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A. J. STAM

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RENEWAL THEORY IN r DIMENSIONS II

by

A. J. Stam

1. Introduction

This paper is a direct continuation of Stam [6], which will be cited as I. The notation and definitions of I will be taken over without reference. The same holds for the assumptions of I, section 1: strict d -dimensionality, finite second moments and nonzero first moment vector.

We now assume

$$(1.1) \quad \mu_1 > 0.$$

The restriction of U_F to the strip $\{\bar{x} : t \leq x \leq t+a\}$ is a finite measure with variation tending to $\mu_1^{-1}a$ as $t \rightarrow \infty$ if X_{11} is nonarithmetic. It will be shown that this measure satisfies a local central limit theorem for $t \rightarrow \infty$, if $E|X_{11}|^\rho < \infty$ and F is nonarithmetic. The limit theorem (theorem 3.1) has the usual form applying to the n -fold convolution of a probability measure, with n replaced by $\mu_1^{-1}t$. See e.g. Spitzer [4], Ch. II.7 and Stone [8]. For arithmetic F a similar result holds (theorem 3.2).

We might have considered any strip $\{\bar{x} : t \leq (\bar{c}, \bar{x}) \leq t+a\}$ with the unit vector \bar{c} such that $(\bar{\mu}, \bar{c}) > 0$. What is done here is choosing a coordinate system with positive x_1 -axis in the direction of \bar{c} .

The global version of the limit theorem, with $\mu_2 = \dots = \mu_d = 0$, was proved in Stam [5]. Theorems 5.3 and 5.4 of I are special cases of the local theorems, viz. $\mu_2 = \dots = \mu_d = 0, x_2 = \dots = x_d = 0$.

Proofs follow the same lines as in I, with the complication that limits for $x_1 \rightarrow \infty$ have to be uniform with respect to x_2, \dots, x_d .

Section 4 contains some results on the order of decrease of $U_F(A+\bar{x})$ as $|\bar{x}| \rightarrow \infty$ if certain moments of F exist.

The following notation is used throughout this paper. Let E be the covariance matrix of the random variables $X_{1j} - \mu_1^{-1}\mu_j X_{11}, j = 2, \dots, d$, and ε_{ih} the (i, h) -element of E^{-1} . We put

$$(1.2) \quad Z(\bar{x}) = \exp \left[-\frac{1}{2} \mu_1^{-1} x_1^{-1} \sum_{i=2}^d \sum_{j=2}^d \varepsilon_{ij} (x_i - \mu_1^{-1} \mu_i x_1) (x_j - \mu_1^{-1} \mu_j x_1) \right],$$

$$(1.3) \quad L(\bar{x}) = \mu_1^{-1} (2\pi)^{-\rho} (\text{Det } E)^{-\frac{1}{2}} Z(\bar{x}),$$

so that $\mu_1^{\rho+1} x_1^{-\rho} L(\bar{x})$ for fixed $x_1 > 0$ is a gaussian probability density on R_{d-1} .

By \mathcal{C}_d we denote the class of continuous functions on R_d with compact support.

2. Preliminary lemmas

LEMMA 2.1. *If for every $g \in K_d$*

$$(2.1) \quad \lim_{|\bar{x}| \rightarrow \infty} \int g(\bar{y} - \bar{x}) \{W_G(d\bar{y}) - W_H(d\bar{y})\} = 0,$$

uniformly in the direction of \bar{x} , then the same is true for every $g \in \mathcal{C}_d$.

The class K_d is defined in I, definition 2.3.

PROOF. It is sufficient to show that to any $g \in \mathcal{C}_d$ and any $\varepsilon > 0$ there is $g_\varepsilon \in K_d$ with

$$(2.2) \quad \int |g(\bar{y} - \bar{x}) - g_\varepsilon(\bar{y} - \bar{x})| W(d\bar{y}) < \frac{1}{2}\varepsilon,$$

uniformly in \bar{x} , where $W = W_G + W_H$. The relation (2.1) then follows by the inequality

$$\begin{aligned} & \left| \int g(\bar{y} - \bar{x}) \{W_G(d\bar{y}) - W_H(d\bar{y})\} \right| \leq \\ & \left| \int g_\varepsilon(\bar{y} - \bar{x}) \{W_G(d\bar{y}) - W_H(d\bar{y})\} \right| + \int |g(\bar{y} - \bar{x}) - g_\varepsilon(\bar{y} - \bar{x})| W(d\bar{y}). \end{aligned}$$

To prove (2.2) we take a probability density $h \in K_d$ and put

$$(2.3) \quad h_a(\bar{z}) = ah(a\bar{z}), \quad a > 0,$$

$$(2.4) \quad g_a(\bar{z}) = \int g(\bar{z} - \bar{x}) h_a(\bar{x}) d\bar{x}.$$

Then $h_a \in K_d$ and $g_a \in K_d$. We have

$$\int |g(\bar{y} - \bar{x}) - g_a(\bar{y} - \bar{x})| W(d\bar{y}) \leq \iint |g(\bar{y} - \bar{x}) - g(\bar{y} - \bar{x} - \bar{i})| W(d\bar{y}) h_a(\bar{i}) d\bar{i}.$$

Since $g \in \mathcal{C}_d$, we have by I, lemma 2.4

$$(2.5) \quad \begin{aligned} \int |g(\bar{y} - \bar{x}) - g_a(\bar{y} - \bar{x})| W(d\bar{y}) & \leq C_1 \int_{|\bar{i}| \geq \delta} h_a(\bar{i}) d\bar{i} \\ & + \int_{|\bar{i}| \leq \delta} \int_{D+\bar{x}} |g(\bar{y} - \bar{x}) - g(\bar{y} - \bar{x} - \bar{i})| W(d\bar{y}) h_a(\bar{i}) d\bar{i}, \end{aligned}$$

where $0 < \delta < 1$ and the bounded set D is taken so that $g(\bar{z}) = 0$, $g(\bar{z} - \bar{i}) = 0$ for $\bar{z} \notin D$ and all \bar{i} with $|\bar{i}| \leq 1$. Since g is uniformly con-

tinuous, we first may take δ so small that the second term on the right in (2.5) is smaller than $\frac{1}{4}\varepsilon$, and then a so large that by (2.3) the first term is smaller than $\frac{1}{4}\varepsilon$.

LEMMA 2.2. *If F is gaussian, the density w_F of W_F satisfies*

$$(2.6) \quad \lim_{x_1 \rightarrow \infty} |w_F(\bar{x}) - L(\bar{x})| = 0,$$

uniformly in x_2, \dots, x_d .

COROLLARY. *Under the conditions of lemma 2.2*

$$(2.7) \quad \lim_{x_1 \rightarrow \infty} \left\{ \int g(\bar{z} - \bar{x}) W_F(d\bar{z}) - L(\bar{x}) \int g(\bar{z}) d\bar{z} \right\} = 0,$$

uniformly in x_2, \dots, x_d , if $g \in \mathcal{C}_d$.

PROOF. By I, lemma 2.2, it is sufficient that (2.6) holds uniformly in a cone $C_\theta = \{\bar{x} : x_1 \geq 0, |x_j - \mu_1^{-1} \mu_j x_1| \leq \theta x_1, j \geq 2\}$. Let

$$Y_{m1} = S_{m1}, Y_{mk} = S_{mk} - \mu_1^{-1} \mu_k S_{m1}, k = 2, \dots, d,$$

with $\bar{S}_m = \bar{X}_1 + \dots + \bar{X}_m$. Then the density f_m of F^m and the joint density q_m of Y_{m1}, \dots, Y_{md} are connected by

$$(2.8) \quad f_m(\bar{x}) = q_m(x_1, x_2 - \mu_1^{-1} \mu_2 x_1, \dots, x_d - \mu_1^{-1} \mu_d x_1).$$

Let P be the covariance matrix of Y_{11}, \dots, Y_{1d} , and π_{ij} be the (i, j) -element of P^{-1} . Put

$$(2.9) \quad \eta = \pi_{11}^{-1} \sum_{j=2}^d \pi_{1j} (x_j - \mu_1^{-1} \mu_j x_1).$$

Since $E\{Y_{m1}\} = m\mu_1$, $E\{Y_{mk}\} = 0$, $k \geq 2$, the relation (2.8) gives

$$\begin{aligned} f_m(\bar{x}) &= (2\pi m)^{-\frac{1}{2}d} (\text{Det } P)^{-\frac{1}{2}} \exp \left[-\frac{\pi_{11}}{2m} (x_1 - m\mu_1 + \eta)^2 \right] \\ &\quad \cdot \exp \left[\frac{\pi_{11}}{2m} \eta^2 - \frac{1}{2m} \sum_{i=2}^d \sum_{j=2}^d \pi_{ij} (x_i - \mu_1^{-1} \mu_i x_1) (x_j - \mu_1^{-1} \mu_j x_1) \right], \\ f_m(\bar{x}) &= (2\pi m)^{-\frac{1}{2}d} (\text{Det } P)^{-\frac{1}{2}} Z_m(\bar{x}) \exp \left[-\frac{\pi_{11}}{2m} (x_1 - m\mu_1 + \eta)^2 \right] \end{aligned}$$

where

$$(2.10) \quad Z_m(\bar{x}) = \exp \left[-\frac{1}{2m} \sum_{i=2}^d \sum_{j=2}^d \varepsilon_{ij} (x_i - \mu_1^{-1} \mu_i x_1) (x_j - \mu_1^{-1} \mu_j x_1) \right].$$

Since $\pi_{11} \text{Det } P = \text{Det } E$,

$$f_m(\bar{x}) = (2\pi m)^{-\rho} (\text{Det } E)^{-\frac{1}{2}} p^{(m)}(x_1 + \eta) Z_m(\bar{x}),$$

where $p^{(m)}$ is the m -fold convolution of the normal density with mean μ_1 and variance π_{11}^{-1} . It is noted that $\pi_{11} > 0$ since P is nonsingular. So

$$(2.11) \quad w_F(\bar{x}) = (2\pi)^{-\rho} (\text{Det } E)^{-\frac{1}{2}} \sum_{m=1}^{\infty} p^{(m)}(x_1 + \eta) Z_m(\bar{x}).$$

In defining the cone C_θ we take θ so small that

$$(2.12) \quad |\eta| \leq \frac{1}{2}x_1, \quad \bar{x} \in C_\theta.$$

We divide C_θ into $C_\theta R_A$ and $C_\theta R_A^c$ with

$$(2.12a) \quad R_A = \{\bar{x} : A^2|x_1| \leq \sum_{j=2}^d (x_j - \mu_1^{-1}\mu_j x_1)^2\}.$$

Put $\lambda = (2\pi)^{-\rho} (\text{Det } E)^{-\frac{1}{2}}$. Since E is nonsingular we have for $\bar{x} \in C_\theta R_A$

$$(2.13) \quad \begin{aligned} w_F(\bar{x}) &\leq \lambda \sum_{m=1}^{\infty} p^{(m)}(x_1 + \eta) \exp(-c_1 A^2 x_1 m^{-1}) \\ &\leq \lambda \sum_{m=1}^{\infty} p^{(m)}(x_1 + \eta) \exp\{-\frac{2}{3}c_1 A^2 m^{-1}(x_1 + \eta)\}, \end{aligned}$$

$$(2.14) \quad L(x) \leq \mu_1^{-1} \lambda \exp(-c_1 \mu_1 A^2),$$

with $c_1 > 0$. Moreover, by the inequality $|\exp(-\alpha) - \exp(-\beta)| \leq |\alpha - \beta|$ $\alpha \geq 0, \beta \geq 0$, we have for $\bar{x} \in C_\theta R_A^c$

$$(2.15) \quad \begin{aligned} |w_F(\bar{x}) - L(\bar{x})| &\leq \lambda \sum_{m=1}^{\infty} p^{(m)}(x_1 + \eta) |Z_m(\bar{x}) - Z(\bar{x})| + \lambda |h(x_1 + \eta) - \mu_1^{-1} Z(\bar{x})| \\ &\leq cA^2 \sum_{m=1}^{\infty} p^{(m)}(x_1 + \eta) \left| \frac{x_1}{m} - \mu_1 \right| + \lambda |h(x_1 + \eta) - \mu_1^{-1}|, \end{aligned}$$

where

$$h(x) = \sum_{m=1}^{\infty} p^{(m)}(x).$$

For given $\varepsilon > 0$ by (2.12), (2.13), (2.14) we may take A so large that $|w_F(\bar{x}) - L(\bar{x})| < \varepsilon$ for $x_1 \geq c_3$ and $\bar{x} \in C_\theta R_A$, since

$$\lim_{A \rightarrow \infty} \sum_{m=1}^{\infty} p^{(m)}(z) \exp(-\frac{2}{3}c_1 A^2 m^{-1}z) = 0,$$

uniformly in z for $z \geq c_3 > 0$. For this A the right-hand side of (2.15) then tends to zero as $x_1 \rightarrow \infty$, uniformly in $C_\theta R_A^c$. For the second term we apply the renewal theorem for densities. The first term is more complicated. It is noted that $|\eta| \leq c_4|x_1|^{\frac{1}{2}}$, where c_4 depends on A . We may define the family of random variables $M_z, z > 0$, with

$$P\{M_z = m\} = p^{(m)}(z)/h(z), \quad m = 1, 2, \dots$$

Then $z^{-1}M_z \rightarrow \mu^{-1}$ in quadratic mean as $z \rightarrow \infty$. We refer to Kalma [1], [2]. A similar technique is used in the proof of theorem 5.3 in I. A direct proof proceeds by dividing the sum over m into three parts:

$$\left| \frac{x_1}{m} - \mu_1 \right| < \varepsilon, \quad \frac{x_1}{m} - \mu_1 \leq -\varepsilon \quad \text{and} \quad \frac{x_1}{m} - \mu_1 \geq \varepsilon.$$

The corollary follows from (2.6) and the fact that

$$\lim_{x_1 \rightarrow \infty} \{L(\bar{x} + \bar{z}) - L(\bar{x})\} = 0,$$

uniformly in x_2, \dots, x_d and uniformly with respect to \bar{z} in bounded sets.

LEMMA 2.3. Let $\{x(t, \bar{\tau}), (t, \bar{\tau}) \in E \subset R_k\}$ be a family of positive random variables such that

$$(2.16) \quad \lim_{t \rightarrow \infty} E[\{x(t, \bar{\tau}) - c\}^2] = 0,$$

uniformly in $\bar{\tau}$, where c is a positive constant. Then for any θ and any $\varepsilon > 0$

$$(2.17) \quad \lim_{t \rightarrow \infty} P\{|x^\theta(t, \bar{\tau}) - c^\theta| \geq \varepsilon\} = 0,$$

uniformly in $\bar{\tau}$. If moreover to any $\delta > 0$ there are $K(\delta)$ and $T(\delta)$ with

$$(2.18) \quad E[x^\theta(t, \bar{\tau})I\{x^\theta(t, \bar{\tau}) \geq K(\delta)\}] < \delta$$

for $t \geq T(\delta)$ and every $\bar{\tau}$, we have

$$(2.19) \quad \lim_{t \rightarrow \infty} E\{x^\theta(t, \bar{\tau})\} = c^\theta,$$

uniformly in $\bar{\tau}$.

REMARK. A sufficient condition for (2.18) is the existence of $s > 1$ with

$$E\{x^{s\theta}(t, \bar{\tau})\} \leq M < \infty, (t, \bar{\tau}) \in E.$$

See Loève [3], § 11.4.

PROOF. The relation (2.17) follows from (2.16) for $\theta = 1$ by Chebychev's inequality and then for any real θ since

$$\{|x^\theta(t, \bar{\tau}) - c^\theta| \geq \varepsilon\} \subset \{|x(t, \bar{\tau}) - c| \geq \eta\},$$

for some positive η independent of t and $\bar{\tau}$.

Now let B be the distribution function of $x^\theta(t, \bar{\tau})$. Then

$$\begin{aligned}
E|x^\theta(t, \bar{\tau}) - c^\theta| &\leq \int_{c^\theta - \eta}^{c^\theta + \eta} |x - c^\theta| B(dx) \\
&\quad + \left\{ \int_0^{c^\theta - \eta} + \int_{c^\theta + \eta}^K + \int_K^\infty \right\} (xB(dx)) + c^\theta P\{|x^\theta(t, \bar{\tau}) - c^\theta| \geq \eta\} \\
&\leq \eta + (K + 2c^\theta) P\{|x^\theta(t, \bar{\tau}) - c^\theta| \geq \eta\} + \int_K^\infty xB(dx).
\end{aligned}$$

We now prove (2.19) by first taking $\eta = \varepsilon/3$, then $K = K(\frac{1}{3}\varepsilon)$ as in (2.18) and finally applying (2.17).

LEMMA 2.4. *If F is nonarithmetic and $g \in \mathcal{C}_d$,*

$$\lim_{x_1 \rightarrow \infty} \left| \int g(\bar{z} - \bar{x}) W_F(d\bar{z}) - L(\bar{x}) \int g(\bar{z}) d\bar{z} \right| = 0,$$

uniformly in x_2, \dots, x_d .

PROOF. From lemma 2.2 (corollary), lemma 2.1 and I, theorem 3.2.

LEMMA 2.5. *Let a Cartesian coordinate system exist, such that the components Z_1, \dots, Z_d of \bar{X}_1 in this system have joint characteristic function ζ with $\zeta(\bar{u}) = 1$ if u_1, \dots, u_d are integer multiples of 2π and $|\zeta(\bar{u})| < 1$ elsewhere. Then*

$$\lim_{x_1 \rightarrow \infty} \{W_F(\{\bar{x}\}) - L(\bar{x})\} = 0,$$

uniformly in x_2, \dots, x_d if \bar{x} is restricted to lattice points of F .

PROOF. From lemma 2.2 and I, theorem 3.4. The rotation of the F -lattice is a consequence of our choice of coordinates.

LEMMA 2.6. *For fixed nonnegative integer k with $E|X_{11}|^k < \infty$, let*

$$(2.20) \quad V_F(A) = \sum_{m=1}^{\infty} m^{\rho-k} F^m(A).$$

Then, if F is nonarithmetic, we have for $g \in \mathcal{C}_d$,

$$(2.21) \quad \lim_{x_1 \rightarrow \infty} \left\{ x_1^k \int g(\bar{z} - \bar{x}) V_F(d\bar{z}) - \mu_1^k L(\bar{x}) \int g(\bar{z}) d\bar{z} \right\} = 0,$$

uniformly in x_2, \dots, x_d .

PROOF. We will show that

$$(2.22) \quad \lim_{x_1 \rightarrow \infty} \left\{ \int z_1^k g(\bar{z} - \bar{x}) V_F(d\bar{z}) - \mu_1 L(x) \int g(\bar{z}) d\bar{z} \right\} = 0,$$

uniformly in x_2, \dots, x_d . The relation (2.21) then follows by the inequality

$$\left| \int (z_1^k - x_1^k) g(\bar{z} - \bar{x}) V_F(d\bar{z}) \right| \leq C_1 x_1^{-1} \int z_1^k |g(\bar{z} - \bar{x})| V_F(d\bar{z}),$$

which is a consequence of the fact that $g \in \mathcal{C}_d$.

In the same way as in the proof of I, theorem 5.3.

$$(2.23) \quad \int z_1^k g(\bar{z} - \bar{x}) V_F(d\bar{z}) = \Phi(\bar{x}) + \int g(\bar{z} - \bar{x}) W_F Q^k(d\bar{z}),$$

with $\lim_{|\bar{x}| \rightarrow \infty} \Phi(\bar{x}) = 0$, uniformly in the direction of \bar{x} , and

$$(2.24) \quad Q(E) = \int_E x_1 F(d\bar{x}).$$

Since Q is a finite signed measure, we may write $Q^k = K' + K''$, where K' is restricted to a bounded set and the variation of K'' is so small that in

$$\begin{aligned} & \int g(\bar{z} - \bar{x}) W_F Q^k(d\bar{z}) - \mu_1^k L(\bar{x}) \int g(\bar{z}) d\bar{z} \\ &= \left\{ \int g(\bar{z} - \bar{x}) W_F Q^k(d\bar{z}) - \int g(\bar{z} - \bar{x}) W_F K'(d\bar{z}) \right\} \\ &+ \left\{ \int g(\bar{z} - \bar{x}) W_F K'(d\bar{z}) - L(\bar{x}) K'(R_d) \int g(\bar{z}) d\bar{z} \right\} \\ &+ L(\bar{x}) \int g(\bar{z}) d\bar{z} \{K'(R_d) - \mu_1^k\} \end{aligned}$$

the first and third term on the right are smaller than $\frac{1}{3}\varepsilon$. For the first term we apply I, lemma 2.4. The second term is written

$$\begin{aligned} & \int \left\{ \int g(\bar{z} + \bar{\zeta} - \bar{x}) W_F(d\bar{z}) - L(\bar{x} - \bar{\zeta}) \int g(\bar{y}) d\bar{y} \right\} K'(d\bar{\zeta}) \\ &+ \int \left\{ L(\bar{x} - \bar{\zeta}) - L(\bar{x}) \right\} K'(d\bar{\zeta}) \cdot \int g(\bar{y}) d\bar{y}. \end{aligned}$$

Here the first term tends to zero as $x_1 \rightarrow \infty$, uniformly in x_2, \dots, x_d , by lemma 2.4, since K' is restricted to a bounded set. The same holds for the second term by (1.3). One should distinguish the sets R_A and R_A^c defined by (2.12a).

3. Local limit theorems for U_F

THEOREM 3.1. *If F is nonarithmetic and $E|X_{11}|^p < \infty$,*

$$(3.1) \quad \lim_{x_1 \rightarrow \infty} \left\{ x_1^p \int g(\bar{z} - \bar{x}) U_F(d\bar{z}) - \mu_1^p L(\bar{x}) \int g(\bar{z}) d\bar{z} \right\} = 0,$$

for $g \in \mathcal{C}_d$, uniformly in x_2, \dots, x_d .

PROOF. When d is odd, the theorem coincides with lemma 2.6 for $k = \rho$.

Now assume that d is even, $d \geq 6$. It is no restriction to assume that $g \geq 0$. First we intend to show

(I) The relation (3.1) holds uniformly in the set R_A^c , with R_A^c defined by (2.12a).

Putting

$$(3.2) \quad \alpha = \int g(\bar{z}) d\bar{z},$$

$$(3.3) \quad q(m, \bar{x}) = \int g(\bar{z} - \bar{x}) F^m(d\bar{z}), \quad m = 1, 2, \dots,$$

we have by lemma 2.6 with $k = 0, 1, 2$,

$$(3.4) \quad \sum_{m=1}^{\infty} m^{\rho} q(m, \bar{x}) = \alpha L(\bar{x}) + \varepsilon_0(\bar{x}),$$

$$(3.5) \quad x_1 \sum_{m=1}^{\infty} m^{\rho-1} q(m, \bar{x}) = \mu_1 \alpha L(\bar{x}) + \varepsilon_1(\bar{x}),$$

$$(3.6) \quad x_1^2 \sum_{m=1}^{\infty} m^{\rho-2} q(m, \bar{x}) = \mu_1^2 \alpha L(\bar{x}) + \varepsilon_2(\bar{x}),$$

with

$$(3.7) \quad \lim_{x_1 \rightarrow \infty} \varepsilon_i(\bar{x}) = 0, \quad i = 0, 1, 2,$$

uniformly in x_2, \dots, x_d . Consider the family of positive integer valued random variables $\{M(\bar{x}), \bar{x} \in R_A^c, x_1 > 0\}$:

$$(3.8) \quad P\{M(\bar{x}) = m\} = \frac{x_1^2 m^{\rho-2} q(m, \bar{x})}{\mu_1^2 \alpha L(\bar{x}) + \varepsilon_2(\bar{x})}, \quad m = 1, 2, \dots$$

Expectation with respect to the distribution (3.8) will be denoted by E_1 . From (3.4)–(3.7) and the inequality

$$(3.9) \quad L(\bar{x}) \geq C_1(A) > 0, \quad \bar{x} \in R_A^c,$$

it follows that

$$(3.10) \quad \lim_{x_1 \rightarrow \infty} E_1[x_1^{-1} M(\bar{x}) - \mu_1^{-1}]^2 = 0,$$

uniformly in x_2, \dots, x_d . By lemma 2.3 and (3.9) this implies

$$(3.11) \quad \lim_{x_1 \rightarrow \infty} E_1\{x_1^{\rho-2} M^{2-\rho}(\bar{x})\} = \mu_1^{\rho-2},$$

uniformly in R_A^c – and hence the desired result (I) – if to every $\delta > 0$ there are $J(\delta)$ and $T(\delta)$ with

$$(3.12) \quad x_1^\rho \sum_{m=1}^{[x_1/J(\delta)]} q(m, \bar{x}) < \delta$$

for all $\bar{x} \in R_A^c$ with $x_1 \geq T(\delta)$. In the same way as I, (5.8), we derive

$$(3.13) \quad \int |z_1|^\rho h(\bar{z}) F^m(d\bar{z}) \leq m^\rho \int h(\bar{z}) F^{m-1} R(d\bar{z})$$

for $h \geq 0$, where

$$(3.14) \quad R(E) = \int_E |x_1|^\rho F(d\bar{x}).$$

So, since $g \in \mathcal{C}_d$, it is sufficient for (3.12) that

$$(3.15) \quad \sum_{m=1}^{[x_1/J(\delta)]} m^\rho \int g(\bar{z} - \bar{x}) F^{m-1} R(d\bar{z}) < \delta.$$

The first term in (3.15) tends to zero as $x_1 \rightarrow \infty$, uniformly in R_A^c , since R is a finite measure. From (3.10), (2.17), (3.8) and (3.9) we have for $J > \mu_1$,

$$(3.16) \quad \lim_{x_1 \rightarrow \infty} E[x_1^{-2} M^2(\bar{x}) I\{x_1^{-1} M(\bar{x}) < J^{-1}\}] = 0,$$

$$\lim_{x_1 \rightarrow \infty} \sum_{m=1}^{[x_1/J]} m^\rho \int g(\bar{z} - \bar{x}) F^m(d\bar{z}) = 0,$$

both uniformly in R_A^c . For $J > \mu_1$, and $\bar{x} \in R_A^c$

$$\sum_{m=2}^{[x_1/J]} m^\rho \int g(\bar{z} - \bar{x}) F^{m-1} R(d\bar{z})$$

$$\leq 2^\rho \int \left\{ \sum_{m=1}^{[x_1/J]} m^\rho \int g(\bar{z} + \bar{\zeta} - \bar{x}) F^m(d\bar{z}) \right\} R(d\bar{\zeta}) \leq 2^\rho \int \eta(\zeta_1 - x_1) R(d\bar{\zeta}),$$

where η is a bounded function by I, lemma 2.4, and $\lim_{t \rightarrow \infty} \eta(t) = 0$ by (3.16). This proves (3.15) and therefore (I).

Now we will prove

(II). To any $\varepsilon > 0$ and $A > 0$ there is $\xi(\varepsilon, A)$ with

$$(3.17) \quad x_1^\rho \int g(\bar{z} - \bar{x}) U_F(d\bar{z}) < \varepsilon + c_1 \exp(-c_0 A^2),$$

for all $\bar{x} \in R_A$ with $x_1 \geq \xi(\varepsilon, A)$, where R_A is given by (2.12a) and c_0, c_1 do not depend on A or ε .

By (3.13), since $g \in \mathcal{C}_d$,

$$(3.18) \quad x_1^\rho \int g(\bar{z} - \bar{x}) U_F(d\bar{z}) \leq C_2 \sum_{m=1}^{\infty} m^\rho \int (g(\bar{z} - \bar{x}) F^{m-1} R(d\bar{z}))$$

$$\leq C_2 \int g(\bar{z} - \bar{x}) R(d\bar{z}) + 2^\rho C_2 \int g(\bar{z} - \bar{x}) W_F R(d\bar{z}).$$

Here the first term tends to zero as $x_1 \rightarrow \infty$, uniformly in x_2, \dots, x_d and by lemma 2.4 the second term is majorized by

$$(3.19) \quad 2^\rho C_2 \alpha \int L(\bar{x} - \bar{\zeta}) R(d\bar{\zeta}) + 2^\rho C_2 \int \theta(x_1 - \zeta_1) R(d\bar{\zeta})$$

where θ is a bounded function by I, lemma 2.4, and $\lim_{t \rightarrow \infty} \theta(t) = 0$. So the second term in (3.19) tends to zero as $x_1 \rightarrow \infty$. The inequality (3.17) now follows by considering the first term of (3.19), using the definition of $L(\bar{x})$ and writing $R = R' + R''$ where the measure R' has total variation smaller than $\frac{1}{2}\varepsilon$ and R'' is restricted to a bounded set.

The theorem now follows from (I), (II) and the definition of $L(\bar{x})$.

For $d = 2$ and $d = 4$ the proof of (I) remains unchanged up to and including (3.10). The relation (3.11) now follows from the remark to lemma 2.3, since $0 < 2 - \rho < 2$. The proof of (II) holds for $d = 4$ but not for $d = 2$ since (3.13) is derived by Minkowski's inequality with exponent ρ .

For $d = 2$ we have

$$x_1^{\frac{1}{2}} \sum_{m=1}^{[x_1]} \int g(\bar{z} - \bar{x}) F^m(d\bar{z}) \leq x_1 \sum_{m=1}^{[x_1]} m^{-\frac{1}{2}} \int g(\bar{z} - \bar{x}) F^m(d\bar{z}),$$

$$x_1^{\frac{1}{2}} \sum_{m=[x_1]+1}^{\infty} \int g(\bar{z} - \bar{x}) F^m(d\bar{z}) \leq \sum_{m=[x_1]+1}^{\infty} m^{\frac{1}{2}} \int g(\bar{z} - \bar{x}) F^m(d\bar{z}),$$

so

$$x_1^{\frac{1}{2}} \int g(\bar{z} - \bar{x}) U_F(d\bar{z}) \leq \int g(\bar{z} - \bar{x}) W_F(dz) + x_1 \sum_{m=1}^{\infty} m^{-\frac{1}{2}} \int g(\bar{z} - \bar{x}) F^m(d\bar{z}),$$

and (II) now follows by lemma 2.4 and lemma 2.6 with $k = 1$.

THEOREM 3.2. *Under the lattice conditions of lemma 2.5, if $E|X_{11}|^\rho < \infty$,*

$$(3.20) \quad \lim_{x_1 \rightarrow \infty} \{x_1^\rho U_F(\{\bar{x}\}) - \mu_1^\rho L(\bar{x})\} = 0,$$

uniformly in x_2, \dots, x_d , if \bar{x} is restricted to lattice points of F .

PROOF. From lemma 2.5, by methods similar to those used in the proof of theorem 3.1. We need the following version of (2.21):

$$\lim_{x_1 \rightarrow \infty} \{x_1^k V_F(\{\bar{x}\}) - \mu_1^k L(\bar{x})\} = 0,$$

uniformly in x_2, \dots, x_d , if \bar{x} is restricted to lattice points of F .

It is noted that a corresponding theorem for densities may be derived by similar techniques, if $\varphi \in L_1$. A similar remark hold for the results of I.

4. Existence of moments of order unequal to ρ

Theorems 5.1 and 5.2 of I give some information about the order of decrease of $U_F(A + \bar{x})$ if $\bar{x} \rightarrow \infty$ in a direction different from $\bar{\mu}$. The following supplementary result will be derived by direct appeal to one-dimensional renewal theory. Second moments need not be finite.

THEOREM 4.1. *Let $S = \{\bar{x} : (\bar{x}, \mu) \leq \theta |\bar{x}| |\bar{\mu}|\}$, with $-1 \leq \theta < 1$. Then, if $E|X_{11}|^p < \infty$, where $p > 1$, we have*

$$(4.1) \quad \int_S (1 + |\bar{x}|)^{p-2} U_F(d\bar{x}) < \infty.$$

PROOF. It is sufficient to show that (1) holds with S replaced by

$$C = \{\bar{x} : (\bar{x}, \bar{\alpha}) \geq \gamma |\bar{x}|\},$$

where $\gamma \in (0, 1)$ and the unit vector $\bar{\alpha}$ are such that $\bar{\mu} \notin C$. We may choose our coordinate system in such a way that $\mu_1 > 0$ and

$$C \subset K = \{\bar{x} : x_1 \leq 0, x_2^2 + \cdots + x_d^2 \leq \beta x_1^2\},$$

where β is a positive constant. By applying the inequality $|\bar{x}| \geq |x_1|$ if $p < 2$ and $|\bar{x}| \leq |x_1| \sqrt{1 + \beta}$ for $\bar{x} \in K$ if $p \geq 2$, we find

$$\int_K (1 + |\bar{x}|)^{p-2} U_F(d\bar{x}) \leq \int_{x_1 \leq 0} (1 + c|x_1|)^{p-2} U_F(dx),$$

with $c = 1$ or $c = \sqrt{1 + \beta}$. Let $U_1 = \sum_1^\infty F_1^m$ be the one-dimensional renewal measure belonging to the probability distribution F_1 of X_{11} . Then

$$\int_{x_1 \leq 0} (1 + c|x_1|)^{p-2} U_F(dx) = \int_{x_1 \leq 0} (1 + c|x_1|)^{p-2} U_1(dx_1) < \infty.$$

(See Stone and Wainger [9], Stam [7].)

Let u_F denote the density of U_F , if present. The density version of theorem 3.1 says that if $E|X_{11}|^p < \infty$,

$$(4.2) \quad \lim_{x_1 \rightarrow \infty} x_1^\rho \{\mu_F(\bar{x}) - \mu_1^{-1} q(\bar{x})\} = 0,$$

uniformly in x_2, \dots, x_d . Here $q(\bar{x})$ for fixed x_1 is a gaussian probability density in x_2, \dots, x_d with covariance matrix proportional to x_1 . The form of (4.2) suggests that ρ may be replaced by p if $E|X_{11}|^p < \infty$, and that a similar remark might apply to (3.1).

For $p > \rho$ this is not true. As an example take X_{11}, \dots, X_{1d} independent, X_{11} negative exponential with parameter 1 and X_{1j} gaussian with zero expectation and unit variance, $j = 2, \dots, d$. For $x_2 = \dots = x_d = 0$ we then should have

$$\lim_{x \rightarrow \infty} x^p \left[\sum_{m=1}^{\infty} \frac{x^{m-1}}{(m-1)!} e^{-x} m^{-\rho} - x^{-\rho} \right] = 0$$

for any $p > 0$. Take $d = 5$. We have

$$\sum_{m=1}^{\infty} \frac{x^{m-1}}{(m-1)!} e^{-x} m^{-2} = \sum_{n=0}^{\infty} \left[\frac{x^n}{(n+2)!} + \frac{x^n}{n!(n+1)^2(n+2)} \right] e^{-x}.$$

Here the first term between brackets gives rise to x^{-2} plus exponential terms and the second term to a contribution of order x^{-3} by the law of large numbers for the Poisson distribution with parameter tending to ∞ .

For $p < \rho$ we would obtain $x_1^p \mu_F(\bar{x}) \rightarrow 0$ and this is correct for $2 < p < \rho$.

THEOREM 4.2. *If $E|X_{11}|^p < \infty$, where $2 < p < \rho$, and F has finite second moments, we have for bounded A , uniformly in x_2, \dots, x_d ,*

$$(4.3) \quad \lim_{x_1 \rightarrow \infty} x_1^p U_F(A + \bar{x}) = 0.$$

PROOF. By the boundedness of A and by (3.13) with ρ replaced by p we have

$$x_1^p F^m(A + \bar{x}) \leq c \int_{A + \bar{x}} z_1^p F^m(d\bar{z}) \leq cm^p F^{m-1} R(A + \bar{x}),$$

where R is defined by (3.14) with ρ replaced by p . So

$$(4.4) \quad x_1^p U_F(A + \bar{x}) \leq cR(A + \bar{x}) + cHR(A + \bar{x}),$$

where $H = \sum_1^{\infty} (m+1)^p F^m$. Since $p < \rho$ we have

$$\lim_{|\bar{y}| \rightarrow \infty} H(A + \bar{y}) = 0.$$

(See the proof of (3.8) in I.) Since R is a finite measure, (4.3) follows from (4.4).

Summary

Let $\bar{X}_1, \bar{X}_2, \dots$ be strictly d -dimensional random vectors with common distribution F , with finite second moments and with $\mu_1 = EX_{11} > 0$. Let $U(A) = \sum_1^{\infty} F^m(A)$, where F^m is the m -fold convolution of F . The restriction of U to the strip $\{\bar{x} : t \leq x_1 \leq t+a\}$ is a finite measure with variation tending to $\mu_1^{-1}a$ if F is nonarithmetic. For $t \rightarrow \infty$ this measure satisfies a central limit theorem. The paper derives the local form of this limit theorem. A version of it for purely arithmetic F also is given. The global form was proved by the author in *Zeitschrift für Wahrsch. th. u. verw. Geb.*, 10 (1968), 81–86. The paper is a continuation of *Comp. Math.* 21 (1969), 383–399.

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Mathematisch Instituut der Rijksuniversiteit,
Hoogbouw WSN,
Universiteitscomplex Paddepoel,
Postbus 800,
Groningen