J. DE VRIES

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EQUIVALENCE OF ALMOST PERIODIC COMPACTIFICATIONS

by

J. de Vries

1. Introduction

1.1. Purpose of this note is to give a simple proof of [1], Theorem 5.5 (see Corollary 2.4 below).

Let S, T be semitopological semigroups ([2], p. 1). We do not assume S and T to have an identity. The Banach space of almost periodic (a.p.) functions on S is denoted by $\mathrm{A}(S)$ and the Banach space of weakly almost periodic functions on S is denoted by $\mathrm{W}(S)$. If $\phi : S \to T$ is a continuous homomorphism, then the induced mapping $f \mapsto f \circ \phi$ of $\mathrm{C}(T)$ into $\mathrm{C}(S)$ is denoted by $\tilde{\phi}$.

Recall that $\tilde{\phi}[\mathrm{C}(T)] \subset \mathrm{W}(S)$ if T is a compact semitopological semigroup, and that $\tilde{\phi}[\mathrm{C}(T)] \subset \mathrm{A}(S)$ if T is a compact topological semigroup ([1], lemma 5.2).

1.2. An ordered pair (ϕ, T) is called an *almost periodic compactification* (*a weakly almost periodic compactification*) of S, if the following conditions are satisfied:

(i) T is a compact topological (semitopological) Hausdorff semigroup.

(ii) $\phi : S \to T$ is a continuous homomorphism such that for each $f \in \mathrm{A}(S)$ ($f \in \mathrm{W}(S)$) there is a unique $\bar{f} \in \mathrm{C}(T)$ with $f = \bar{f} \circ \phi$.

Note that (ii) is equivalent with

(ii)′ $\phi : S \to T$ is a continuous homomorphism with dense image in T, and $\tilde{\phi}[\mathrm{C}(T)] = \mathrm{A}(S)$ ($\tilde{\phi}[\mathrm{C}(T)] = \mathrm{W}(S)$).

(The proof depends on 1.1 and the fact, that T is a completely regular topological space.)

1.3. Let S be a semitopological semigroup. Two a.p. compactifications (w.a.p. compactifications) (ϕ_1, T_1) and (ϕ_2, T_2) are called *equivalent*, if there is a topological isomorphism ψ of T_1 onto T_2 such that $\psi \circ \phi_1 = \phi_2$.

We shall prove, that two a.p. compactifications (w.a.p. compactifications) of a semitopological semigroup are equivalent ([1], Corollary 5.6); in fact, this is an easy corollary of the 'universal property' of the a.p. and w.a.p. compactifications, explained in Theorem 2.2 of this note. From this, Theorem 5.5 of [1] is also easily derived.

All semitopological semigroups are not supposed to have an identity (so the existence of the compactifications is not a priori guaranteed).

2. The main theorem

2.1. LEMMA. *Let X be a uniform space, Y a compact Hausdorff topological space. A function $f: X \to Y$ is uniformly continuous (Y with its unique uniformity) if and only if $g \circ f: X \to [0, 1]$ is uniformly continuous for all continuous $g: Y \to [0, 1]$.*

PROOF: It is known that Y may be regarded as a (closed) subset of a topological product of copies of the interval $[0, 1]$ such that the restrictions to Y of the canonical projections are the continuous functions of Y into $[0, 1]$. The lemma now follows from [3], Ch. 6, Theorem 10.

2.2. THEOREM. *Let (ϕ, S^a) be an a.p. compactification of S and (ψ, S^w) a weakly a.p. compactification. Then (ϕ, S^a) and (ψ, S^w) have the following 'universal' property:*

If T is a topological (resp. semitopological) compact Hausdorff semigroup and $\xi: S \to T$ a continuous homomorphism, then there exists a unique continuous homomorphism $\xi^a: S^a \to T$ (resp. $\xi^w: S^w \to T$) such that $\xi = \xi^a \circ \phi$ (resp. $\xi = \xi^w \circ \psi$).

PROOF: Unicity follows from the fact that T is a Hausdorff topological space and that ϕ and ψ have dense images.

We now prove the existence of ξ^w in the case that T is a semitopological compact Hausdorff semigroup (the existence of ξ^a in the case that T is a topological compact Hausdorff semigroup can be proved in a similar way). First, we note that

$$(1) \qquad \forall s, t \in S: \psi(s) = \psi(t) \Rightarrow \xi(s) = \xi(t)$$

Indeed: $C(T)$ separates the points of T, so

$$\xi(s) \neq \xi(t) \Rightarrow \exists g \in C(T): g(\xi(s)) \neq g(\xi(t)).$$

Since $g \circ \xi \in W(S)$ it follows from 1.2 that in this case $\psi(s) \neq \psi(t)$.

Defining $\psi': S \to \operatorname{Im}\psi$ by $\psi = i \circ \psi'$ ($i: \operatorname{Im}\psi \to S^w$ is the inclusion map) it follows from (1) that there is a homomorphism $\xi': \operatorname{Im}\psi \to T$ such that $\xi = \xi' \circ \psi$.

Now it is sufficient to show that ξ' is uniformly continuous ($\operatorname{Im}\psi$ provided with the relative uniformity of S^w): T is complete as a uniform space ([3], Ch. 6, Theorem 32), so in this case ξ' has a uniformly continuous extension $\xi^w : S^w \to T$ ([3], Ch. 6, Theorem 26). It is easy to see that this continuous extension ξ^w of the homomorphism ξ' is itself a homomorphism, using separate continuity of multiplication in S^w and T.

Uniform continuity of ξ' will follow from 2.1 if we succeed in proving the following assertion: if $g : T \to [0, 1]$ is any continuous map, than $g \circ \xi' : \operatorname{Im}\psi \to [0, 1]$ is uniformly continuous.

By 1.1, $g \circ \xi \in \mathrm{W}(S)$, so there is a $\bar{g} \in \mathrm{C}(S^w)$ such that $g \circ \xi = \bar{g} \circ \psi$, that is

$$g \circ \xi' \circ \psi' = \bar{g} \circ i \circ \psi'$$

$$\begin{tikzcd}
S \arrow[r, "\psi'"] \arrow[dr, "\xi"'] & \operatorname{Im}\psi \arrow[r, "i"] \arrow[d, "\xi'"] & S^w \arrow[d, "\bar{g}"] \\
& T \arrow[r, "g"'] & {[0,1]}
\end{tikzcd}$$

Since S is mapped onto $\operatorname{Im}\psi$ by ψ', it follows that

$$g \circ \xi' = \bar{g} \circ i.$$

Now uniform continuity of $g \circ \xi'$ follows from uniform continuity of i and $\bar{g}$.

2.3. Corollary ([1], Corollary 5.6). *Two a.p. compactifications (weakly a.p. compactifications) of a semitopological semigroup are equivalent.*

Proof: Let (ϕ_1, T_1) and (ϕ_2, T_2) be weakly a.p. compactifications of S. By 2.2 there are continuous homomorphisms $\phi_1^w : T_2 \to T_1$ and $\phi_2^w : T_1 \to T_2$ such that

$$\phi_2^w \circ \phi_1 = \phi_2 \quad \text{and} \quad \phi_1^w \circ \phi_2 = \phi_1.$$

We have to prove, that ϕ_1 (or ϕ_2) is a topological isomorphism. It is easy to see, that

$$(\phi_1^w \circ \phi_2^w) \circ \phi_1 = \phi_1 = I_1 \circ \phi_1$$

so by unicity it follows that $\phi_1^w \circ \phi_2^w = I_1$.

Similarly $\phi_2^w \circ \phi_1^w = I_2$ (I_i denotes the identity mapping of T_i for $i = 1, 2$). From this the result follows.

2.4. Corollary ([1], Theorem 5.5). *Let S and S_1 be semitopological semigroups, (ϕ, S') and (ϕ_1, S_1') a.p. compactifications (resp. w.a.p.*

compactifications) of S *and* S_1 *and* $\psi : S \to S_1$ *a continuous homomorphism. Then there is an unique continuous homomorphism* $\psi' : S' \to S_1'$ *for which* $\psi' \circ \phi = \phi_1 \circ \psi$.

PROOF: Existence: take $\psi' = (\phi_1 \circ \psi)^a$ (respectively $\psi' = (\phi_1 \circ \psi)^w$; notation as in 2.2).

Unicity follows from unicity in 2.2: if $\psi'' : S' \to S_1'$ satisfies $\psi'' \circ \phi = \phi_1 \circ \psi$, then $\psi'' = (\phi_1 \circ \psi)^a$ (respectively $\psi'' = (\phi_1 \circ \psi)^w$).

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Free University, Amsterdam.