ by $H_2 = k^{-1} \circ H_1 \circ k$. Thus, H_2 is a homeomorphism and furthermore, since $(H_1 \circ k)(F \times_{r_0} W') = F \times_{1-\varphi} W'$, we have $H_2(F \times_{r_0} W') = F \times_{(1-\varphi) r_0} W'$ since f satisfies the condition 3) of 4.1. Define $H : F \times_{r_0} U \to F \times_r U$, where $r = (1-\varphi) r_0$ on W' and $r = r_0$ on $(U - W) \cup A$, by $H = H_2$ above W' and $H = id$ above $(U - W) \cup A$. The function r is continuous since φ goes to zero near $\text{Bd}_{U'} W'$ and $r_0 = 0$ on A. We now show that H is continuous. Since φ goes to zero near $\text{Bd}_{U'} W'$, then the identity above $(U - W) - A$ and H_2 above W' are compatible. To show that these are compatible with the identity above A we check the coordinatewise continuity of H. The continuity of r_0 together with condition 2) of 4.1 gives the continuity of H on the first or F coordinate, and g being unbounded near A yields the continuity of H on the second, or U, coordinate. Thus H is continuous and one-to-one. We must show that H^{-1} is continuous.

Since $(U - W) \cup A$ is closed in U it is sufficient to show that if $\{x^\alpha\}$ is a generalized sequence of points in $F \times_{r_0} (W - A)$ where $\{H(x^\alpha)\}$ converges to $H(x) = x \in F \times_r [(U - W) \cup A]$, then $X^\alpha \to x$. For each α, let $x^\alpha = (u^\alpha, v^\alpha)$, $H(x^\alpha) = (y^\alpha, z^\alpha)$, and $x = (y, z)$. We have two cases; the first is when $x \in F \times_{r_0} A$. Here, $y = 0$ and $z \in A$. Since $z^\alpha \to z$ and $g(z^\alpha) \to \infty$, we also have $v^\alpha \to z$. The continuity of r_0 implies $r_0(v^\alpha) \to 0$ and thus $u^\alpha \to 0$ since $u^\alpha \in f(F, r_0(v^\alpha))$ and f satisfies condition 2) of 4.1. Thus, in this case $x^\alpha \to x$. The other case is when $x \in (U - W) - A$. Here we have $\varphi^*(x^\alpha) \to 0$ since $\varphi^* H(x^\alpha)$ does. Furthermore, since the h of 5.3 is continuous and $h(w, 0, u) = w$ for all $w \in F_0^\omega$ and $u \in [1, \infty)$, we have $x^\alpha \to x$ since $H(x^\alpha)$ does. Thus, H^{-1} is continuous and hence H is a homeomorphism.

5.5 COROLLARY. *If M is a paracompact manifold modeled on a metrizable TVS, $F \cong F^\omega$, then $M \times F \cong M$.*

Proof. This follows immediately from Theorems 4.2 and 5.4 and since every metric space is perfectly normal.

Lemma 5.3 and Theorem 5.4 were stated and proved for topological vector spaces as a matter of convenience and in a form of sufficient generality to accommodate all known applications. We now show how to abstract the properties of TVS's that are essential to these results.

If X is a space and f, $g : X \to X$ are maps, then f is *invertibly isotopic* to g if there exists an invertible isotopy $h : X \times I \to X$ such that $h_0 = f$ and $h_1 = g$. A space X has the *reflective isotopy property* if the homeomorphism from X^ω to itself that interchanges the first and second coordinates is invertibly isotopic to the identity map. This name is due to James E. West [16].

5.6 Theorem (*R.Y.T. Wong* [17])[1]. *If X is a Hausdorff space, then a necessary and sufficient condition that each homeomorphism on X^ω is invertibly isotopic to the identity map is that X has the reflective isotopy property.*

5.7 Corollary. *If X is a space with the reflective isotopy property, then any two homeomorphisms on X^ω are invertibly isotopic.*

5.8 Lemma. *Let X be a space satisfying the reflective isotopy property and let D be a closed disk. Then every invertible isotopy of $X^\omega \times \mathrm{Bd}\,(D)$ into X^ω may be extended to an invertible isotopy of $X^\omega \times D$ into X^ω.*

Proof. Corollary 2 of Theorem 2 in [14] says that if X satisfies the reflective isotopy property and A is any space, then every invertible isotopy of $X^\omega \times A$ into X^ω may be extended to an invertible isotopy of $X^\omega \times$ cone (A) into X^ω. Since $D \cong$ cone $(\mathrm{Bd}\, D)$ the result follows.

5.9 Lemma. *Let X be a space with the reflective isotopy property, $F = X^\omega, f : F \times I \to F$ be a map, and 0 be a point of F such that $(F, 0, f)$ is an S-space. There exists a function $h : F_0^\omega \times I \times [1, \infty) \to F_0^\omega$ that satisfies the conditions of* 5.3.

Proof. Following the proof of 5.3 we will define h' first. For $i \geqq 0$ and $n \geqq 1$, let

$D_{i,n} = [1-2^{-i},\ 1-2^{-(i+1)}] \times [n,\ n+1]$ and for $x = (x_0,\ x_1, x_2, \cdots) \in F_1^\omega$, let

$$h'(x, t, n) = \begin{cases} x \quad \text{if} \quad t = 0 \\ (x_{n+i}, x_1, \cdots, x_n, x_0, x_{n+1}, \cdots, x_{n+i-1}, x_{n+i+1}, \cdots) \text{ if } i > 0 \\ (0, x_1, \cdots, x_n, x_0, x_{n+1}, x_{n+2}, \cdots) \text{ if } t = 1. \end{cases}$$

[1] Wong didn't state the theorem in quite this generality, but it is clear that his proof also applies to this more general version.

Thus h' has been defined for all the vertices of all the $D_{i,n}$. Now it is easy to see that h' can be extended inductively, by the use of 5.7 and 5.8, to the union of all the $D_{i,n}$ where care is taken on the individual edges and disks to apply the appropriate lemma only to the coordinate spaces where the coordinates are changed and to leave the other coordinates fixed. For example, if $x \in F_0^\omega$, the h' image of x at the vertices of the right-hand edge, $\{1/2\} \times [1, 2]$ of $D_{0,1}$ are $h'(x, 1/2, 1) = (x_2, x_1, x_0, x_3, x_4, \cdots)$ and $h'(x, 1/2, 2) = (x_3, x_1, x_2, x_0, x_4, \cdots)$. Here the coordinates differ only in the 0th, 2nd, and 3rd coordinate places. Thus, 5.7 would be applied to the space $F_0 \times F_2 \times F_3$ and the induced homeomorphisms, which are $(x_0, x_2, x_3) \to (x_2, x_0, x_3)$ and $(x_0, x_2, x_3) \to (x_3, x_2, x_0)$, to obtain an invertible isotopy $(F_0 \times F_2 \times F_3) \times \{1/2\} \times [1, 2] \to F_0 \times F_2 \times F_3$ and this would be combined with the identity isotopy on $(\pi_{i \neq 0,2,3} F_i) \times \{1/2\} \times 1, 2]$ to obtain an invertible isotopy $h' : F_0^\omega \times \{1/2\} \times [1, 2] \to F_0^\omega$ which leaves for $k \neq 0, 2, 3$, the kth coordinate fixed. This example illustrates case 1) in the proof of 5.3. Cases 2) and 3) have similar analogies where the 4×4 matrix of case 2) corresponds to four coordinates that are changed and the 2×2 and 3×3 matrices of case 3) corresponding to a pair of coordinates that are interchanged and three other coordinates that are independently permuted such as in case 1). Thus, we have an invertible isotopy $h' : F_0^\omega \times [0, 1) \times [1, \infty) \to F_0^\omega$ with the analogous properties of the corresponding map in the proof of 5.3.

Now, modify the map h' with the map f to obtain $h : F_0^\omega \times [0, 1) \times [1, \infty) \to F_0^\omega$ as was done in the proof of 5.3. Then h has a unique extension to $F_0^\omega \times I \times [1, \infty)$ that is continuous and yields an embedding with all the required properties as seen in the proof of 5.3.

The previous lemma together with the proof of Theorem 5.4 yields the following.

5.10 THEOREM. *Let X be a space with the reflective isotopy property such that $F = X^\omega$ is perfectly normal and an S-space. If M is a paracompact manifold modeled on F, then $M \times F \cong M$.*

As shown by West in [16], every TVS has the reflective isotopy property and thus Theorem 5.4 follows from Theorem 5.10. The following Corollary is essentially Theorem II of [3].

5.11 COROLLARY. *If M is a paracompact manifold modeled on I^ω, then $M \times I^\omega \cong M$.*

PROOF. Wong [17] proves that I has the reflective isotopy property. Furthermore, I^ω is metrizable and thus perfectly normal and is easily seen to be an S-space.

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