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Local central limit theorem for first entrance of a random walk into a half space


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LOCAL CENTRAL LIMIT THEOREM FOR FIRST ENTRANCE
OF A RANDOM WALK INTO A HALF SPACE

by

A. J. Stam

1. Introduction, notations

Throughout this paper the following assumptions apply. Let $X_k = (X_{k1}, \cdots, X_{kd})$, $k = 1, 2, \cdots$, be independent strictly $d$-dimensional random vectors with common probability distribution $F$ and characteristic function $\varphi$. (The bar distinguishes vectors from scalars and strict $d$-dimensionality means that the support of $F$ is not contained in a hyperplane of dimension lower than $d$.) The second moments of the $X_i$ will be finite and the first moment vector $\bar{\mu}$ nonzero. We put $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$,

(1.1) \quad U(A) = \sum_{m=1}^{\infty} F^m(A),

where the exponent denotes convolution. The distribution function of $X_{11}$ if $F_1$.

We consider the first entrance of the random walk $\{S_n\}$ into the half space $\{\bar{x} : a_1 x_1 + \cdots + a_d x_d \geq t\}$, where $t > 0$. It is essential that the half line $\bar{x} = c\bar{\mu}$, $c > 0$, intersects the boundary of the half space. For convenience of notation we assume that the $x_1$-axis of our coordinate system has been chosen in the direction of $\bar{a}$. This implies that we have to assume throughout this paper

(1.2) \quad \mu_1 > 0.

Now let $N(t) = \min \{n : S_{n1} \geq t\}$, and let $R_t$ be the joint probability distribution of

$Z_1(t) - t$, $Z_2(t)$, $\cdots$, $Z_d(t)$,

where $Z(t) = S_{N(t)}$. It will be shown in section 3 that $R_t$ for $t \to \infty$ satisfies a local central limit theorem, if either $F$ is nonarithmetic – i.e. $\{\bar{u} : \varphi(\bar{u}) = 1\} = \{0\}$ – or $X_{1k}$ is arithmetic with span 1, $k = 1, \cdots, d$.

The approximating probability measure is the product of the well known limiting distribution of $Z_1(t) - t$ and a normal distribution for $Z_2(t)$, $\cdots$, $Z_d(t)$. The corresponding ‘marginal’ result for $Z_2(t)$, $\cdots$, $Z_d(t)$ also is derived.
We will need the strict ascending ladder process with respect to the $x_1$-coordinate, i.e. the random walk $\bar{S}_{n_1}, \bar{S}_{n_2}, \cdots$ in $R_d$, where $n_1, n_2, \cdots$ are the times at which a strict ascending ladder point occurs in the random walk $S_{11}, S_{21}, S_{31}, \cdots$. We put
\begin{equation}
\bar{Y} = \bar{S}_{n_1}.
\end{equation}

By Wald’s identity for expectations we have, since $E\{n_1\} < \infty$ by (1.2),
\begin{equation}
\bar{v} \overset{df}{=} E\{\bar{Y}\} = \mu E\{n_1\}.
\end{equation}

By $H_1$ we denote the probability distribution of $Y_1$.

Let $E$ denote the covariance matrix of the random variables $X_{1j} - \mu_i^{-1}\mu_jX_{11}, j = 2, \cdots, d$ and $\varepsilon_{ij}$ the $(i,j)$-element of $E^{-1}$. We put
\begin{equation}
Z(x_1, \cdots, x_d) = \exp\left[-\frac{1}{2}\mu_1 x_1^{-1} \sum_{i=2}^{d} \sum_{j=2}^{d} \varepsilon_{ij}(x_i - \mu_i^{-1}\mu_i x_i)(x_j - \mu_j^{-1}\mu_j x_i)\right],
\end{equation}
\begin{equation}
L(x_1, \cdots, x_d) = \mu_1^{-1}(2\pi)^{-\rho}(\text{Det } E)^{-\frac{1}{2}}Z(x_1, \cdots, x_d),
\end{equation}
where
\begin{equation}
\rho = \frac{1}{2}(d-1).
\end{equation}
If $x_1$ is kept fixed, $\mu_1^{\rho+1}x_1^{-\rho}L(x_1, x_2, \cdots, x_d)$ considered as a function of $x_2, \cdots, x_d$, is a $(d-1)$-dimensional normal probability density. By $C_d$ we denote the class of continuous functions on $R_d$ with compact support. The indicator function of a set $A$ is written $I_A$.

Proofs are based on the results obtained in Stam [1].

2. Preliminary lemmas

Lemma 2.1. If $F$ is nonarithmetic and $E|X_{11}|^p < \infty$, then for $g \in C_d$
\begin{equation}
\lim_{x_1 \to \infty} \left\{x_1^p \int g(\bar{z} - \bar{x})U(d\bar{z}) - \mu_1^p L(\bar{x}) \int g(\bar{z}) d\bar{z}\right\} = 0,
\end{equation}
uniformly in $x_2, \cdots, x_d$.

This is theorem 3.1 of Stam [1], II. We also need theorem 3.2 of the same paper:

Lemma 2.2. If there is a Cartesian coordinate system such that the components of $\bar{X}_1$ in this system are arithmetic with span 1 and their joint characteristic function $\zeta$ satisfies the condition: $\zeta(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of $2\pi$ and $|\zeta(\bar{u})| < 1$ elsewhere and if $E|X_{11}|^p < \infty$, then
uniformly in $x_2, \ldots, x_d$, if $\bar{x}$ is restricted to lattice points of $U$.

**Lemma 2.3.** If $F$ satisfies the conditions of lemma 2.1 and $g(\bar{x}) = I_{a,b}(x_1)g_1(\bar{x})$ with $g_1 \in C_d$, then (2.1) holds for $g$.

**Proof.** We may write $g = h + h_1$ with $h \in C_d$ and $|h_1| \leq h_2 \in C_d$. Then

\[
\lim_{x_1 \to \infty} \{x_1^d U(\{\bar{x}\}) - \mu_x^d L(\bar{x})\} = 0,
\]

uniformly in $x_2, \ldots, x_d$, if $\bar{x}$ is restricted to lattice points of $U$.

Since $L(\bar{x})$ is bounded, we may choose $h, h_1$ and $h_2$ so that

\[
\mu_x^d L(\bar{x}) \int h_2(\bar{z}) d\bar{z} < \varepsilon/4.
\]

Then

\[
\mu_x^d L(\bar{x}) \int h_2(\bar{z}) d\bar{z} \leq \mu_x^d L(\bar{x}) \int h_2(\bar{z}) d\bar{z}
\]

and the lemma follows from (2.2), (2.3), (2.4) and lemma 2.1.

**Lemma 2.4.** The random variables $Y_1, \ldots, Y_d$ of (1.3) have finite second moments. If $\mu_j = 0$, $j \geq 2$,

\[
\text{cov} (Y_j, Y_k) = E\{ n_j \} \text{cov} (X_{1j}, X_{1k}), \quad j, k = 2, \ldots, d.
\]

See theorems 1.2, 1.4, 1.5 of Nevels [2].

**Lemma 2.5.** The covariance matrix of the random variables $Y_j - v_1^{-1} v_j Y_1$, $j = 2, \ldots, d$, is $E\{ n_1 \} \cdot E$, where $E$ is defined as in section 1.

**Proof.** By (1.4) we have $v_1^{-1} v_j = \mu_1^{-1} \mu_j$. So

\[
Y_j - v_1^{-1} v_j Y_1 = \sum_{k=1}^{n_1} W_{kj},
\]

where $W_{kj} = X_{kj} - \mu_1^{-1} \mu_j X_{k1}$ has expectation zero. The lemma follows from lemma 2.5 by considering the random walk with steps $(X_{k1}, W_{k2}, \ldots, W_{kd})$.

**Lemma 2.6.** If $E|X_{11}|^2 < \infty$, where $\lambda > 0$, then $E|Y_1|^4 < \infty$.

**Proof.** See Nevels [2], theorem 1.1.
3. Local limit theorems for $R_t$

**Theorem 3.1.** If $F$ is nonarithmetic and $E|X_{11}|^p < \infty$, we have for $g \in C_d$

$$
\lim_{t \to \infty} t^p \left| \int g(x_1, x_2 - a_2, \ldots, x_d - a_d) R_t(d\bar{x}) - \int g(x_1, x_2 - a_2, \ldots, x_d - a_d) \beta(x_1) q_t(x_2, \ldots, x_d) d\bar{x} \right| = 0,
$$

uniformly in $a_2, \ldots, a_d$. Here

$$(3.1) \quad \beta(x_1) = 0, \quad x_1 \leq 0, \quad \beta(x_1) = v_1^{-1}\{1 - H_1(x_1)\}, \quad x_1 > 0,$$

and $q_t$ is the $(d-1)$-dimensional normal density with covariance matrix $\mu_1^{-1}tE$ and means $\mu_j^{-1}t$, $j = 2, \ldots, d$.

**Proof.** First we assume that $X_{11} \geq 0$ with probability 1. Since $g \in C_d$, it is sufficient to show that

$$
\lim_{t \to \infty} t^p \int g(x_1, x_2 - a_2, \ldots, x_d - a_d) R_t(d\bar{x}) - t^p q_t(a_2, \ldots, a_d) \int \beta(x_1) g(d\bar{x}) d\bar{x} = 0,
$$

uniformly in $a_2, \ldots, a_d$. We have

$$
\begin{align*}
&\quad \quad \quad \quad \quad \quad t^p \int g(x_1, x_2 - a_2, \ldots, x_d - a_d) R_t(d\bar{x}) \\
&= t^p \int I_{[t, \infty)}(x_1) g(x_1 - t, x_2 - a_2, \ldots, x_d - a_d) F(d\bar{x}) \\
&\quad + t^p \sum_{m=1}^{\infty} \int I_{(-\infty, t]}(x_1) I_{[t, \infty)}(x_1 + \xi_1) g(x_1 + \xi_1 - t, x_2 + \xi_2 - a_2, \ldots, x_d + \xi_d - a_d) F^m(d\bar{x}) F(d\xi).
\end{align*}
$$

Here the first term tends to zero for $t \to \infty$, uniformly in $a_2, \ldots, a_d$, since $E|X_{11}|^p < \infty$. The second term may be written

$$
(3.4) \quad T_2 = \int \Lambda(\bar{\xi}, t, \bar{a}) F(d\xi),
$$

where $\bar{a} = (0, a_2, \ldots, a_d)$ and

$$
\begin{align*}
\Lambda(\bar{\xi}, t, \bar{a}) &= t^p \int I_{(-\infty, 0]}(x_1 - t) g(x_1 + \xi_1 - t, x_2 + \xi_2 - a_2, \ldots, x_d + \xi_d - a_d) U(d\bar{x}).
\end{align*}
$$

By lemma 2.3, applied to the function $I_{(-\infty, 0]}(x_1) g(\bar{x} + \xi)$ with $\xi$ fixed,
\[(3.5) \quad A(\xi, t, \bar{a}) = \eta(\xi, t, \bar{a}) + \mu_{\xi} L(t, a_2, \cdots, a_d) \int_{I_{[-\xi, 0]}} g(\bar{z} + \xi) \, d\bar{z},\]

where \(\lim_{t \to \infty} \eta(\xi, t, \bar{a}) = 0\), uniformly in \(a_2, \cdots, a_d\) for fixed \(\xi\). Equivalently
\[(3.6) \quad \lim_{t \to \infty} \zeta(\xi, t) = 0,
\]

for fixed \(\xi\), where \(\zeta(\xi, t) = \sup_{a} \eta(\xi, t, \bar{a})\). We now write
\[(3.7) \quad T_2 = T_3 + T_4,
\]

\[(3.8) \quad T_3 = \int_{I_{[\xi t, \infty]}} A(\xi, t, \bar{a}) F(d\xi),
\]

\[(3.9) \quad T_4 = \int_{I_{[0, \frac{1}{2}t]}} A(\xi, t, \bar{a}) F(d\xi).
\]

Since \(\int g(\bar{z} - \bar{y}) U(d\bar{z})\) is bounded in \(\bar{y}\), we have by (3.4a) and the assumption that \(E|X_1|^p < \infty\),
\[(3.10) \quad 0 \leq \zeta < \frac{1}{2} t.
\]

So (3.4a) and lemma 2.3 show that \(I_{[0, \frac{1}{2}t]}(\xi_1) \eta(\xi, t, \bar{a})\) and therefore also \(I_{[0, \frac{1}{2}t]}(\xi_1) \zeta(\xi, t, \bar{a})\) is bounded by a constant. So
\[(3.11) \quad 0 \leq \zeta < \frac{1}{2} t.
\]

So by (1.5) and (1.6)
\[(3.12) \quad 0 \leq \zeta < \frac{1}{2} t.
\]

And (3.2) follows from (3.3), (3.4), (3.7), (3.8), (3.10) and (3.11).
If $P\{X_{11} < 0\} > 0$, we apply the part of the theorem proved above, to the random walk arising by sampling the $S_n$-process at the strict ladder times of the process $\{S_n\}$. It is noted that the first entrance of $\{S_n\}$ into the half space $\{x_1 \geq t\}$ necessarily is a ladder point of $\{S_n\}$. The theorem now follows by lemma 2.5 and (1.4). Lemma 2.6 guarantees that the condition on the absolute moment of order $\rho$ of the $x_1$-component is satisfied.

**Theorem 3.2.** If $F$ is nonarithmetic and $E|X_{11}|^\rho < \infty$, we have for $h \in C_{d-1}$

$$
\lim_{t \to \infty} t^\rho \left| \int h(x_2-a_2, \ldots, x_d-a_d) R_t(d\bar{x}) \\
- \int h(x_2-a_2, \ldots, x_d-a_d) \eta_t(x_2, \ldots, x_d) dx_2 \cdots dx_d \right| = 0,
$$

uniformly in $a_2, \ldots, a_d$. Here $\eta_t$ is the same as in theorem 3.1.

**Proof.** Since $h \in C_{d-1}$, it is sufficient to show that, uniformly in $a_2, \ldots, a_d$,

$$
\lim_{t \to \infty} t^\rho \left| \int h(x_2-a_2, \ldots, x_d-a_d) R_t(d\bar{x}) - \eta_t(a_2, \ldots, a_d) \\
\times \int h(x_2, \ldots, x_d) dx_2 \cdots dx_d \right| = 0.
$$

First we assume that $X_{11} \geq 0$. We then start the proof of (3.2) anew at (3.3), where for $g(x_1, \ldots, x_d)$ we now take $h(x_2, \ldots, x_d)$. We obtain (3.4), (3.5), (3.6), since lemma 2.3 applies to the function $I_{(-\xi_1, 0)}(\xi)$ $h(x_2+\xi_2, \ldots, x_d+\xi_d)$ with $\xi$ fixed. To obtain (3.8) and (3.9) we have to take into account the factor $I_{(-\xi_1, 0)}(x_1-t)$ in (3.4a). This means that in the integral in (3.4a) the variable $x_1$ is restricted to the interval $[t-\xi_1, t)$. We then have in $T_3$

$$
(3.13) \quad A(\xi, t, \bar{a}) \leq t^\rho \int_{(0,t)}(x_1) h(x_2+\xi_2-a_2, \ldots, x_d+\xi_d-a_d) U(d\bar{x}).
$$

By lemma 2.3, for $m \geq 1$,

$$
\int I_{[m, m+1)}(x_1) h(x_2+\xi_2-a_2, \ldots, x_d+\xi_d-a_d) U(d\bar{x}) \leq c_2 m^{-\rho},
$$

so

$$
(3.14) \quad A(\xi, t, \bar{a}) \leq t^\rho \{c_0 + c_2 \sum_{m=1}^{[t+1]} m^{-\rho}\}.
$$

Therefore $T_3 \to 0$, uniformly, since $E|X_{11}|^\rho < \infty$. For $\rho = \frac{1}{2}$ and $\rho = 1$ we have to appeal to the existence of first and second moments. To apply the Lebesgue dominated convergence theorem to $T_4$ we note that the
second term on the right in (3.5) is bounded by $c_3|\xi_1|$ with $c_3$ a constant. In the same way as (3.14) we obtain

$$I_{(0, t)}(\xi)A(\xi, t, \bar{a}) \leq c_4 \sum_{m=-r}^{t+1} m^{-p} \leq c_5|\xi_1|. $$

So $|\zeta(\xi, t)| \leq c_6|\xi_1|$ and (3.9) follows by the existence of first moments. The relation (3.10) also follows and (3.11) is replaced by

$$\int \gamma(\xi, t, \bar{a})F(d\xi) = t^p q_1(a_2, \ldots, a_d) \int h(y_2, \ldots, y_d)dy_2 \cdots dy_d. $$

The relation (3.12) now follows from the counterparts of (3.3), (3.4), (3.7), (3.8), (3.10) and (3.11), if $X_{11} \geq 0$. The proof is concluded in the same way as the proof of theorem 3.1.

**Theorem 3.3.** Let $F$ satisfy the conditions of lemma 2.2. For $t > 0$ let $\bar{a}(t)$ be a $d$-vector such that $0 \leq a_1(t) \leq K$ and $t + \bar{a}(t)$ belongs to the $F$-lattice. Then

$$\lim_{t \to \infty} t^p|R_t(\bar{a}(t)) - v_1^{-1} H_1(E_t)q_1(a_2(t), \ldots, a_d(t))| = 0,$$

uniformly in $\bar{a}(t)$ for fixed $K$. Here $E_t$ denotes the open interval $(a_1(t), \infty)$ and $q_1$ the same normal density as in theorem 3.1.

**Corollary.** If $X_{11}, \ldots, X_{1d}$ are integer valued such that $\varphi(\bar{a}) = 1$ if $u_1, \ldots, u_d$ are integer multiples of $2\pi$ and $|\varphi(\bar{a})| < 1$ elsewhere, and if $E|X_{11}|^p < \infty$, then

$$\lim_{h \to \infty} h^p|R_h(\bar{a}) - v_1^{-1} H_1((k_1, \infty))q_h(k_2, \ldots, k_d)| = 0,$$

uniformly in $k_2, \ldots, k_d$, if $h, k_1, \ldots, k_d$ are integers with $h > 0, k_1 \geq 0$.

**Proof.** First assume $X_{11} \geq 0$ with probability 1. We have

$$t^pR_t(\bar{a}(t)) = t^p P\{S_1 = t + \bar{a}(t)\} + T_2,$$

where the first term is dealt with by the existence of $EX_{11}^p$

$$T_2 = t^p \sum_{m=1}^{\infty} P\{S_{m1} < t, S_{m+1} = t + \bar{a}(t)\},$$

$$T_2 = t^p \sum_{m=1}^{\infty} \sum_{\xi} P\{X_{m+1} = \xi\}P\{S_{m1} < t, S_m = t + \bar{a}(t) - \xi\},$$

where $\xi$ runs through points of the $F$-lattice. Because of the second factor we may write

$$T_2 = t^p \sum_{\xi_1 > a_1(t)} F(\xi)U(t + \bar{a}(t) - \xi_1).$$
By lemma 2.2 we have for fixed $\xi$

$$t^n U\{t + a(t) - \xi\} = \mu_t^n \mathcal{L}(t, a_2(t), \cdots, a_d(t)) + \eta,$$

where $\eta \to 0$ as $t \to \infty$, uniformly in $a(t)$ if $0 \leq a_1(t) \leq K$, if $\xi$ is kept fixed. The proof now proceeds in the same way as with theorem 3.1. We write $T_2 = T_3 + T_4$ where the sum is taken over the sets $\{\xi_1 \geq 1/t\}$ and $\{a_1(t) < \xi_1 < 1/t\}$, respectively. Handling of $T_3$ and $T_4$ requires the same estimations as in the proof of theorem 3.1.

The lattice counterpart of theorem 3.2 is restricted to integer valued $X_{11}, \cdots, X_{1d}$, since under the more general assumptions of theorem 3.3 the lattice description of $Z_2(t), \cdots, Z_d(t)$ is difficult.

**Theorem 3.4.** If $X_{11}, \cdots, X_{1d}$ are integer valued, such that $\varphi(\bar{u}) = 1$ if $u_1, \cdots, u_d$ are integer multiples of $2\pi$ and $|\varphi(\bar{u})| < 1$ elsewhere, and if $E|X_{11}|^p < \infty$, then

$$\lim_{h \to \infty} h^p |P\{Z_2(h) = k_2, \cdots, Z_d(h) = k_d\} - q_h(k_2, \cdots, k_d)| = 0.$$

uniformly in $k_2, \cdots, k_d$. Here $h, k_2, \cdots, k_d$ are integers and $q_t$ is the same normal density as in theorem 3.1.

**Proof.** First take $P\{X_{11} \geq 0\} = 1$. We have

$$h^p P\{Z_2(h) = k_2, \cdots, Z_d(h) = k_d\} = h^p P\{X_{11} \geq h, X_{12} = k_2, \cdots, X_{1d} = k_d\} + T_2,$$

where the first term tends to zero uniformly in $(k_2, \cdots, k_d)$ as $h \to \infty$ since $E|X_{11}|^p < \infty$ and

$$T_2 = h^p \sum_{m=1}^{\infty} P\{S_{m1} < h, S_{m+1,1} \geq h, S_{m+1,r} = k_r, \quad r = 2, \cdots, d\}$$

$$= h^p \sum_{m=1}^{\infty} \sum' \sum'' F^m\{i_1, \cdots, i_d\} F\{j_1, \cdots, j_d\},$$

where $\sum'$ and $\sum''$ are subject to the restrictions $i_1 < h$, $i_1 + j_1 \geq h$, $i_r + j_r = k_r$, $r = 2, \cdots, d$. So

$$T_2 = h^p \sum_{j_1, \cdots, j_d} F\{j_1, \cdots, j_d\} \sum_{i_1 = h-j_1}^{h-1} \sum_{i_2, \cdots, i_d} U\{i_1, k_2 - j_2, \cdots, k_d - j_d\}.$$

By lemma 2.2 we have for fixed $j_1, \cdots, j_d$ and $h-j_1 \leq i_1 < h-1$

$$U\{i_1, k_2 - j_2, \cdots, k_d - j_d\} = \mu_t^n L(h, k_2, \cdots, k_d) + \eta,$$

with $\lim_{h \to \infty} \eta = 0$, uniformly in $k_2, \cdots, k_d$. 

[8]
The relation (3.15) now follows with (1.5) and (1.6) if passing to the limit in (3.16) under the sum over $j_1, \cdots, j_d$ is justified. This is done by the same methods as in the proof of theorem 3.2.

If $P\{X_{11} < 0\} > 0$ we consider the random walk at the ladder times of the process $\{S_{n1}\}$.

**Summary**

Let $X_1, X_2, \cdots$ be independent strictly $d$-dimensional random vectors, with common distribution, with finite second moments and positive $x_1$-component of the first-moment vector. Let $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$, $N(t) = \min \{n: S_{n1} \geq t\}$ and $Z(t) = S_{N(t)}$.

If $E|X_{11}|^\rho < \infty$, where $\rho = \frac{1}{2}(d-1)$, the joint distribution of $Z_1(t) - t$, $Z_2(t), \cdots, Z_d(t)$ satisfies a local central limit theorem for $t \to \infty$. The approximating probability measure is the product of the well known limiting distribution for $Z_1(t) - t$ and a normal distribution for $Z_2(t), \cdots, Z_d(t)$. The difference is $o(t^{-\rho})$ as in a local central limit theorem for sums of independent $(d-1)$-vectors.

The theorem is stated and proved for nonarithmetic $F$ and for $F$ restricted to a (rotated) cubic lattice with span 1. A special case of the global version was proved by the author in Zeitschr. für Wahrsch. th. u. verw. Geb. 10 (1968), 81–86.

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A. J. Stam


K. Nevels


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