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ON TOPOLOGICAL NEIGHBOURHOODS

by

C. P. Rourke and B. J. Sanderson

This paper is concerned with the 'normal bundle' problem for topological manifolds: Suppose M^n is a proper, locally flat submanifold of Q^{n+r}; then what structure can be put on the neighbourhood of M in Q? If $r \leqq 2$ the problem has been solved by Kirby [22], who has shown that there is an essentially unique normal disc bundle, while if $r \geqq 3$ then our counterexample [38] showed that the notion of fibre bundle is too strong a concept. The notion of topological block bundle [37; § 1] seems inapplicable since M might possibly be untriangulable and a triangulation of M would be unnatural structure for the problem. The answer we propose here is the 'stable microbundle pair'. The idea of using microbundle pairs to classify neighbourhoods was introduced by Haefliger [9, 10] in the *pl* case and we showed [35; § 5] that his theory essentially coincides with our theory of *pl* block bundles.

An *r-microbundle pair* is a pair $\varepsilon^N \subset \zeta^{N+r}$ where ε^N denotes the trivial microbundle of rank N. Two are *equivalent* if they are isomorphic after possibly adding further trivial bundles to both elements and the isomorphism restricts to the identity on the trivial subbundles. The equivalence classes form a good 'theory' with classifying space $BTop_r = \lim_{n\to\infty}(BTop_{r+n,n})$. To the manifold pair $M \subset Q$ we associate the pair $\tau_M \oplus \nu_M \subset \tau_Q|M \oplus \nu_M$, where ν_M denotes any stable inverse to τ_M. Our main theorem (in § 3) asserts that this association classifies the germ of neighbourhood of M in Q except possibly in the case $n = 1$, $q = 3$ (and $n = 2$, $q = 4$ if $\partial M \neq \emptyset$); these ommisions are due to the unsolved 4-dimensional annulus problem.

The main work of the proof is a stability theorem for $\pi_i\ (Top_{r+n,n})$ which is contained in § 2. This we reduce by means of immersion theory to a statement about straightening handles in the sense of Kirby and Siebenmann [21, 23], keeping a *pl* subhandle fixed, which is proved in § 1. The proof follows the Kirby-Siebenmann proof for the absolute case using the relative surgery techniques of [32]. In §§ 4, 5 and 6 we give some applications of the main theorem.

In § 4 are theorems about existence and uniqueness of normal block bundles in the case that M has a triangulation not necessarily combinatorial. The results hold for any type of block bundle (open, closed or micro) and are the same as [33; § 4] (existence and uniqueness up to isotopy) in the following cases: $r \geqq 5$ or $\leqq 2$, $r = 4$ and M is 1-connected, $r = 3$ and M is 2-connected (the omitted cases are again due to 4-dimensional problems).

In § 5 we prove stable existence and uniqueness of normal micro and disc bundles. The dimensions are the same as obtained in the *pl* case by Haefliger and Wall [12] (improved slightly by Morlet [31] and Scott [40]). However we need codimension $\geqq 5$ for our results on microbundles, and 6 for disc bundles.

In § 6 we prove analogues of smoothing theory for submanifolds. There are two cases:

(a) M and Q are both pl. manifolds and we seek to isotope M to a pl. submanifold. This is always possible in an essentially unique way if $r \geqq 3$; and if $r \leqq 2$, $n+r \geqq 5$ there is a well-defined obstruction. (The codim $\geqq 3$ result was originally announced by Bryant and Seebeck [2] using a result of Homma [15] unfortunately the proof of Homma's result appears to contain some gaps. Several other alternative proofs have been given.)

(b) Q is a pl. manifold. Here we have the analogue of the Lashof-Rothenberg result [28]. M can be isotoped to a pl. submanifold if and only if the classifying map $M \to BTop_r$ for the germ of neighbourhood lifts to BPL_r. If $r \geqq 3$ the problem is identical to the absolute problem of finding a pl. structure and if $r \leqq 2$ the map lifts in an essentially unique way by the result of Kirby mentioned above.

We are indebted to A. Haefliger for his unpublished preprint [9] and for a private communication containing his arguments for classifying germs of pl. neighbourhoods. We are also indebted to R. C. Kirby for a copy of his excellent and detailed notes [21] on triangulating manifolds.

We plan a further paper which will contain the technical details of defining transversality for topological manifolds (Hudson showed [17] that a local definition is inadequate). This is done by examining Whitney sums (defined in § 3 of this paper) along the lines of [34; § 3] and [39]; we then define M to be 'germ transversal' to W in Q if along $M \cap W$ the three germs of neighbourhoods form the Whitney sum decomposition. A relative transversality theorem in case $\dim M \cap W \geqq 5$ can then be proved using local pl. structures which exist by Kirby and Siebenmann's results.

0. Preliminaries

We use the same basic scheme of notation as in [35; § 0]. R^n denotes Euclidean n-space and I^n the double unit cube $[-1, +1]^n$. $\alpha I^n =$

$[-\alpha, \alpha]^n$, $S^{n-1} = \partial I^n$. $\Delta^n \subset R^n$ is the standard n-simplex with vertices $v_0, v_1 \cdots v_n$. There are natural inclusions $R^n \subset R^{n+r}$, $I^n \subset I^{n+r}$; and identifications $R^n \times R^r = R^{n+r}$, $I^n \times I^r = I^{n+r}$.

Microbundles

We refer to Milnor [30] for basic results on microbundles. We recall that if M is an unbounded manifold then τ_M is the microbundle with total space $M \times M$, zerosection Δ_M and projection π_1 (the projection on the first factor). We often write $\tau(M)$ for τ_M. If M is bounded then we define $\tau_M = \tau(M_+)|M$ where $M_+ = M \cup$ open collar. Suppose ξ^n, η^{n+r} are microbundles. We say ξ is a *subbundle* of η and write $\xi \subset \eta$ if $B(\xi) = B(\eta)$, $E(\xi) \subset E(\eta)$, at least in some neighbourhood of $B(\xi)$, and for each $x \in B(\xi)$ there exists microbundle charts $h : U \times R^n \to E(\xi)$, $g : U \times R^{n+r} \to E(\eta)$ with $x \in \operatorname{int} U$, so that

$$\begin{array}{ccc} U \times R^n & \xrightarrow{h} & E(\xi) \\ \cap & & \cap \\ U \times R^{n+r} & \xrightarrow{g} & E(\eta) \end{array} \qquad \text{commutes.}$$

The *trival* bundle ε^n of rank n is defined by the diagram

$$B = B \times \{0\} \subset B \times R^n \xrightarrow{\pi_1} B$$

An *inverse* to ξ is a pair (η, t) where η is a bundle with the same base as ξ and t: $E(\xi \oplus \eta) \to E(\varepsilon^N)$ is a trivialisation. Inverses are unique up to stable isomorphism of η and bundle homotopy of t, see [30].

Δ-sets and groups

We refer to [36, 37] for the theory of semisimplicial complexes and groups without degeneracies. The Δ-group Top_n (resp. PL_n) has as typical k-simplex a germ of homeomorphisms (resp. pl. homeomorphisms)

$$\sigma : \Delta^k \times R^n \to \Delta^k \times R^n$$

defined in a neighbourhood of $\Delta^k \times \{0\}$ and satisfying

(i) $\qquad \sigma|\Delta^k \times \{0\} = \mathrm{id}$

(ii) $\qquad \sigma$ commutes with projection on Δ^k.

Applying the classifying functor of [37; § 1] we get classifying spaces $BTop_n$, BPL_n which are *Kan Δ*-sets. $BTop_n$ classifies n-microbundles with base a CW complex and there is a universal microbundle $\gamma^n/BTop_n$. (We recall from [36] that there is a natural bijection between homotopy classes of Δ-maps $[S(X), Y]$ and continuous maps $[X, |Y|]$, where X has the homotopy type of a CW-complex, Y is a *Kan Δ*-set and $S(X)$ is the singular complex. We will denote both these sets by $[X, Y]$). All topological manifolds have the homotopy type of CW complexes and thus $BTop_n$ classifies n-microbundles over manifolds. Similar remarks apply to BPL_n.

We define Δ-subgroups Top^n_{r+n}, $Top_{r+n,n}$ of Top_{r+n} by the conditions $(\mathrm{iii})_1$ and $(\mathrm{iii})_2$ respectively

$$(\mathrm{iii})_1 \qquad \sigma(\Delta^k \times R^n) = \Delta^k \times R^n$$

$$(\mathrm{iii})_2 \qquad \sigma(x) = x \quad \text{all} \quad x \in \Delta^k \times R^n$$

where here R^n is identified with $\{0\} \times R^n \subset R^r \times R^n = R^{r+n}$. PL^n_{r+n}, $PL^n_{r+n,n}$ are defined similarly. There are natural inclusions of all the pl. groups in the corresponding topological groups and of $Top_{r+n,n}$ in Top^n_{r+n} etc.

We now define two suspension maps s and s' (both injective) in the diagram

$$\begin{tikzcd}
Top_{r+n+1,n} \arrow[r, "s"] & Top_{r+n+2,n+1} \\
Top_{r+n,n} \arrow[u, "s'"] \arrow[r, "s"'] & Top_{r+n+1,n+1} \arrow[u, "s'"'] \arrow[ul, dashed, "f"']
\end{tikzcd}$$

where $s(\sigma) : \Delta^k \times R^{r+n} \times R^1$ is defined to be $\sigma \times \mathrm{id}$ and

$$s'(\sigma) : \Delta^k \times R^r \times R^1 \times R^n$$

is obtained from $s(\sigma)$ by reordering the last coordinate into the $(r+1)$-st place and the j-th coordinate to the $(j+1)$-st place for $j = r+1, \cdots n$. f is the natural inclusion. The outside square commutes while the triangles do not; however, it is easy to see that they commute up to homotopy which is all we require.

We obtain a large diagram of inclusions:

$$\begin{tikzcd}[column sep=small]
Top \arrow[r, phantom, "\subset"] & \cdots & & & & \arrow[r, phantom, "\subset"] & Top \\
\cup \\
\vdots & & & & & & \cup \\
\cup & & & & & & \vdots \\
& & & & & & \cup \\
Top_{r+1} \arrow[r, phantom, "\subset \cdots \subset"] & Top_{r+n+1,n} \arrow[r, phantom, "\subset"] & Top_{r+n+2,n+1} \arrow[r, phantom, "\cdots \subset"] & & & & Top_{r+1} \\
\cup & \cup & \cup & & & & \cup \\
Top_r \arrow[r, phantom, "\subset \cdots \subset"] & Top_{r+n,n} \arrow[r, phantom, "\subset"] & Top_{r+n+1,n+1} \arrow[uul, "\nwarrow" description, phantom] \arrow[r, phantom, "\cdots \subset"] & & & & Top_r \\
\cup & & & & & & \vdots \\
\vdots & & & & & & \cup \\
\cup \\
Top_0 \arrow[r, phantom, "\subset"] & \cdots & & & & \arrow[r, phantom, "\subset"] & Top_0
\end{tikzcd}$$

where the vertical inclusions are s', the horizontal ones s, and we have defined $Top = \cup\, Top_r$, $\boldsymbol{Top}_r = \cup\, Top_{r+n,n}$ and $\boldsymbol{Top} = \cup\, \boldsymbol{Top}_r$. By homotopy commutativity we have $Top \subset \boldsymbol{Top}$ a homotopy equivalence. Again there are similar definitions for the pl. groups.

If X is a Δ-set then we denote by $X^{(k)}$ the k-skeleton of X.

Main tools

Apart from microbundles the principal tools will be isotopy extension and immersion theory, in both pl. and topological categories. An isotopy of M and Q is *locally trivial* if it is locally the restriction of an isotopy of an open subset of Q in Q. All embeddings of manifolds will be locally flat and all isotopies locally trivial. The isotopy extension theorem (for locally trivial isotopies) is proved by Hudson–Zeeman [20] in the pl. case and Edwards–Kirby [4] in the topological case. We also need the theorem for cubes of isotopies. This is Hudson [16] in the pl. case, while the topological case follows by combining his methods with those of Edwards and Kirby, see also Kirby [21]. From this last theorem we have a *Kan* fibration

$$Top_{r+n,n} \subset Top^n_{r+n} \xrightarrow{p} Top_n$$

where p restricts to the last n coordinates, and the fibre is $Top_{r+n,n}$.

We will need a doubly relative version of immersion theory, this is stated in Corollary 2 of the appendix to this paper. The pl. version is stated but not proved in Haefliger–Poenaru [11], see the last three lines of § 2. However it follows easily from what they do prove by analogous (rather simpler) arguments to those used in our appendix. Incomplete versions of topological immersion theory have been given by Lees [29], Lashof [27] and Gauld [5].

1. Relative handle straightening

Definition of the set $H_k(n, i)$ for $k \geqq 0$, $n \geqq i \geqq 0$.

A representative is a pair (h, V) where V is a pl. manifold and $h : \Delta^k \times R^n \to V$ a homeomorphism such that $h|\partial\Delta^k \times R^n \cup \Delta^k \times R^i$ is pl. and $h|\Delta^k \times R^i$ is pl. locally flat. Two such (h_1, V_1) and (h_2, V_2) are *equivalent* if there is a pl. homeomorphism $q : V_1 \to V_2$ defined in a neighbourhood of $h_1(\Delta^k \times \{0\})$ such that

$$\begin{tikzcd} \Delta^k \times R^n \arrow[r, "h_1"] \arrow[dr, "h_2"'] & V_1 \arrow[d, "q"] \\ & V_2 \end{tikzcd}$$

commutes up to a topological isotopy which is fixed on $\partial\Delta^k \times R^n \cup \Delta^k \times R^i$ and defined in a neighbourhood of $\Delta^k \times \{0\} \times I$.

Now identify Δ^k with I^k by an orientation preserving pl. homeomorphism then an addition in $H_k(n, i)$ is defined for $k > 0$ by identifying I^k with each of $I^{k-1} \times [-1, 0]$ and $I^{k-1} \times [0, 1]$ and gluing the two representatives along $I^{k-1} \times \{0\}$. This addition makes $H_k(n, i)$ into a group with zero represented by (id., $\Delta^k \times R^n$). This follows from Proposition 1.1 below and the definition of addition in $\pi_k(Top_{m,q}, PL_{m,q})$. By ignoring conditions on $\Delta^k \times R$. we have a set $H_k(n)$; however in this case the 'equivalence' relation is not transitive since the composition of q_1 and q_2 might not be defined and we take the transitive closure of this relation. This 'absolute' set is essentially the set of handle problems considered by Kirby and Siebenmann [23] and the first halves of 1.1 and 1.2 are theirs.

There is a forgetful function $f: H_k(n, i) \to H_k(n)$ and a suspension $s : H_k(n, i) \to H_k(n+1, i+1)$ defined by $s(h, V) = (h \times \text{id}, V \times R^1)$.

PROPOSITION 1.1. *There are bijections*

$$H_k(n) \to \pi_k(Top_m, PL_m)$$

and

$$H_k(n, i) \to \pi_k(Top_{m,q}, PL_{m,q})$$

which commute with the suspension and forgetful functions, where $m = k+n$ *and* $q = k+i$.

The proof of 1.1 is postponed to § 2.

THEOREM 1.2. $H_k(n, i) = 0$ *if* $n-i \geqq 2$.
If $n+k \geqq 5$ *then*

$$H_k(n) \cong \begin{cases} 0 & k \neq 3 \\ \boldsymbol{Z}_2 & k = 3 \end{cases}$$

and if further $n-i \geqq 3$, *then* $H_k(n, i) \cong H_k(n)$ *and we have a commutative square of isomorphisms*

$$\begin{tikzcd} H_k(n+1) & H_k(n+1, i+1) \arrow[l, "f"'] \\ H_k(n) \arrow[u, "s"] & H_k(n, i) \arrow[l, "f"'] \arrow[u, "s"] \end{tikzcd}$$

The cases $n-i \leqq 2$ of the theorem follow easily from Kirby's results [22] on codimension 2 embeddings. The proof for $n-i \geqq 3$ is in two parts; first we show that $H_k(n, i) = 0$ or $\boldsymbol{Z}_2$ when $k = 3$ and $H_k(n, i) = 0$, if $k \neq 3$. This is done by relativising the Kirby–Siebenmann 'main diagram' [21; 5.1]. Then secondly we show by 'unwrapping' [21; 5.2] that $H_k(n, i)$ is actually $\boldsymbol{Z}_2$ in the case $k = 3$.

Relativisation of the main diagram

Let $h : \Delta^k \times R^n \to V^m$ be a particular relative handle straightening problem. It will be convenient to denote the pl. submanifold $h(\Delta^k \times R^i)$ by V^q and write $h : \Delta^k \times R^{n,i} \to V^{m,q}$ a map of pairs. Consider diagram 1:

$$
\begin{tikzcd}
\Delta^k \times 2I^{n,i} \arrow[rr, "H", "\text{id on } \partial \text{ and } I^k \times 2I^i"'] & & \Delta^k \times 2I^{n,i} \\
\Delta^k \times R^{n,i} \arrow[u, "\Gamma"'] \arrow[rr, "G", "\text{bounded; id. on } \partial \text{ and } \Delta^k \times R^i"'] \arrow[d, "e"] & & \Delta^k \times R^{n,i} \arrow[u, "\Gamma"] \arrow[d, "e"] \\
\Delta^k \times T^{n,i} \arrow[rr, "\text{id on } \partial \text{ and } \Delta^k \times T^i"] \arrow[dr, "\hat{g}"'] \arrow[dd, "p"'] & & \Delta^k \times T^{n,i} \arrow[dl, "{g' \text{ pl.} = \hat{g} \text{ on } \partial \text{ and } \Delta^k \times T^i}"] \\
& W^{m,q} \arrow[d, "p'"] & \\
\Delta^k \times T^{n,i} \arrow[r, "g"] & W^{m,q} & \\
\Delta^k \times T_0^{n,i} \arrow[u, hook] \arrow[r, "\text{id}"] \arrow[d, "\alpha"] & W_0^{m,q} \arrow[u, hook] \arrow[d, "h\alpha"] & \\
\Delta^k \times R^{n,i} \arrow[r, "h"] & V^{m,q} &
\end{tikzcd}
$$

$$
m = k+n \qquad q = i+n
$$

Diagram 1

All maps are pl. on boundaries and indicated submanifolds, and by relative collaring [6] we may also assume that all maps are pl. in a neighbourhood of the boundary. $T_0^{n,i} = T^{n,i}$ minus a disc pair; α is an immersion of $\Delta^k \times T_0^n$ in $\Delta^k \times R^n$ which respects boundary and immerses $\Delta^k \times T_0^i$ in $\Delta^k \times R$.; $W_0^{m,q}$ is $\Delta^k \times T_0^{n,i}$ with PL structure induced from $h\alpha$. All the maps on the left commute with a standard inclusion of $\Delta^k \times I^{n,i}$ in $\Delta^k \times T_0^{n,i}$. The construction of the diagram is exactly as in [21; pages 71–74] except for two points

a) *The construction of g*. Let $C \subset \Delta^k$ be an open pl. collar on $\partial\Delta^k$ defined so that $h|C \times R^n$ is pl. Now define

$$
U^{m,q} = \Delta^k \times T_0^{n,i} \cup C \times T^{n,i}
$$
$$
W_c^{m,q} = U^{m,q} \cup_{\text{'id'}} W_0^{m,q}
$$

Let $\imath : U^{m,q} \to W_c^{m,q}$ be the identification map (see figure 1). Next identify the one-point compactification of $U^{m,q}$ with $\Delta^k \times T^{n,i}$ by a homeomorphism which is pl. on U and the identity on $\Delta^k \times I^{n,i}$. Finally define g to be the one-pair compactification of $\imath$. We claim that $W^{m,q}$ has a pl. structure extending that of $W_c^{m,q}$ and so that $g|\Delta^k \times T^i$ is pl. Looking at the end of $W_c^{m,q}$ we see that it is enough to prove a relative form of the hauptvermutung for $S^{m-1} \times \boldsymbol{R}$, stated in proposition 1.3 below, and proved at the end of the section.

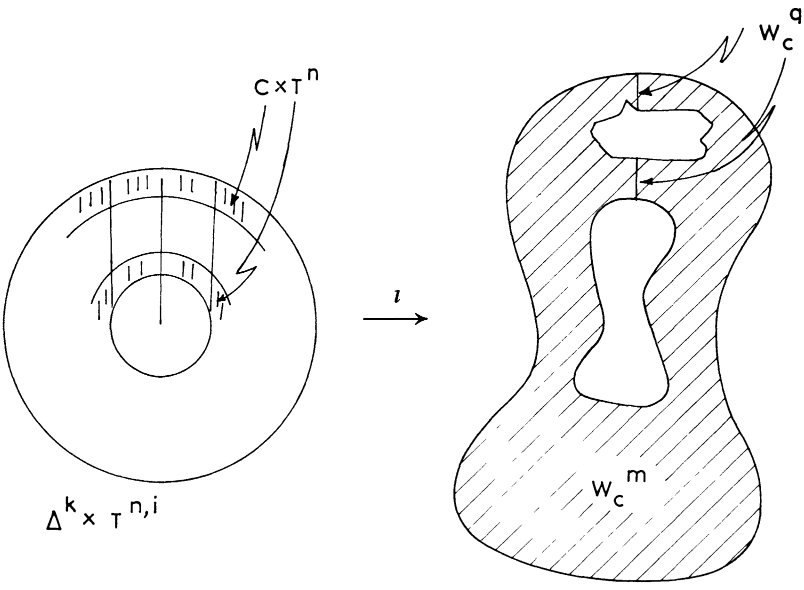

Figure 1

b) *The construction of g'.*

We need $g' = g$ on $\Delta^k \times T^i$ as well as on $\partial(\Delta^k \times T^n)$ as in [21]. When it is possible to find g' at all the extra condition can also be satisfied. This follows from a result of [32], stated in 1.4 below:

PROPOSITION 1.3. *Suppose (W^m, W^q) is a pl. manifold pair and that there is a homeomorphism $h : (W^m, W^q) \to (S^{m-1} \times R, S^{q-1} \times R)$ such that $h|W^q$ is pl. Then there exists a pl. homeomorphism $h' : W^m \to S^{m-1} \times R$, extending $h|W^q$, provided $m \geqq 5$ and $m-q \geqq 3$.*

PROPOSITION 1.4. *Suppose $h : Q^{m,q} \to W^{m,q}$ is a homeomorphism of pairs of compact pl. manifolds such that $h|\partial Q^m \cup Q^q$ is pl. Suppose further that h is homotopic rel ∂Q^m to a pl. homeomorphism, then h is homotopic rel $\partial Q^m \cup Q^q$ to a pl. homeomorphism provided $m \geqq 5$ and $m-q \geqq 3$.*

PROOF OF THEOREM 1.2. Let (h, V) be the relative problem considered above and suppose $n-i \geqq 3$ and $n+k \geqq 5$ and further that (h, V) is straightenable as an absolute problem. Then by 1.3 and 1.4 we can construct the complete relative main diagram for h. H is the identity on $\partial(\Delta^k \times 2I^n)$ and $\Delta^k \times 2I^i$ and hence is isotopic to the identity rel these subsets by an Alexander isotopy. Restricting to a neighbourhood of $\Delta^k \times \{0\}$ which is embedded in $\Delta^k \times R^n$ throughout the isotopy shows that h is straightenable as a relative problem. Now suppose (h, V) is unstraightenable as an absolute problem, it is therefore unstraightenable as a relative problem; but by adding any unstraightenablerelative problem we get a straightenable absolute (and hence relative) problem. Thus $H_k(n, i) = 0$ if $k \neq 3$, $k+n \geqq 5$, $n-i \geqq 3$ and 0 or $\boldsymbol{Z}_2$ if $k = 3$. It remains to show that $H_3(n, i) = \boldsymbol{Z}_2$ in this range.

Consider the commutative diagram (diagram 2):

$$\begin{tikzcd}
\vdots & & & & \\
H_3(5) \arrow[u] & & & & \\
H_3(4) \arrow[u, "\cong"'] & H_3(5,0) \arrow[ul] \arrow[r] & \cdots & & \\
H_3(3) \arrow[u, "\cong"'] & H_3(4,0) \arrow[ul] \arrow[u] \arrow[r] & H_3(5,1) \arrow[ul] \arrow[r] & {} & \\
\boldsymbol{Z}_2 \cong H_3(2) \arrow[u, "\cong"'] \arrow[r, dashed, "\alpha"'] & H_3(3,0) \arrow[ul] \arrow[u] \arrow[r] & H_3(4,1) \arrow[ul] \arrow[u] \arrow[r] & H_3(5,2) \arrow[ul] \arrow[r] & \cdots
\end{tikzcd}$$

Diagram 2

The indicated isomorphisms come from 1.1 and [21; theorem 12]. Horizontal maps are suspension and diagonal maps forgetful functions. They are homomorphisms by 1.1. We have shown that all the groups are 0 or $\boldsymbol{Z}_2$, we will construct a function α which makes the diagram commute, it then follows that all the groups are $\boldsymbol{Z}_2$ and the homomorphisms are isomorphisms.

DEFINITION OF α. We reverse the unwrapping construction [21; p. 79]. Let $h : \Delta^3 \times R^2 \to V$ represent an element of $H_3(2)$. Identify $\Delta^3 \times I^2 \subset \Delta^3 \times R^2$ with $\Delta^3 \times I^2 \times \{1\} \subset \partial(\Delta^3 \times I^3)$ and let $V_0 = h(\Delta^3 \times I^3)$. Glue $\partial\Delta^3 \times I^3$ to V_0 via h on $\partial\Delta^3 \times I^2 \times \{1\}$ this forms W_0, which has pl.

interior since h is pl. on this subset, and we have a homeomorphism $h' : \Delta^3 \times I^2 \times \{1\} \cup \partial\Delta^3 \times I^3 \to W_0$ which is pl. on $\partial\Delta^3 \times I^3$. Now $\mathrm{int}(W_0)$ is a contractible pl. manifold and hence R^5 (Stallings [42]). In $\mathrm{int}(W_0)$ choose a pl. ball B^5 sufficiently large to contain

$$h'(\Delta^3 \times \{0\} \times \{1\} \cup \partial\Delta^3 \times \{0\} \times [0, 1])$$

in its interior, and denote $C^5 = (h')^{-1}B^5 \subset \partial(\Delta^3 \times I^3)$. Now extend $h'|C^5$ to $g : B^6 = \Delta^3 \times I^3 \to B^5 \times I$ by two conical extensions. First extend over ∂B^6 using the fact that $\partial B^6 - \mathrm{int}\, C^5$ is a ball by the Schoenflies thecrem and second extend over B^6 using the standard cone structure on B^6. See figure 2. Observe that $g|\Delta^3 \times \{0\}$ is pl. since $h'|\partial\Delta^3 \times \{0\}$ was pl. Finally identify R^3 with $\mathrm{int}\, \varepsilon I^3$ and restrict g to $\Delta^3 \times \varepsilon I^3$ to complete the definition of $\alpha(h)$, ε being chosen so that $g|\partial\Delta^3 \times \varepsilon I^3$ is pl. It is easy to check (cf [21; p. 83]) that $\alpha(h)$ is equivalent to the suspension of h, as required.

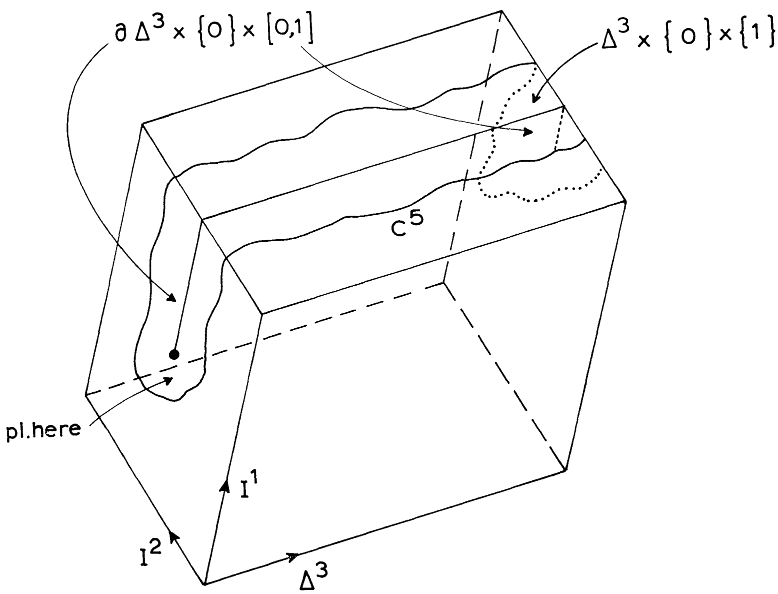

Figure 2

This completes the proof of 1.2. We note for future reference (§ 4) that we could have constructed g to be pl. on all of $\partial\Delta^3 \times I^3$ by the regular neighbourhood theorem, and also that the range of g is a pl. ball.

Proof of 1.3. *The case* $m \geqq 6$. Let ξ be a normal block bundle for $W^q \subset W^m$. We have ε the trivial normal block bundle for $S^{q-1} \subset S^{m-1}$ and we can homotope h rel W^q to a map $h'' : W^m \to S^{m-1} \times R$ so that

$h''|E(\xi)$ is a block homotopy equivalence of ξ with $\varepsilon \times R$ (see [37; § 3] for definition). This follows by an easy induction argument using homotopy extension and the fact that h has local degree 1. Then $h''|E(\xi)$ determines a map $g : W^q \to \tilde{G}_r/\tilde{PL}_r$, $r = m-q$, which we claim is nulhomotopic. Consider $s \circ g : W^q \to G/PL$, the suspension of g, this factors via Top/PL by a standard argument since we started with a homeomorphism. $s \circ g$ is therefore nulhomotopic since the natural map of Top/PL in G/PL is zero on homotopy, by [23] and the fact that $\pi_3(G/PL) = 0$ (see also Wall [43]). Hence g is nulhomotopic by stability of $\tilde{G}_r/\tilde{PL}_r$ (see [35; 1.10]).

It follows that h'' is homotopic rel W^q to h''' which restricts to a block bundle isomorphism of ξ with $\varepsilon \times R$. This provides a pl. product structure on $E(\xi)$ which extends to all of W^m by Siebenmann's relative collaring theorem [41]. We now have a pl. isomorphism

$$(W^m, W^q) \cong (M^{m-1}, S^{q-1}) \times R$$

which extends $h|W^q$. But M^{m-1} is a pl. sphere by the Poincaré theorem and the pair (M^{m-1}, S^{q-1}) is unknotted by Zeeman [45]. The construction of the desired h' is now easy.

The case $m = 5$. The case $q = 1$ presents little difficulty so we concentrate on the case $q = 2$. It is easy to verify that any pl. self-homeomorphism of $S^1 \times R$ extends to $S^4 \times R$ if $q \leqq 2$ and it suffices to find some pl. homeomorphism of pairs $W^{5,2} \cong S^{4,1} \times R$. By Wall [43] W^5 is pl. homeomorphic with $S^4 \times R$ and we assume that $W^5 = S^4 \times R$; we have to unknot $M^2 \equiv h^{-1}(S^1 \times R)$ in $S^4 \times R$. We show how to isotope M to meet each sphere $S^4 \times \{n\}$, $n \in \boldsymbol{Z}$, in an essential circle and the result follows from the 2-dimensional pl. annulus theorem and unknotting $S^1 \times I$ in $S^4 \times I$ (Hudson and Lickorish' concordence extension theorem [19]). The method for each S^4 is the same. By transversality $S^4 \cap M$ can be taken to be a finite number of circles. We show how to pipe two neighboring circles together and the result follows by induction. Choose points p, q on each circle and arcs α, β in M and S^4 joining them and not meeting other intersections. Then the circle $\alpha \cup \beta$ spans a 2-disc D which meets M and S^4 only in $\alpha \cup \beta$. A regular neighbourhood of D is a 5-ball. B^5 meeting S^4 and M in unknotted subdiscs B^4, B^2 which in turn meet in 2 arcs *ab*, *cd* say. It is a trivial matter to isotope B^2 in B^5 rel boundry so as to replace *ab*, *cd* by arcs *ac*, *bd* (or *ad*, *bc*) and this has the effect of piping the two original circles together.

2. The stability theorems

Before proving 1.1 it is convenient to define a new set $H'_k(n, i)$. A representative is a homeomorphism $h : R^k \times R^n \to V$ where V is a pl. manifold, $h|cl(R^k - \Delta^k) \times R^n$ is pl. and $h|\Delta^k \times R^i$ is pl. locally flat. $(h, V) \approx (g, W)$ if there is a pl. homeomorphism $q : V \to W$ defined in a neighbourhood of $\Delta^k \times \{0\}$ such that

$$\begin{tikzcd} R^k \times R^n \arrow[rr, "h"] \arrow[dr, "g"'] & & V \arrow[dl, "q"] \\ & W & \end{tikzcd}$$

commutes up to a topological isotopy which is pl. on $cl(R^k - \Delta^k) \times R^n \cup \Delta^k \times R^i$ and defined in a neighbourhood of $\Delta^k \times \{0\} \times I$. $(h, V) \sim (g, W)$ if $(h, V) = (h_1, V_1) \approx (h_2, V_2) \approx \cdots \approx (h_1, V_1) = (g, W)$. The set of equivalence classes forms $H'_k(n, i)$; $H'_k(n)$ is defined similarly. There are obvious surjections $\psi_1 : H_k(n, i) \to H'_k(n, i)$ and $\psi_2 : H_k(n) \to H'_k(n)$ defined by 'adding a collar'.

PROPOSITION 2.1. *ψ_1 and ψ_2 are bijections.*

PROOF. We have to show injectivity. Suppose $(h, V) \approx (g, W)$; let q be as above and s_t the isotopy of qh to g; so we have $s_0 = qh$ and $s_1 = g$. Now by two applications of the pl. covering isotopy theorem we can find a pl. ambient isotopy s'_t of W so that $s'_0 = \text{id}$ and $s'_t qh|L = s_t|L$ where $L = \partial\Delta^k \times R^n \cup \Delta^k \times R^i$. Define $\bar{q} = s'_1 q$ and $\bar{s}_t = s'_1 \circ (s'_t)^{-1} \circ s_t$ then $\bar{q}$ is pl. and $\bar{s}_t$ is an isotopy between $\bar{q}h$ and g which is fixed on L. This shows that the restriction of (h, V) and (g, W) to $\Delta^k \times R^n$ are equivalent in $H_k(n, i)$, as required.

Now define $I_k(n, i)$ to be the set of regular homotopy classes of orientation preserving immersions $h : R^k \times R^n \to R^{k+n}$ such that h is a pl. immersion of $R^k \times R^i$ and of a neighbourhood of $cl(R^k - \Delta^k) \times R^n$. The regular homotopies are via such immersions but defined only in a neighbourhood of $(\Delta^k \times \{0\}) \times I$. Similarly $I_k(n)$ is defined by ignoring the condition of $\Delta^k \times R^i$. Now h induces a pl. manifold structure on $R^k \times R^n$; denote this pl. manifold by $(R^k \times R^n)_h$. We then get a relative handle problem h_t, with $h_0 = h$. It is easy to see that ('id', $(R^k \times R^n)_h$) $\approx$ ('id', $(R^k \times R^n)_{h_\varepsilon}$) for small ε and hence, by compactness of I, the ('id', $(R^k \times R^n)_h$) $\sim$ ('id', $(R^k \times R^n)_{h_1}$). We therefore have well-defined functions $\varphi_1 : I_k(n, i) \to H_k(n, i)$ and $\varphi_2 : I_k(n) \to H'_k(n)$.

PROPOSITION 2.2. *φ_1 and φ_2 are bijections.*

PROOF. *Surjectivity*. Let (h, V) represent an element of $H'_k(n, i)$ then by a collaring argument we may suppose h is pl. in a neighbourhood of $\partial\Delta^k \times R^n$. Now V is a contractible pl. manifold and therefore pl. immerses in R^{k+n} by an immersion α such that αh is orientation preserving.

Injectivity. Suppose ('id', $(R^k \times R^n)_{h_0}$) $\approx$ ('id', $(R^k \times R^n)_{h_1}$) we have to construct a regular homotopy between h_0 and h_1. Let q, s_t be given by the definition of $\approx$ (notation as in 2.1). We construct the regular homotopy in two stages.

Stage 1. By collaring s_t may be taken to be pl. in a neighbourhood of $\partial \Delta \times R^n$ then h, s_t defines an allowable regular homotopy between $h_1 \circ q$ and h_1.

Stage 2. h_0 and $h_1 \circ q$ are both orientation-preserving pl. immersions of $(R^k \times R^n)_{h_0}$ in R^{k+n} and are therefore regularly homotopic since both manifolds are contractible.

PROOF OF 1.1. We have functions

$$d_1 : I_k(n, i) \to \pi_k(Top_{m,q}, PL_{m,q})$$
$$d_2 : I_k(n) \to \pi_k(Top_m, PL_m)$$

defined by restricting the differential of an immersion to $\Delta^k \times \{0\}$ and it follows from the pl. and topological immersion theorems that these are bijections. The result now follows using 2.1 and 2.2 and the commutativity of ψ_i, φ_i and d_i with suspension and forgetful functions.

THEOREM 2.3. *Suppose $r \leqq 2$ or $k+r \geqq 5$ then inclusion induces an isomorphism.*

$$i_* : \pi_i(Top_{r+k,k}) \to \pi_i(Top_r) \qquad \text{for } i \leqq k$$

PROOF. Consider the diagram

$$\begin{array}{ccccccccc}
\pi_{i+1}(\;,\;) & \to & \pi_i(PL_{r+k,k}) & \to & \pi_i(Top_{r+k,k}) & \to & \pi_i(Top_{r+k,k}, PL_{r+k,k}) & \to & \pi_{i-1}(PL_{r+k,k}) \\
\downarrow i_1 & & \downarrow i_2 & & \downarrow i_* & & \downarrow i_3 & & \downarrow i_4 \\
\pi_{i+1}(\;,\;) & \to & \pi_i(\boldsymbol{PL}_r) & \to & \pi_i(\boldsymbol{Top}_r) & \to & \pi_i(\boldsymbol{Top}_r, \boldsymbol{PL}_r) & \to & \pi_{i-1}(\boldsymbol{PL}_r)
\end{array}$$

i_2 and i_4 are isomorphisms for $i \leqq k$ by [9; 8.5] see also [35; 5.4]. i_3 is an isomorphism by 1.1 and 1.2. To apply the 5-lemma we need i_1 epimorphic. This is true for $i \neq 2$ since $\pi_{i+1}(\boldsymbol{Top}_r, \boldsymbol{PL}_r) = 0$ by 1.1 and 1.2. For $i = 2$ however we have $\pi_2(PL_{r+k,k}) \cong \pi_2(\boldsymbol{PL}_r) \cong \pi_2(\tilde{PL}_r) \cong \pi_2(0_r) \cong 0$ (see [35; 5.5] and [8; 6.6]) so i_* is an isomorphism by an easier argument.

THEOREM 2.4. *Suppose $r \geqq 3$. Then $\boldsymbol{Top}_r/\boldsymbol{PL}_r \to Top/PL$ is a homotopy equivalence.*

PROOF. This is immediate from 1.1 and 1.2.

COROLLARY 2.5. *Suppose $r \geqq 3$. Then $\boldsymbol{G}_r/\boldsymbol{Top}_r \to G/Top$ is a homotopy* *equivalence. Here* $\mathbf{G}_r$ *is the homotopy analogue* $\mathbf{Top}_r$ (*cf* [35; § 0]), *which has the homotopy type of the monoid* G_r *of self-homotopy-equivalences of* S^{r-1}.

PROOF. Use 2.4 and [35; 1.10].

3. Classification of *r*-neighbourhoods of manifolds

For simplicity we work first with an unbounded manifold M^n. The bounded case introduces technical difficulties which will be dealt with at the end of the section. Let $i : M^n \to N^{n+r}$ be a locally flat embedding in the unbounded manifold N. The pair (i, N) is called an *r-neighbourhood* of M. Two such (i, N), (i', N') are equivalent if there is an embedding $h : N \to N'$ defined in a neighbourhood of $i(M)$ and such that $h \circ i = i'$. The set of equivalence classes is denoted $\mathcal{N}_r(M)$ and called the set of germs of r-neighbourhoods of M.

To the r-neighbourhood (i, N) we associate the microbundle pair $(i^*\tau(N), \tau(M))$. The isomorphism class rel $\tau(M)$ of this pair depends only on the germ of (i, N) and this gives us a well-defined function

$$K : \mathcal{N}_r(M) \to [M, BTop^n_{n+r}]_{\tau(M)}$$

where $[M, BTop^n_{n+r}]_{\tau(M)}$ denotes homotopy classes of sections of the fibration over M induced from the fibration

$$BTop_{n+q,n} \to BTop^n_{n+q} \to BTop_n$$

by the classifying map $\tau(M) : M \to BTop_n$.

PROPOSITION 3.1. *K is a bijection.*

PROOF. *K is injective*: Suppose $K(i, N) = K(i', N')$ then there is an isomorphism $h : (i^*\tau(N), \tau(M)) \to (i'^*\tau(N'), \tau(M))$ of microbundle pairs such that $h|\tau(M) = \mathrm{id}$. By the immersion theorem there is an immersion $h' : (N, M) \to (N', M)$ defined in a neighbourhood of M such that $h'|M = i'$. This last condition implies that h is an embedding in some smaller neighbourhood and thus $(i, N) \sim (i', N')$.

K is surjective: Suppose given a microbundle pair $(\xi^{n+r}, \tau(M))$. We have to construct an $(n+r)$-manifold M with $M \subset N$ and an isomorphism $(\tau(N)|M, \tau(M)) \to (\xi^{n+r}, \tau(M))$ rel $\tau(M)$. We will construct N inductively over an open cover of M using the immersion theorem to match overlaps:

Let $\{U_i\}$, $\{U_i'\}$, $i = 1, 2 \cdots$, be countable locally finite open covers of M such that $\bar{U}_i \subset U_i'$ and $(\xi, \tau)|U_i'$ is trivial for all i. Let

$$U = \bigcup \{U_i; i = 1, \cdots, p-1\}$$

and suppose inductively that there is an $(n+r)$-manifold $V \supset U$ and an isomorphism $h : (\xi|U, \tau(U)) \to (\tau(V)|U, \tau(U))$ which restricts to id. on $\tau(U)$. Let $t : U'_p \times (R^{n+r}, R^n) \to (\xi|U'_p, \tau(U'_p))$ be a trivialisation, and let $Z = U \cap U'_p$. We can define a representation $\varphi : \tau(Z \times R^r)|Z \to \tau(V)|Z$ by the following diagram, in which the maps are suitably restricted:

$$\begin{tikzcd}
\tau(Z \times R^r)|Z = \tau(Z) \times R^r \arrow[r, dashed, "\varphi"] \arrow[d, "(t^{-1}|\tau(Z)) \times \mathrm{id.}"] & \tau(V)|Z \arrow[d, "h"] \\
Z \times R^n \times R^n = Z \times R^{r+r} \arrow[r, "t"'] & \xi|Z
\end{tikzcd}$$

and it is trivially checked that $\varphi|\tau(Z) = \mathrm{id}$. Thus by the immersion theorem φ is homotopic rel $\tau(Z)$ to the differential of an immersion $\varphi' : N(Z) \to V$, $\varphi'|Z = \mathrm{id}$, where $N(Z)$ is a neighbourhood of Z in $Z \times R^r$. Now φ' is an embedding in $N(Z) \cap N(U_p)$ for some neighbourhood $N(U_p)$ of U_p in $U'_p \times R_r$. Now define $V' = V \cup N(U_p)$ identified by $\varphi'|$ and $U' = \cup \{U_i; i = 1, \cdots, p\}$ then it is easily verified that $V' \supset U'$ has the inductive property.

Now let (i, N) be an r-neighbourhood of M and choose an inverse v to $\tau(M)$ (that is to say $\tau(M) \oplus v$ has a preferred trivialization). Then we can identify the pair $(i^*\tau(N) \oplus v, \tau(M) \oplus v)$ with $(i^*\tau(N) \oplus v, \varepsilon^N)$ using this trivialisation. The last pair determines a classifying map

$$c(i, N) : M \to BTop_r$$

Now the isomorphism class rel ε^N of the above pair depends only on the germ of (i, N) and thus we have a well defined function

$$c : \mathscr{R}_r(M) \to [M, BTop_r]$$

THEOREM 3.2. *Suppose $n+r \geqq 5$ or $r \leqq 2$, then c is a bijection.*

PROOF. Consider the following diagram.

$$\begin{tikzcd}
 & BTop_{n+r,n} \arrow[r, "s_1"] \arrow[d] & BTop_r \arrow[d, "i"] \\
 & BTop^n_{n+r} \arrow[r, "s_2"] \arrow[d, "p_1"] & BTop'_r \arrow[d, "p_2"] \arrow[u, dashed, bend left, "t"] \\
M \arrow[r, "\tau(M)"'] \arrow[ur, dashed, "K(i, N)"] & BTop_n \arrow[r, "s_3"] & BTop
\end{tikzcd}$$

Here s_i are suspensions. The vertical sequences are fibrations. We now use the main stability theorem 2.3 to deduce that s_1 induces isomorphisms on π_k for $k \leqq n+1$ and hence by 3.1 that $\mathscr{N}_r(M)$ is in 1–1 correspondence with lifts of $s_3 \circ \tau(M)$ over p_2. But p_2 is fibre homotopy trivial; this is seen as follows. There is a retraction $t : BTop'_r \to BTop_r$ constructed inductively over skeletons by means of inverses $\zeta^{(k)}$ for $p_2^*(\gamma^{(k)})|BTop'^{(k)}_r$ where $\gamma^{(k)}$ is the universal bundle over $BTop^{(k)} t^{(k)}$ is then the classifying map for the pair $(\xi^{(k)} \oplus \zeta^{(k)}, \varepsilon)$ where $(\xi^{(k)}, p^*\gamma^{(k)})$ is the universal bundle over $BTop'_r$.

It is easily checked that $t \circ i \simeq \mathrm{id}$. Thus p_2 is fibre homotopy trivial by [3] and it follows that $\mathscr{N}_r(M)$ is in 1–1 correspondence with $[M, BTop_r]$ by the correspondence $(i, N) \to t \circ s_2 \circ K(i, N)$. It remains to observe that this is homotopic to $c(i, N)$ by stable uniqueness of inverses.

The bounded case

Let M be bounded and $M_+ = M \cup \{\text{open collar}\}$ and recall that $\tau(M) = \tau(M_+)|M$. We then have an isomorphism $\tau(\partial M) \oplus \varepsilon^1 \to \tau(M)|\partial M$ given by choosing an inward collar, and a commutative diagram

$$\begin{tikzcd}
\partial M \arrow[r, phantom, "\subset"] \arrow[d, "\tau(\partial M)"] & M \arrow[d, "\tau(M)"] \\
BTop_{n-1} \arrow[r, "s'_3"] & BTop_n
\end{tikzcd}$$

An *r-neighbourhood* of M is a pair (i, N) where N is a bounded $(n+r)$-manifold, $i : M \to N$ an embedding and $i^{-1}(\partial N) = \partial M$. Two such are *equivalent* if there is an embedding $h : N \to N'$ defined in a neighbourhood of $i(M)$ such that $h^{-1}(\partial N') = \partial N$ and $h\, i = i'$. The set of equivalence classes is again denoted $\mathscr{N}_r(M)$. To the pair (i, N) we associate the bundle diagram

$$\begin{array}{ccc}
\tau(\partial M) & \subset & \tau(M) \\
\cap & & \cap \\
i^*\tau(\partial N) & \subset & i^*\tau(N)
\end{array}$$

which depends up to isomorphism rel $\tau(M)$ only on the germ of (i, N). This gives a function K from $\mathscr{N}_r(M)$ to the set of homotopy classes of commuting diagrams:

$$\begin{tikzcd}
& BTop^{n-1}_{n+r-1} \arrow[rr, "s'_2"] \arrow[dd] & & BTop^n_{n+r} \arrow[dd] \\
\partial M \arrow[ur, dashed] \arrow[rr] \arrow[dr, "\tau(\partial M)"'] & & M \arrow[ur, dashed] \arrow[dr, "\tau(M)"] & \\
& BTop_{n-1} \arrow[rr, "s'_3"'] & & BTop_n
\end{tikzcd}$$

Proposition 3.1b. *K is a bijection.*

Proof. Follow the proof of the unbounded case but split into two cases. Deal first with the boundary then relativise the bounded argument to deal with the interior keeping boundaries fixed. Details are left to the reader.

Now suppose (j, L) is a fixed r-neighbourhood of ∂M and consider r-neighbourhoods which extend (j, L) under the equivalence of germ of homeomorphism rel L. Call this set $\mathcal{N}_r(M \operatorname{rel} L)$ then the proof of 3.1b shows

Proposition 3.1c. *$\mathcal{N}_r(M \operatorname{rel} L)$ is in 1–1 correspondence with lifts of $\tau(M)$ in $BTop^n_{n+r}$ which extend $s'_2 \circ K(j, L)$.*

We now stabilise 3.1b *and* c *along the lines of* 3.2. *As before choose a fixed inverse* v *to* $\tau(M)$ *then we get functions*

$$c_1 : \mathcal{N}_r(M) \to [M, BTop_r]$$

$$c_2 : \mathcal{N}_r(M, \operatorname{rel} L) \to [M, BTop_r]_L$$

where the last set denotes homotopy classes of maps which extend $c(j, L)$ on ∂M.

Theorem 3.2b. *c_1 is a bijection provided $n+r \geqq 6$ or $r \leqq 2$.*

Theorem 3.2c. *c_2 is a bijection provided $n+r \geqq 5$ or $r \leqq 2$.*

Proof. Consider the diagram

$$\begin{tikzcd}
BTop_{n+r,n-1} \arrow[r, "s'_1"] \arrow[d] & BTop_{n+r,n} \arrow[r, "s_1"] \arrow[d] & BTop_r \arrow[d, shift left, "i"] \\
BTop^{n-1}_{n+r-1} \arrow[r, "s'_2"] \arrow[d] & BTop^n_{n+r} \arrow[r, "s_2"] \arrow[d] & BTop'_r \arrow[u, shift left, bend left, "t"] \arrow[d] \\
BTop_{n-1} \arrow[r, "s'_3"'] & BTop_n \arrow[r, "s_3"] & BTop \\
\partial M \arrow[uu, dashed, bend left, "K(j,L)"] \arrow[u, "\tau(\partial M)"'] \arrow[r, phantom, "\subset"] & M \arrow[uu, dashed, "K(i,N)"] \arrow[u, "\tau(M)"'] &
\end{tikzcd}$$

The results follows from 3.1b or 3.1c the triviality of the right-hand fibration and the stability properties of s'_1 and s_1 (2.3). Note that we get one better dimension for 3.2c than 3.2b since we are considering a fixed map $K(j, L)$ and thus do not need stability for s'_1.

Whitney sums, product theorem and induced neighbourhoods

We deduce three simple consequences of the main theorem (3.2).

Suppose $(i_1, N_1)(i_2, N_2)$ are neighbourhoods of M of codimension r_1, r_2 respectively. We can form their *Whitney sum* (i_3, N_3) of codimension r_1+r_2 uniquely up to equivalence provided $n+r_1+r_2 \geqq 5$ (or 6 if $\partial M \neq \emptyset$): form the bundle pair $(i^*\tau(N_1) \oplus \nu \oplus i^*(N_2) \oplus \nu, \varepsilon \oplus \varepsilon)$ and then define (i_3, N_3) to be a member of the class classified by this pair.

Secondly, suppose $M \cong M_1 \times I$ and (i, N) an r-neighbourhood of M. Then provided $n+r \geqq 6$ (or 7 if $\partial M_1 \neq \emptyset$) we can find a neighbourhood (i_1, N_1) of M_1 such that (i, N) is equivalent to $(i_1 + \mathrm{id}, N_1 \times I)$: Define (i_1, N_1) to be a neighbourhood classified by $c(i, N)|M_1$ and then the result follows since $c(i, N)$ and $c(i_1 \times \mathrm{id}; N_1 \times I)$ agree on M_1.

Thirdly, suppose (i, N) is an r-neighbourhood of M and $f: W \to M$ a map of manifolds let $\omega = \dim W$ then provided only $r+\omega \geqq 5$ ($\geqq 6$ if $\partial W \neq \emptyset$) we can define the *induced* neighbourhood $f^*(i, N)$ of ω uniquely up to equivalence as the neighbourhood classified by $c(i, N) \circ f$.

4. Normal block bundles

We deal first with micro block bundles. Analogous results for open and closed block ubndles will be deduced afterwards. $T\tilde{o}p_r(\mu)$ is the topological analogue of $\tilde{PL}_r(\mu)$ [33], a typical k-simplex is a germ of block and zero-preserving homeomorphisms $\sigma : \Delta^k \times R^r$ W defined in a neighbourhood of $\Delta^k \times \{0\}$. *An r-microblock bundle* will mean a $T\tilde{o}p_r(\mu)$-block bundle in the sense of [37; § 1]. If K is the base of ξ then we write ξ/K and identify $|K|$ with the zero section in $E(\xi)$. Isomorphism classes of block bundles are classified by homotopy classes of maps in $BT\tilde{o}p_r(\mu)$ (for more detail see [37; § 1]).

We will define a map

$$\chi : BT\tilde{o}p_r(\mu) \to BTop_r$$

inductively over skeleta. We need

LEMMA 4.1. *$\pi_k(T\tilde{o}p_r(\mu))$ is countable if $r+k \geqq 5$ or $r \leqq 2$.*

PROOF. There is a surjection $H_k(r, 0) \to \pi_k(T\tilde{o}p_r(\mu), \tilde{PL}_r(\mu))$ (see 4.5 below) and the result follows from [32; 2.6] and 1.2.

Assume $r \geqq 5$ or $\leqq 2$ and use 4.1 to replace $BT\tilde{o}p_r(\mu)^{(k)}$ by a parallelized open manifold T_k by imbedding a locally finite simplicial complex (cf [33; § 2]) in some large dimensional Euclidiean space and consdering the interior of a regular neighbourhood. Then the pair (id, $E(\gamma^r|T_k)$ is an r-neighbourhood of T_k where γ^r is the universal bundle, and thus determines a map $\chi^{(k)} : T_k \to BTop_r$. We observe that by choosing T_k to be a parallelized open manifold we can take $\chi^{(k)}$ to classify the pair

$(\tau(E(\gamma))), \tau(T_k)$ without adding an inverse. $\chi^{(k)}$ is defined up to homotopy independently of the choices made and hence we can assume $\chi^{(k+1)}$ extends $\chi^{(k)}$ and thus define a limit map χ. There is a similarly defined map for $r+n \geqq 5$ or $r \leqq 2$.

$$\chi' : BT\tilde{o}p_{r+n,n}(\mu) \to BTop_r$$

Finally if $r = 3, 4$ define χ to be the composition.

$$BT\tilde{o}p_r \to BT\tilde{o}p_{5,5-r} \xrightarrow{\chi'} BTop_r$$

PROPOSITION 4.2. (i) *Let* $|K| = M$ *be a triangulated topological manifold and let* ξ^r/K *be a block bundle. Let* $l : K \to BT\tilde{o}p_r$ *classify* ξ *then* $\chi \circ l \simeq c(\mathrm{id}, E(\xi))$.

(ii) *The diagram*

$$\begin{tikzcd} BTop_{r+n,n} \arrow[r] \arrow[d] & BTop_r \arrow[d] \\ BT\tilde{o}p_{r+n,n}(\mu) \arrow[r, "s"'] \arrow[ur, "\chi'"] & BT\tilde{o}p_r(\mu) \end{tikzcd}$$

commutes up to homotopy.

(iii) $BTop_r \to BT\tilde{o}p_r(\mu)$ *is a homotopy equivalence.*

PROOF. For (i) observe that we can assume $l: K \to T_k$ is an embedding and ν a normal microbundle on this embedding, the result is then clear. This proves the result for $r \neq 3, 4$ but in these cases note that s determines an isomorphism of $(\tau(E(\xi)), \tau(M))$ with $(\tau(E(\xi)) \oplus \varepsilon, \tau(M) \oplus \varepsilon)$ and the result follows on applying the above argument to χ'. (ii) is easily checked from definitions and (iii) follows at once from stabilizing (ii).

REMARK. K was not assumed to be a combinational manifold. This remark holds throughout the section.

Now let $M^n \subset Q^{n+r}$ be a proper locally flat submanifold (i.e. $\partial Q \cap M = \partial M$) and $|K| = M$ a triangulation and denote $L \subset K$ the subcomplex underlying ∂M.

A *normal micro-block bundle* on M in Q is a micro-block bundle ξ^r/K with $E(\xi) \subset Q$ and $E(\xi) \cap \partial Q = E(\xi|L)$. These conditions imply that $E(\xi)$ is a topological neighbourhood of M in Q. Normal bundles ξ, η are *isotopic* if there is an isotopy of $E(\xi)$ in Q through normal bundles starting with the identity and ending with an isomorphism of ξ with η.

For simplicity now assume $\partial M = \emptyset$. Analogous results for the bounded case will be stated below. An *almost* normal micro-block bundle on M in Q is a pair (ξ, f) where ξ^r/K is a micro-block bundle and $f: (\tau(E(\xi))|M, \tau(M)) \to (\tau(Q)|M, \tau(M))$ is an isomorphism of pairs with $f|\tau(M) = \mathrm{id}$.

Two such are *isotopic* if there is a bundle isomorphism $g : E(\xi_1) \to E(\xi_2)$ such that

$$\begin{tikzcd} \tau(E(\xi_1))|M \arrow[d, "dg|"'] \arrow[dr, "f_1"] & \\ \tau(E(\xi_2))|M \arrow[r, "f_2"'] & \tau(Q)|M \end{tikzcd}$$

commutes up to homotopy of bundle maps rel $\tau(M)$.

PROPOSITION 4.3. *Isotopy classes of normal micro block bundles on M in Q correspond bijectively with isotopy classes of almost normal bundles.*

PROOF. By the immersion theorem an almost normal bundle gives an immersed normal bundle which can be replaced by an embedded one by 'radially' shrinking into a small neighbourhood of M. Similar considerations apply to homotopies.

THEOREM 4.4. *Suppose $n+r \geqq 5$ or $r \leqq 2$ then isotopy classes of normal micro block bundles on M in Q correspond bijectively with lifts of $c(\mathrm{id}, Q)$ over χ.*

REMARK. Since χ is not a fibration we need to replace it by one to make sense of 'lifting' over χ. This amounts to considering *homotopylifts* (i.e. a map $f : M \to BT\tilde{o}p_r$ together with a mohotopy F_t of $\chi \circ f$ to c) and homotopy classes of such lifts.

PROOF. By 4.3 and stability (2.3), isotopy classes of normal bundles correspond bijectively with stable isotopy classes of stable almost normal bundles. We now examine the proof of 4.2 (i) in greater detail. Assume $r \neq 3, 4$ and fix a sufficiently large skeleton T_k of $BT\tilde{o}p_r$, as in the definition of χ, and assume that T_k is an open subset of Euclidean space of sufficiently large dimension for M to unknot and to have normal bundle ν say. Next choose a fixed isomorphism of the pair $(\tau(Q)|M \oplus \nu, \varepsilon)$ with $c^*(\gamma, \varepsilon)$ where the latter is the classifying pair. Now a lift of $c(\mathrm{id}, Q)$ over χ gives a block bundle ξ^r/K and a microbundle pair $\eta/K \times I$ such that $\eta|K \times \{0\}$ has an expliict isomorphism with $(\tau(E(\xi))|M \oplus \nu, \varepsilon)$ and $\eta|K \times \{1\}$ one with $(\tau(Q)|M \oplus \nu, \varepsilon)$ both restricting to the identity on ε. Then by the product theorem for microbundle pairs η is a product and we have an isomorphism of $(\tau(E(\xi))|M \oplus \nu, \varepsilon)$ with $(\tau(Q)|M \oplus \nu, \varepsilon)$ determined up to bundle homotopy rel ε. I.e. a unique stable isotopy of almost normal block bundles. Similarly a homotopy over $c(\mathrm{id}, \theta)$ gives a bundle $\zeta/K \oplus I$ and a bundle map $q : \tau(E(\zeta))|M \times I \oplus \nu \times I \to \tau(Q)|M \oplus \nu$ which is the natural projection on trivial subbundles. To convert this into the required homotopy of abstract bundles we need a product theorem for block bundles (see [37; 2.4]). There is a homeomorphism $h : E(\zeta | K \times \{0\}) \times I \to E(\zeta)$ such that $h = \text{id}$ on $\{0\}-\text{end} \cup K \times I$ and $h|\{1\}-\text{end}$ is an isomorphism g say. The $q \circ dh$ is the homotopy of bundle maps which makes the diagram above 4.3 homotopy commute. Finally the cases $r = 3, 4$ are dealt with by applying the above argument to χ' exactly as in the proof of 4.2 (i).

Bounded case. We state the versions of 4.4 for bounded manifolds. The proofs are by relativising the absolute case as in second half of § 3. Details will be left to the reader.

Theorem 4.4b. *Suppose $\partial M = \emptyset$ and $r+n \geqq 6$ or $r \leqq 2$ then isotopy classes of normal micro-block bundles on M in Q correspond bijectively with lifts of c (id, Q) over χ.*

Theorem 4.4c. *Suppose η/L is a given normal micro block bundle on ∂M in ∂Q and $n+r \geqq 5$ or $r \leqq 2$. Then isotopy classes* rel $E(\eta)$ *of normal bundles on M in Q which extend η corresponds bijectively with lifts of $c(\text{id}, Q)$ over χ which extend the classifying map of η.*

In order to apply theorem 4.4 we need to examine the homotopy properties of χ. By 4.2 parts (ii) and (iii) this is equivalent to examining the properties of $s : BT\tilde{o}p_r(\mu) \to BT\tilde{o}p_r(\mu)$. As usual we reduce the analysis to PL results:

We refer to [36] for a general definition of the homotopy groups of a pair of Δ-set. We may regard the group $\pi_k(T\tilde{o}p_{n,i}(\mu), \tilde{PL}_{n,i}(\mu))$ as the set of concordence classes of germs of homeomorphisms $\sigma : \Delta^k \times R^n \hookleftarrow$ defined in a neighbourhood of $\Delta^k \times \{0\}$ such that $\sigma|\Delta^k \times R. = \text{id}$ and $h|\partial\Delta^k \times R^n$ is pl. By considering isotopy classes, rather than concordance classes, we get a set $\pi_k^{\text{iso}}(T\tilde{o}p_{n,i}(\mu), PL_{n,i}(\mu))$.

Proposition 4.5. *The natural map*

$$\varphi : \pi_k^{\text{iso}}(T\tilde{o}p_{n,i}(\mu), \tilde{PL}_{n,i}(\mu)) \to H_k(n, i)$$

is a bijection.

Proof. Let (h, V) represent an element of $H_k(n, i)$. Extend $h|\Delta^k \times R^i$ to a proper immersion $h_1 : \Delta^k \times R^n \to V$ by the pl. immersion theorem. Now H_1 embeds a sufficiently small neighbourhood of $\Delta^k \times \{0\}$ and $(h, V) \sim (h_1^{-1} \circ h, I^k \times R^n)$. This shows that φ is onto. A similar argument shows that φ is injective.

Now by 4.5 and 1.1 the differential

$$d : \pi_k^{\text{iso}}(T\tilde{o}p_{n,i}(\mu), \tilde{PL}_{n,i}(\mu)) \to \pi_k(Top_{m,q}, PL_{m,q})$$

is a bijection, where $m = k+n$ and $q = k+i$. We can define a homomorphism $d^1 : \pi_k(T\tilde{o}p_{n,i}(\mu), \tilde{PL}_{n,i}(\mu)) \to \pi_k(Top_{m+1,q+1}, PL_{m+1,q+1})$ by observing that the differential of a concordiance defines a homotopy between the suspensions of the differentials at the ends. We then have a commutative diagram

$$\begin{array}{ccc} \pi_k^{\text{iso}}(T\tilde{o}p_{n,i}(\mu), \tilde{PL}_{n,i}(\mu)) & \xrightarrow{i_1} & \pi_k(T\tilde{o}p_{n,i}(\mu), \tilde{PL}_{n,i}(\mu)) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ \pi_k(Top_{m,q}, PL_{m,q}) & \xrightarrow{i_2} & \pi_k(Top_{m+1,q+1}, PL_{m+1,q+1}) \end{array}$$

where i_1 and i_2 are the natural maps.

PROPOSITION 4.6. *All the maps in the above diagram are bijections if* $k+n \geqq 5$ *or* $n-1 \leqq 2$.

PROOF. d is always a bijection as observed above and i_1 is trivially always onto. Now i_2 is a bijection in the range of dimensions considered by 1.1 and 1.2. The result follows by commutativity.

COROLLARY 4.7. *The suspensions*

$$s_1 : \pi_k(T\tilde{o}p_{n,i}(\mu), PL_{n,i}(\mu)) \to \pi_k(T\tilde{o}p_{n+1,i+1}(\mu), \tilde{PL}_{n+1,i+1}(\mu))$$
$$s_2 : \pi_k(T\tilde{o}p_{n,i}(\mu)) \to \pi_k(T\tilde{o}p_{n+1,i+1}(\mu))$$

are isomorphisms provided $n+k \geqq 5$ *or* $n-i \leqq 2$.

PROOF. The diagram above 4.6 commutes with suspensions and the result follows for s_1 by 1.1 and 1.2. The result for s_2 now follows by 5-lemma exactly as in proof of 2.3.

COROLLARY 4.8. $\chi : BT\tilde{o}p_r(\mu) \to BTop_r$ *induces isomorphism on* π_k *for* $k+r \geqq 6$ *or* $r \leqq 2$.

PROOF. Use 4.7 and the remarks below 4.4c.

Combining 4.8 with 4.4 we deduce:

THEOREM 4.9. (a) *Let* $M^n \subset Q^{n+r}$ *be a locally flat submanifold and* $|K| = M$ *a triangulation suppose* $\partial M = \emptyset$ *and* $n+r \geqq 5$ *or* $r \leqq 2$ *then if* $r \geqq 5$ *or* $\leqq 2$ *M has a normal micro block bundle which is unique up to isotopy. If* $r = 4$ *the same result holds provided M is* 1*-connected and if* $r = 3$ *provided M is* 2*-connected.*

(b) *Same hypotheses but* $\partial M \neq \emptyset$ *and* $n+r \geqq 6$ *or* $r \leqq 2$. *Same conclusions provided both* ∂M *and* $(M, \partial M)$ *satisfy the connectivity conditions for* $r = 3, 4$.

(c) *Same hypotheses as* (b) *but* η/L *a given normal block micro bundle on* ∂M *in* ∂Q *and* $n+r \geqq 5$ *or* $r \leqq 2$. *Then* η *extends up to isotopy* rel $E(\eta)$

to a normal micro block bundle on M in Q provided if $r = 4, 3$ that M, ∂M is 1, 2*-connected.*

Open and closed block bundles

Now let $T\tilde{o}p_q$ and $T\tilde{o}p_q(R)$ be the topological analogues of $\tilde{PL}_q$, $\tilde{PL}_q(R)$ [35; § 0]. These are the groups for closed block bundles and open block bundles respectively. There are natural homeomorphisms, see [35; § 0]:

$$\begin{tikzcd} T\tilde{o}p_q \arrow[rr, "\text{germ } (I)"] \arrow[dr, "\text{collar}"'] & & T\tilde{o}p_q(\mu) \\ & T\tilde{o}p_q(R) \arrow[ur, "\text{germ}"'] & \end{tikzcd}$$

Note that in the pl. case all these maps are homotopy equivalences. We may define maps, in the same way as χ,

$$\chi_I : BT\tilde{o}p_q \to BTop_q$$
$$\chi_R : BT\tilde{o}p_q(R) \to BTop_q$$

and the analogues of 4.3 and 4.4 hold for closed and open block bundles, indeed the proofs were written in such a way that they require ano lteration. Now χ_R χ_I and χ commute up to homotopy with the diagram (1) and to deduce results on normal open and closed bundles we need to examine the properties of the maps in (1).

THEOREM 4.10 *The maps in* (1) *induce iosmorphisms on* $\pi_n(\ )$ *for* $q \leqq 2$ *or* $n+q \geqq 5$.

PROOF. We prove the result for germ (I). A similar proof works for R and the result follows by commutativity. We show that germ $(\mathrm{I})_*$: $\pi_n(PL_q, Top_q) \to \pi_n(PL_q(\mu), Top_q(\mu))$ is an isomorphism in the range considered and the result follows by the 5-lemms and the $\tilde{PL}$ result. Now let $h : I^n \times I^q \hookleftarrow$ represent an element of $\pi_n(T\tilde{o}p_q, PL_q)$ then by relative handle straightening and isotopy extension we may assume that h is pl. in a neighbourhood of $I^n \times \{0\}$ provided $n+q \geqq 5$, $n \neq 3$ or $q \leqq 2$. So we have h pl. on $I^n \times \varepsilon I^2$ for some ε and by the regular neighbourhood theorem we may assume that h preserves this set setwise. But $I^n \times (I^q - \varepsilon \dot{I}^q)$ is a collar on $I^n \times \varepsilon \dot{I}^q$ and it is thus easy to change h by a concordance to make it a product on this collar and hence pl. This shows $\pi_n(T\tilde{o}p_q, \tilde{PL}_q) = 0$ $n+q \geqq 5$, $n \neq 3$ *or* $q \leqq 2$ and it remains to show that $\pi_3(T\tilde{o}p_q, \tilde{PL}_q) = \boldsymbol{Z}_2$ for $q \geqq 3$ and that germ $(\mathrm{I})_*$ is non-trivial on π_3. The above argument and the usual considertaions shows that $\pi_3(T\tilde{o}p_q, \tilde{PL}_q)$ is 0 or $\boldsymbol{Z}_2$, and it suffices to construct one non-trivial element which usspends to the non-trivial element of $\pi_3(Top/PL)$. The lowest dimension concerned is $\pi_3(\widetilde{Top}_3, \widetilde{PL}_3)$ and we only have to observe that the non-straightenable handle constructed in § 1 was in fact a closed handle and the restriction to $\partial \Delta^3 \times I^3$ was pl. It is then easy to see that it represents an element of $\pi_3(\widetilde{Top}_3, \widetilde{PL}_3)$.

COROLLARY 4.11. *Let K be a simplicial complex. There are natural bijections between isomorphism classes of q-dimensional open, closed and micro block bundles provided $q \leqq 2$ or $\geqq 5$, and provided further that if $q = 4$ then K is 1-connected and if $q = 2$ that K is 2-connected.*

COROLLARY 4.12. *The results of theorem* 4.9 *are true for open and closed block bundles.*

REMARKS. (1) The uniqueness part of 4.12 implies that there are 'regular neighbourhood' theorems in the topological catergoy in restricted dimensions and for restricted subsets.

(2) We do not have a complete analogue of the Kister–Mazur theorem for block bundles. The problem is the lack of engulfing techniques in dimension 4.

5. Normal open, micro and closed disc bundles

Let $Top_n(R)$, $Top_n(I)$ be the groups for open and closed disc bundles. Then there are natural maps $Top_n(R) \to Top_n$, $Top_n(I) \to Top_n$ on taking germs. The first is a homotopy equivalence (Kister [24]), while the second is not (Browder [1]). We get natural maps $Top_n(R) \to \mathbf{Top}_n$, $Top_n \to \mathbf{Top}_n$ which induce maps of classifying spaces $\alpha : BTop_n(R) \to \mathbf{BTop}_n$, $\beta : BTop_n \to \mathbf{BTop}_n$, $\gamma : BTop_n(I) \to \mathbf{BTop}_n$. Now let $M^n \subset Q^{n+q}$ be a locally flat submanifold, then by rather simpler arguments than in Theorem 4.4 we have:

THEOREM 5.1. *Isotopy classes of normal open, micro and disc bundles on M in Q correspond bijectively with liftings of $c(\mathrm{id}, Q)$ under α, β or γ respectively, under the same dimensional restrictions as* 4.4.

In order to apply Theorem 5.1 to prove existence of normal bundles we need to compare Top_n, etc., with $\mathbf{Top}_n$, as usual we work with the pl. groups.

PROPOSITION 5.2. *The n-th homotopy group of the squares*

$$\begin{array}{ccc} PL_q & \to & Top_q \\ \downarrow & 1_q & \downarrow \\ \mathbf{PL}_q & \to & \mathbf{Top}_q \end{array} \qquad \begin{array}{ccc} PL_q(I) & \to & Top_q(I) \\ \downarrow & 2_q & \downarrow \\ \mathbf{PL}_q & \to & \mathbf{Top}_q \end{array}$$

is zero if $n \leqq q+1$, $q \geqq 5$ or $n < q+1$, $q \geqq 6$ respectively.

PROOFS. For the first square use the long exact sequence

$$\to \pi_n(Top_q, PL_q) \to \pi_n(\boldsymbol{Top}_q, \boldsymbol{PL}_q) \to \pi_n(1_q) \to$$

and 1.1, 1.2. Note that we do not need to know $\pi_{q+1}(Top_q, PL_q) \to \pi_{q+1}(\boldsymbol{Top}_q, \boldsymbol{PL}_q)$ is an isomorphism, since the latter group is always zero if $q \geqq 5$.

Now consider the fibrations (up to homotopy)

$$\begin{array}{ccccc} PL_{q-1} & \to & PL_q(I) & \to & S^{q-1} \\ \downarrow & 3_q & \downarrow & & " \\ Top_{q-1} & \to & Top_q(I) & \to & S^{q-1} \end{array}$$

given by the topological and pl. Kister–Mazur theorems [14, 24, 25]. It follows that $\pi_n(3_q) = 0$, all n, q. The second half now follows from the long exact sequence

$$\pi_n(3_q) \to \pi_n(1_{q-1}) \to \pi_n(2_q) \to$$

and the first half. Note that we have identified 1_{q-1} with $(\boldsymbol{Top}_q; Top_{q-1}, \boldsymbol{PL}_q; PL_{q-1})$ by 2.4.

PROPOSITION 5.3. *There is a long exact sequence for* $q \geqq 3$.

$$\to \pi_n(\boldsymbol{Top}_q, Top_q) \to \pi_n(\boldsymbol{Top}_{q+1}, Top_{q+1}(I)) \to \pi_n(F_q, G_q)$$

PROOF. This follows by exactly the same argument as the pl. sequence [35; 3.6] using the isomorphisms $\pi_n(\boldsymbol{Top}_{q+1}, \boldsymbol{Top}_q) \cong \pi_n(\boldsymbol{PL}_{q+1}, \boldsymbol{PL}_q) \cong \pi_n(S^q) \oplus \pi_n(F_q, G_q)$ (2.4 and [35; 2.15]).

COROLLARY 5.4. (i) $\pi_n(\boldsymbol{Top}_q, Top_q(I)) = 0$ *if* $n < q$, $q \geqq 6$.
(ii) $\pi_n(\boldsymbol{Top}_q, Top_q(R)) = 0$ $n \leqq q$, $q \geqq 5$.

PROOF. The first half follows from the pl. result (see [12; 4.2] and [33; § 5]) and 5.2, and the second from 5.3 and the vanishing of $\pi_n(F_q, G_q)$ metastably (James see [8]).

COROLLARY 5.5. *With the notation of* 5.1, *if* $q \geqq 5$ *and* $n \leqq q$ *then* M *has a normal open or micro bundle*; *if* $n < q$ *it is unique up to isotopy. If* $q \geqq 3$ *and* $n < q$, M *has a normal disc bundle and if* $n < q-1$ *it is unique up to isotopy.*

REMARK. We have one better dimension for open bundles than [12], this was obtained in the pl. case by Morlet [31] and Scott [40].

6. Approximation and triangulation

We consider two problems.

Approximation problem

(1) $f: M^n \to Q^{n+r}$ is a locally flat embedding of M in Q both pl.

manifolds, $f^{-1}(\partial Q) = \partial M$ and $f|\partial M$ is pl. It is required to ambient ε-isotope f rel ∂Q to a pl. embedding.

Triangulation problem

(2) $f : M^n \to Q^{n+r}$ is a locally flat embedding and Q only is pl. $f^{-1}(\partial Q) = \partial M$ and $f(\partial M)$ is a pl. submanifold. It is required to ambient ε-isotopy Q rel ∂Q to carry $f(M)$ onto a pl. submanifold.

The solutions we give to these problems (6.1 and 6.4 resp.) are the analogues of the smoothing theorems of Haefliger [10], Lashof–Rothenberg [28] and ourselves [33; 6.6], the coefficients being in $\pi_i(Top;\ PL, Top_r;\ PL_r)$ and $\pi_i(Top_r, PL_r)$ respectively. However, since these coefficients are all computable by stability and Kirby–Siebenmann we get simple solutions to the problems:

THEOREM 6.1 (*Approximation theorem*). *Data as in the approximation problem. If $r \geqq 3$ the required isotopy always exists. If $r \leqq 2$ and $n+r \geqq 5$ then there is an obstruction in $H^3(M, \partial M; \mathbf{Z}_2)$ which vanishes iff the required isotopy exists.*

PROOF. $r \geqq 3$. The case $n = 1$, $r = 3$ is known (Homma–Gluck see [7]) so we may assume $n+r \geqq 5$. The bounded case is a direct relativisation of the unbounded case using the bounded versions of §§ 3 and 4 so for simplicity assume $\partial M = \emptyset$. Let $\xi = c(f, Q) : M \to BTop_q$ classify the neighbourhood of M in Q and consider the diagram

$$\begin{tikzcd}
 & Top_r/PL_r \arrow[r, "\simeq", "2.4"'] \arrow[d] & Top/PL \arrow[d] \\
B\tilde{PL}_r \arrow[r, "\psi", "\simeq"'] \arrow[d] & BPL_r \arrow[r] \arrow[d] & BPL \arrow[d] \\
B\tilde{Top}_r \arrow[r, "\chi"] & BTop_r \arrow[r] & BTop \\
 & M \arrow[uu, dashed, bend left, "\eta"] \arrow[u, "\xi"'] \arrow[uur, dashed, "\eta'"] \arrow[ur, bend right, "\xi'"'] &
\end{tikzcd}$$

Here ψ is the pl. analogue of χ, it is a homotopy equivalence by [35; 5.5]. Consider a lift η of ξ in BPL_r. This determines a lift of ξ over χ and hence by 4.4 a normal block bundle on M in Q with a pl. structure and hence a local pl. structure on Q near M. On the other hand, denoting the stabilization of ξ by ξ', we have, by the main result of Kirby–Siebenmann [23], a natural bijection between liftings of $\xi' \oplus \tau_M = \tau_Q|M$

in BPL and local pl. structures on Q near M. Since τ_M has a preferred lifting by the pl. structure of M, there is a bijection also with lifts of ξ' in BPL. We next observe that these two processes are compatible i.e. the local pl. structures determined by lifting ξ or ξ' agree iff the diagram commutes. This is an easy consequence of the classifying property of ψ (pl. analogue of 4.2 (i)).

Finally we appeal to stability (2.4) to assert that there is a bijection between liftings of ξ and ξ' and hence we can choose a lifting η of ξ which stablises to the lifting η' of ξ' given by the original pl. structure on Q. Hence we have a local pl. structure in Q extending the given structure on M and agreeing up to equivalence with the given structure on Q. The equivalence (cf Kirby–Siebenmann [23]) is ε-isotopy and the result follows.

$r \leqq 2$. In this case the first half of the above argument holds but we have $\boldsymbol{Top_r/PL_r}$ contractible by 2.1, Wall [44] and collaring arguments for $r = 1$ (essentially Kirby [22] again). Hence there is essentially only one lift η of ξ in $\boldsymbol{BPL_r}$ and using the homotopy type of Top/PL a single obstruction in $H^3(M; \boldsymbol{Z}_2)$ to commutativity. This proves 'only if'; 'if' is clear since a ε-isotopy of f to a pl. embedding determines a normal block bundle on f [33] and hence a lift of ξ in $\widetilde{BPL}_r \simeq \boldsymbol{BPL_r}$.

As a direct corollary of the first half of 6.1 we have

COROLLARY 6.2. *Denote by* Emb(M, Q) *resp.* Emb$^{PL}(M, Q)$ *the sets of isotopy classes* (*resp.* pl. *isotopy classes*) *of locally flat* (*resp.* pl. *locally flat*) *embeddings of* M^n *in* Q^{n+r}. *Then if* $r \geqq 3$ *the natural function*

$$\mathrm{Emb}^{PL}(M, Q) \to \mathrm{Emb}\,(M, Q)$$

is a bijection.

PROOF. Onto is clear from the approximation theorem. But topological isotopy ⇒ pl. concordance by the relative statement ⇒ pl. isotopy by Hudson [18].

We can also relativise results on the Hauptvermutung using the proof of 6.1 which showed

COROLLARY 6.3. *Two different* pl. *structures on* Q *which agree on* M *and are equivalent, are equivalent* rel M.

We now turn to the triangulation problem:

THEOREM 6.4 (*Triangulation theorem*). *Data as in the triangulation problem. If* $n \geqq 5$ *and* $r \leqq 2$ *the required isotopy always exists. If* $r \geqq 3$ *then there is an obstruction in* $H^4(M, \partial M; \boldsymbol{Z}_2)$ *which vanishes iff the required isotopy exists.*

PROOF. Use the diagram in 6.1. We show that the required isotopy exists iff ξ lists in BPL_r and the refult follows from the homotopy type of Top_r/PL_r. 'Only if' is clear by the argument at the end of proof of 6.1. So suppose ξ lifts to $\eta : M \to BPL_r$ and let η', ξ' be the stabilizations of η, ξ. Now $\tau_Q|M = \tau_M \oplus \xi'$ has a preferred lift in BPL given by the pl. structure of Q and we can choose a lift of τ_M in BPL so that $\tau_M^{PL} \oplus \eta' = \tau_Q|M$. This determines a pl. structure on M by the main result of [23] and the argument is now exactly as in 6.1.

Appendix: Topological immersions

The arguments given here were contained in a paper by M. A. Armstrong and C. P. Rourke written in February 1968. Crucial use is made of the topological covering isotopy theorem, the paper mentioned above also contained an incomplete proof of this theorem. We refer to Edwards and Kirby [4] for a complete proof. We need a version for cubes of isotopies. This follows from the methods of Edwards and Kirby and Hudson [16], see also Kirby [21]. Familiarity with the paper of Haefliger and Poenaru [11] is assumed and we have made our notation agree with theirs as far as possible.

A1. Definitions and statements of results

Throughout the appendix we use the following notation: V and Q are topological manifolds of dimensions n and $n+r$ respectively and $\partial Q = \emptyset$. $V' \subset V$ is a codim 0 submanifold and $M^m \subset V^n$ is a submanifold of codimension > 0 with $M \cap \partial V = \partial M$, both M and V' are closed subsets of V and $M' = V' \cap M$ is a codim 0 submanifold of M. All embeddings are locally flat in the appropriate senses.

Suppose $\partial V = \emptyset$, then an *immersion* $V \to Q$ is a map which satisfies, for each point $x \in V$ there are charts $h : U \to V$, $g : W \to Q$ with $x \in \operatorname{im} h$ so that

$$\begin{array}{ccc} U & \xrightarrow{h} & V \\ \cap & & \downarrow f \\ W & \xrightarrow{g} & Q \end{array}$$

commutes, where U, W are open sets in R^n, R^{n+r}.

A *simplex of immersions* $f: V \times \Delta^k \to Q \times \Delta^k$ is a map satisfying $f(V \times \{t\}) \subset Q \times \{t\}$ and for each pair $(x, t) \in V \times \Delta^k$ there are charts $h : U \times T \to V \times \Delta^k$ $g : W \times T \to Q \times \Delta^k$ so that

$$\begin{array}{ccc} U \times T & \to & V \times \Delta^k \\ \cap & & \downarrow f \\ W \times T & \xrightarrow{g} & Q \times \Delta^k \end{array}$$

commutes where U, W are open sets in R^n, R^{n+r} respectively, T is an open neighbourhood of t in Δ^k and the whole diagram commutes with projection on Δ^k.

The Δ-set $\mathrm{Im}(V, Q)$ has as typical k-simplex a simplex of immersions of V in Q with face maps defined by restriction.

Now let $\partial V \neq \emptyset$, then we can embed V in an unbounded manifold V_+ by adding an open collar to ∂V. An *immersion* of V in Q is a germ of immersions of V_+ in Q defined in some neighbourhood of V in V_+. Two being equivalent if they agree in some smaller neighbourhood. A *simplex* of immersions is defined similarly and we get a restriction map

$$t : \mathrm{Im}(V, Q) \to \mathrm{Im}(V', Q)$$

determined by choosing a collar on V' in V.

Now let ξ, η be microbundles. A *fibre map* of ξ in η is a pair (f, F) where $f : B(\xi) \to B(\eta)$ is a map and $F : E(\xi) \to E(\eta)$ is a map defined in a neighbourhood of $B(\xi)$ which restricts to f on $B(\xi)$ and carries fibres to fibres, i.e. the diagram

$$\begin{array}{ccc} B(\xi) & \xrightarrow{f} & B(\eta) \\ \cap & & \cap \\ E(\xi) & \xrightarrow{F} & E(\eta) \\ \downarrow{\scriptstyle \pi(\xi)} & & \downarrow{\scriptstyle \pi(\eta)} \\ B(\xi) & \xrightarrow{f} & B(\eta) \end{array}$$

commutes.

For example if $f : V_+ \to Q$ is any map defined in a neighbourhood of V then $df = (f, f \times f)$ is a fibre map of τ_V in τ_Q. Recall that $\tau_V = \tau_{V_+}|V$ and τ_{V_+} is defined by the diagram

$$V_+ \xrightarrow{\Delta} V_+ \times V_+ \xrightarrow{\pi_1} V_+ .$$

(f, F) is a *representation* if it is locally trivial in the following sense. For each $x \in B(\xi)$ there are charts $H : U \times R^n \to E(\xi)$, $g : W \times R^{n+r} \to E(\eta)$ with $x \in \mathrm{im}\, h$ so that

$$\begin{array}{ccc} U \times R^n & \xrightarrow{h} & E(\xi) \\ \downarrow{\scriptstyle \varphi| \times \mathrm{id}} & & \downarrow{\scriptstyle F} \\ W \times R^{n+r} & \xrightarrow{g} & E(\eta) \end{array}$$

commutes where $\varphi : U \to W = g^{-1} \circ F \circ h$.

Equivalently a representation is a fibre map which factors as

$$E(\xi) \xrightarrow{i} E(F^*(\eta)) \xrightarrow{f} E(\eta)$$

where i is the inclusion of a subbundle (cf § 0 of this paper).

As an example, if f is an immersion then df is a representation. A *simplex* of representations of ξ in η is a representation of $\xi \times \Delta^k$ in η. We thus have a Δ-set $R(\xi, \eta)$ and a Δ-map

$$d : \mathrm{Im}(V, Q) \to R(\tau_V, \tau_Q).$$

It is easy to check that both Δ-sets satisfy the Kan condition.

THEOREM 1. *d is a homotopy equivalence provided that if $r = 0$ no component of V is closed.*

THEOREM 2. *$t : \mathrm{Im}(V, Q) \to \mathrm{Im}(V', Q)$ is a* Kan *fibration onto its image provided that if $r = 0$ each component of $V - V'$ meets either ∂V or an end of V.*

Now let $\varphi : V' \to Q$ be a fixed immersion and denote by $\mathrm{Im}_\varphi(V, Q)$, $R_\varphi(\tau_V, \tau_Q)$ the subsets which agree with φ (resp. $d\varphi$) in some neighbourhood of V'.

COROLLARY 1. *$d : \mathrm{Im}_\varphi(V, Q) \to R_\varphi(\tau_V, \tau_Q)$ is a homotopy equivalence with the proviso of theorem* 2.

PROOF. Consider the commutative diagram of Δ-maps

$$\begin{array}{ccc} \mathrm{Im}(V, Q) & \xrightarrow{d} & R(\tau_V, \tau_Q) \\ \downarrow t & & \downarrow u \\ \mathrm{Im}(V', Q) & \xrightarrow{d} & R(\tau_{V'}, \tau_Q) \end{array}$$

where u is given by restriction. u is a Kan fibration onto its image by an easy homotopy extension argument. It follows from the 5-lemma that the corresponding fibres of t and u are homotopy equivalent in particular those lying over φ.

THEOREM 3. *The restriction map*

$$t : \mathrm{Im}_\varphi(V, Q) \to \mathrm{Im}_\psi(M, Q)$$

is a fibration onto its image where $\psi = \varphi | M'$ and provided if $r = 0$ each component of $V - (V' \cup M)$ meets ∂V or an end of V.

Now let $\chi : M \to Q$ be a fixed immersion with $\chi | M' = \psi$ and denote by $\mathrm{Im}_{\varphi,\chi}(V, Q)$, $R_{\varphi,\chi}(\tau_V, \tau_Q)$ the subsets which agree with φ on V' and χ on M.

COROLLARY 2. *$d : \mathrm{Im}_{\varphi,\chi}(V, Q) \to R_{\varphi,\chi}(\tau_V, \tau_Q)$ is a homotopy equivalence with the proviso of theorem* 3.

PROOF. Consider the diagram

$$\begin{array}{ccc} \mathrm{Im}_{\varphi}(V, Q) & \xrightarrow{d} & R_{\varphi}(\tau_V, \tau_Q) \\ \downarrow t & & \downarrow u \\ \mathrm{Im}_{\psi}(M, Q) & \xrightarrow{d} & R_{\psi}(\tau_M, \tau_Q) \end{array}$$

u is again defined by restriction. That u is a fibration onto its image follows from the isotopy extension theorem. The result now follows, as in Corollary 1; from the 5-lemma.

A2. Topological immersions of PL manifolds

In this section we prove theorems 1 and 2 in the case that V is a compact pl. manifold and V' is a pl. submanifold, these are stated in lemmas 2 and 3 below. The proofs in this section follow those of [11] very closely and we will give details only when the arguments differ substantially.

LEMMA 1. *Theorem* 1 *holds if* V *is a disc.*
The proof of Lemma 1 *is identical to* [11; § 5].

LEMMA 2. *Theorem* 1 *holds if* V *is a compact* pl. *manifold.*

LEMMA 3. *Theorem* 2 *holds if* V *is a compact* pl. *manifold and* V' *a submanifold.*

COROLLARY. *Collorary* 1 *holds if* V *is a compact* pl. *manifold and* V' *a* pl. *submanifold.*

PROOF. Apply the proof of corollary 1.

Lemma 3 is the main burden of the section. We show that Lemma 3 $\Rightarrow$ Lemma 2 and finally prove lemma 3.

LEMMA 4. *Corollary* 1 *holds if* $V = I^q \times I^s$ *and* V' *is a neighbourhood of* $\partial I^q \times I^s$.

Lemma 4 follows from lemmas 1 and 3 by the argument on [11; p. 81].

PROOF OF LEMMA 2. We use induction on the number of handles of some handle decomposition of V. Suppose $V = V_0 \cup H$ where H is a handle and the lemma holds for V_0.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Im}\,(V, Q) & \xrightarrow{d} & R(\tau_V, \tau_Q) \\ \downarrow & & \downarrow \\ \mathrm{Im}\,(V_0, Q) & \xrightarrow{d_0} & R(\tau_{V_0}, \tau_Q) \end{array}$$

the restriction of d to each fibre is a homotopy equivalence by Lemma 4. Therefore, by the five lemma, d is a homotopy equivalence.

It remains to prove Lemma 3.

Induced neighbourhoods

We recall from [11] that an *induced neighbourhood* of an immersion $f: V \to Q$ is a triple (U, i, φ), where U is a manifold of the same dimension as Q, i is a closed embedding of V in the interior U and φ is an immersion of U in Q extending $f \circ i^{-1}|i(V)$. The treatment of induced neighbourhoods [11; § 7]. was in a topological setting so there s no need for us to repeat it. In particular, they always exist and give a unique germ of neighbourhood of V.

Now suppose V is a disc B^n. An induced neighbourhood (U, i, φ) is *standard* if the pairs (U, iB) and $(2I^{n+r}, I^n)$ are homeomorphic.

LEMMA 5. *Standard induced neighbourhoods exist.*

PROOF. (See also Lacher [26]). We show that any induced neighbourhood U of $f: B \to Q$ contains a standard neighbourhood. By local flatness, we can choose a smaller disc $D^n \subset B$ which has a standard neighbourhood W in U and we can assume that $(B^n, D^n) \cong (I^n, \frac{1}{2}I^n)$. The isotopy which expands $\frac{1}{2}I^n$ onto I^n is locally trivial hence extends to a homeomorphism of U carrying W onto the required standard neighbourhood.

PROOF OF LEMMA 3. As in [11], we are given a cube of immersions $F_0 : I^k \times V \to Q$ and on extension $F' : I \times I^k \times V' \to Q$. We have to find a further extension $F : I \times I^k \times V \to Q$. We first remark that we can forget about the complement of a neighbourhood of $V - V'$ in V and that we may assume that V is obtained from V' by adding handles. Since the composition of two fibrations is a fibration, we may assume that V is obtained from V' by adding a single handle. Combining these two remarks, we may assume that $V = I^q \times I^s$ and $V' = I^q \times I^s - (\frac{3}{4})I^q \times I^s$. An induced neighbourhood U of an immersion $f: V' \to Q$ is *standard* if as pairs

$$(U, iV') \cong (2I^q \times 2I^s \times 2I^r - (\tfrac{1}{3})I^q \times 2I^s \times I^r, I^q \times I^s \times \{0\} - (\tfrac{3}{4})I^q \times I^s \times \{0\}).$$

LEMMA 6. *For all pairs* $(t, \tau) \in I \times I^k$, *the immersion* $F'_{t,\tau} : V' \to Q$, *given by* $F'|\{t\} \times \{\tau\} \times V'$, *has a standard induced neighbourhood.*

REMARK. Lemma 6 is necessary in order to avoid the usage of the pl. regular neighbourhood theorem made in [11; p. 90 second paragraph].

PROOF. By lemma 5, for $t = 0$, the required standard neighbourhood is easily found inside a standard neighbourhood of $(F_{0,\tau}, V)$. In general, the restriction of F' to $I \times \{\tau\} \times V'$ is a regular homotopy between $F'_{t,\tau}$ and $F'_{0,\tau}$ for any $t \in I$. Suppose t is the greatest lower bound of values for which a standard induced neighbourhood of $F'_{s,\tau}$ does not exist. Let

(U, i, φ) be any induced neighbourhood of $F'_{t,\tau}$. Then there is an $\varepsilon > 0$ and an interval of embeddings in the interior of U,

$$P : [t-\varepsilon, t+\varepsilon] \times \{\tau\} \times V' \to U,$$

extending i, and such that $F'| = \varphi P$. Now $F'_{t-\varepsilon,\tau}$ has a standard induced neighbourhood and, by uniqueness of gems of induced neighbourhoods, we may assume (by choosing a smaller standard neighbourhood) that it is contained in the interior of U. Covering the isotopy P then yields standard neighbourhoods of $F_{s,\tau}$ for all $s \in [t-\varepsilon, t+\varepsilon]$, contradiction. Hence standard neighbourhoods exist for all t, as required.

Now, as in [11; p. 88], we appeal to compactness of $I \times I^k$ to reduce to the case when the following condition is satisfied:

CONDITION c. There is a standard neighbourhood (Ω, i, φ) induced by $F'_{0,0} : V' \to Q$ and a cube of embeddings $P' : I \times I^k \times V' \to \Omega$ such that $F' = \varphi P'$ and $P'_{0,0} = iF'_{0,0}$.

From now on, the argument is virtually identical to the corresponding parts of [11]. One first proves the lemma at the bottom of page 88, which relies only on covering siotopies, and one can take the resulting neighbourhood V'' of V' in V to be $I^q \times I^s - (\frac{1}{2})I^q \times I^s$. Assuming $r > 0$, the formulae on page 90 of [11], then provides the required extension F of F'-covering isotopies is again used essentially here. The case $r = 0$ is also dealt with in the same way as in [11].

A3. Proof of the theorems

PROOF OF THEOREM 1. We show, making crucial use of the results of § A2, that d is a bijection on homotopy classes referred to any basepoint, the result then follows from the Whitehead theorem for Δ-sets [36].

$$d_* : \pi_i(\mathrm{Im}(V, Q)) \to \pi_i(R(\tau_V, \tau_Q)) \text{ is suriective.}$$

PROOF. We are given a sphere of representations of τ_V in τ_Q and we have to construct a corresponding sphere of immersions of V in Q. For simplicity suppose first that $r > 0$ and choose a locally finite cover $\{B_j\}$ of V in $V_+ = V \cup$ collar, where each B_j is an n-ball with locally flat boundary.

INDUCTION HYPOTHESIS. There is a sphere of immersions $F: S^i \times U_j \to Q$ such that dF is hcmotopic to the restriction of the given sphere of representations, where U_j is an open neighbourhood of $y_j \equiv \bigcup_{\alpha=1}^{j} B_\alpha$ in V_+.

The induction starts with $U_1 = B_1 \cup$ open collar of $\dot{B}_1$ using Lemma 1.

INDUCTION STEP. Let $B = B_{j+1}$ and $A = B \cup$ closed collar of $\dot{B}$ in V. Regard A as a pl.-manifold. Denote $X = A \cap y_j$.

LEMMA 7. *There is a* pl. *manifold* W *of* A *such taht* $X \subset$ (*interior of* W *in* A) $\subset W \subset U_j \cap A$.

PROOF. X is a closed subset of the metric space A and $U_j \cap A$ is an open subset. Therefore there is a $\delta > 0$ such that $N_\delta(X, A) \subset U_j \cap A$, where $N_\delta(X, A) = \{y | y \in A, d(y, X) < \delta\}$. Triangulate A by a complex K of mesh less than $\delta/2$. Define the subcomplex $L \subset K$ to consist of all (closed) simplexes which meet X. Define $W =$ simplicial neighbourhood $M(L, K'')$ where K'' denotes a second derived complex. Then $W \subset N_{\delta/2}(L, A) \subset N_\delta(X, A) \subset U_j \cap A$ and $X \subset L \subset$ interior of W in A, as required.

Now $F|S^i \times W$ is a sphere of immersions corresponding to the given sphere of representations of τ_W in τ_Q and this sphere of representations extends to a sphere of representations of τ_A in τ_Q. I.e. we have (based) maps of S^i into three of the spaces in the following commutative diagram

$$\begin{array}{ccc} \mathrm{Im}\,(A, Q) & \xrightarrow{d_A} & R(\tau_A, \tau_Q) \\ \downarrow & & \downarrow \\ \mathrm{Im}\,(W, Q) & \xrightarrow{d_W} & R(\tau_W, \tau_Q) \end{array}$$

and d_A, d_M are homotopy equivalences by Lemma 2. It follows that there is a sphere of immersions F_1 of A in Q which extends $F|S^i \times W$ and corresponds to the given sphere of representations (up to homotopy). Now $F: S^i \times U_j \to Q$ and $F_1: S^i \times A \to Q$ agree on the open neighbourhood $\dot{W}$ of $B \cap y_j$ and hence agree on some open neighbourhood U_{j+1} of y_{j+1}, completing the induction step. Finally we remark that this induction defines a limit sphere of immersions of V in Q since the cover is locally finite.

The case $r = 0$. In this case we can only appeal to Lemma 2 if each component of $A - W$ meets ∂A. This need not happen, so remove a point from each component which does not meet ∂A. This is equivalent to removing the interior of a disc and adding back an open collar. We can therefore appeal to Lemma 2 to find a sphere of immersions of $A - \{\text{finite set}\}$ which extends the given immersions of W. In the limit we get a sphere of immersions of $V - X$ where X is a locally finite set. We will show that id : $V \to V$ is isotopic to an embedding $V \to V - X$ and we thus get the required sphere of immersions of V. Without loss of generality assume V is connected. There are two cases (a) V is compact, in which case X is finite and $\partial V \neq \emptyset$ by hypothesis. In this case the required isotopy is easily found, since we may isotope X into a collar of ∂V. (b) V is non-compact; then V has an end and we can embed R_+ locally-flatly in V so that the image contains X and the end of R_+ goes to the end of V. Now R_+ has a standard neighbourhood in V (cf. Lacher [26] for better results) and the required isotopy is now easily found.

$$d_* : \pi_i(\mathrm{Im}(V, Q)) \to \pi_i(R(\tau_V, \tau_Q)) \textit{ is injective.}$$

We are given an annulus $S^i \times I$ of representtaions of τ_V in τ_Q and two spheres $S^i \times \{0\} \cup S^i \times \{1\}$ of immersions corresponding up to homotopy. For convenience extend (productwise) both the immersions and the representations to $S^i \times [-1, 0] \cup S^i \times [1, 2]$. Choose B_j as in the last proof and use the same notation.

INDUCTION HYPOTHESIS. There is an annulus $S^i \times [-1, 2] \times U_j$ of immersions in Q, which agree witht he given immersion on $S^i \times ([-1, 0] \cup [1, 2]) \times U_j$ and corresponds up to (based) homotopy with the given representation.

The induction starts trivially with $U_0 = \emptyset$ and for the induction step one considers the same diagram as in the last proof. We have based maps of $S^i \times [-1, 2]$ into three of the spaces and a map of $S^i \times ([-1, 0] \cup [1, 2])$ into the fourth, all commuting up to homotopy. It follows that there is a annulus $S^i \times [-1, 2] \times A$ of immersions extending the given immersions on $S^i \times ([-1, 0] \cup [1, 2] \times A \cup S^i \times [-1, 2] \times W$. As before this implies that we have defined an immersion on $S^i \times [-1, 2] \times U_{j+1}$ for some open neighbourhood U_{j+1} of y_{j+1}, satisfying all the requirements.

This completes the proof of the case $r > 0$ if $r = 0$ we again get only a annulus of immersions of $V - X$ but by homotopy extension and the isotopy used before this implies the required annulus of immersions of V.

PROOF OF THEOREM 2. Again for simplicity assume $r > 0$ and choose balls B_j as in the previous proofs. We are given a cube $F_0 : I^k \times V \to Q$ of immersions and an extension $F' : I \times I^k \times V' \to Q$. We have to construct a further extension $F : I \times I^k \times V \to Q$.

INDUCTION HYPOTHESIS. There is a cube of immersions $P_j : I \times I^k \times U_j \to Q$ extending $F_0| \cup F'$ where U_j is an open neighbourhood of $Y_j = V' \cup \bigcup_{\alpha=1}^{j} B_j$ in $V_+ = V \cup$ collar.

(Note that y_j and U_j are defined in a different way than in the last proof.)

The induction staarts since F' is lready defined on a small neighbourhood of V' in V_+ (see the definition of immersion of a bounded manifold). For the induction step define $B = B_{j+1}$, $A = B \cup$ collar and $X = A \cap Y_j$ and choose W as in lemma 7. By lemma 3 there is an extension of $P_j | I \times I^k \times V \cup F_0 | I^k \times A$ to a cube of immersions $P_j : I \times I^k \times A \to A$. P_j and P'_j agree in a neighbourhood of $B \cap Y_j$ and hence define P_{j+1} in some neighbourhood U_{j+1} of Y_{j+1}, completing the induction.

The case $r = 0$ is now dealt with as in previous proofs.

PROOF OF THEOREM 3. We are given a cube of immersions $F_0 : I^k \times V \to Q$ and compatible extensions $F' : I \times I^k \times V' \to Q$, $G : I \times I^k \times M \to Q$ we have to construct a further extension $F : I \times I^k \times V \to Q$. B_i are chosen as in the previous proofs.

INDUCTION HYPOTHESIS. There is a cube of immersions $P_j : I \times I^k \times U_j \to Q$ extending $F_0 \cup F' \cup G$ where U_j is an open neighbourhood of $Y_j \equiv V' \cup M \cup \bigcup_{\alpha=1}^{j} B_i$ in V_+.

The details of the induction step are exactly as in the last proof so it is necessary only to start the induction i.e. to extend to some open neighbourhood of $M \cup V'$. As before F' is already defined in a neighbourhood of V' in V_+. Now by compactness of $I^k \times I$ we can assume that $F' \cup G$ factors via an embedding and the required extension then follows from the isotopy extension theorem, see the proof of the lemma on p. 88 of [11].

REFERENCES

W. BROWDER
[1] Open and closed disc bundles. Ann. of Maths. 84 (1966), 218–230.

J. L. BRYANT AND C. L. SEEBECK
[2]Locally nice embeddings in codimension three. Bull A.M.S. 74 (1968), 378–380.

A. DOLD
[3] Partitions of unity in the theory of fibrations. Ann. of Maths. 78 (1963), 223–255.

R. EDWARDS AND R. C. KIRBY
[4] On deformations of spaces of embeddings, (to appear).

D. GAULD
[5] Mersions of topological manifolds. Ph.D. thesis U.C.L.A. (1969).

L. C. GLASER AND T. M. PRICE
[6] Unknotting locally flat cell pairs. Ill. J. Maths. 10 (1966), 425–430.

H. GLUCK
[7] Unknotting S^1 in S^4. Bull. A.M.S. 69 (1963), 91–94.

A. HAEFLIGER
[8] Differentiable embeddings of S^n in S^{n+q} for $q > 2$. Ann. of Maths. 83 (1966), 402–436.

A. HAEFLIGER
[9] Lissage des immersions, II (preprint).

A. HAEFLIGER
[10] Knotted spheres and related geometric topics. Proceedings of I.C.M. (Moscow) 1966.

A. HAEFLIGER AND V. POENARU
[11] Classification des immersions combinatoire. Publ. I.H.E.S. 23 (1964), 75–91.

A. HAEFLIGER AND C. T. C. WALL
[12] *PL* bundles in the stable range. Topology 4 (1965), 209–214.

M. W. HIRSCH
[13] On tubular neighbourhoods of manifolds I. Proc. Cam. Phil. Soc. 62 (1966), 177–179.

M. W. HIRSCH
[14] On non-linear cell bundles. Ann. of Maths. 84 (1966), 373–385.

T. HOMMA
[15] *PL* approximations of embeddings of manifolds. Notes, Florida State University, (1965).

J. F. P. HUDSON
[16] Extending *PL* isotopies. Proc. Lon. Maths. Soc. 16 (1966), 651–668.

J. F. P. HUDSON
[17] On transversality. Proc. Camb. Phil. Soc. (to appear).

J. F. P. HUDSON
[18] Concordance and isotopy of *PL* embeddings. Bull. A.M.S. 72 (1966), 534–535.

J. F. P. HUDSON AND W. B. R. LICKORISH
[19] Extending *PL* concordances, (to appear).

J. F. P. HUDSON AND E. C. ZEEMAN
[20] On combinatorial isotopy. Publ. I.H.E.S. 19 (1964), 69–94.

R. C. KIRBY
[21] Triangulating topological manifolds. Lecture notes U.C.L.A. (1969).

R. C. KIRBY
[22] Locally flat submanifolds of codim 2. Proc. Georgia Topology Conf. (1969).

R. C. KIRBY AND L. C. SIEBENMANN
[23] On the triangulation problem and the hauptvermutung. Bull. A.M.S. (1969).

J. KISTER
[24] Microbundles are fibre bundles. Ann. of Maths. 80 (1964), 190–199.

N. KUIPER AND R. LASHOF
[25] Microbundles and bundles: I. Invent Maths. 1 (1966), 1–17.

R. C. LACHER
[26] Locally flat strings and half strings.

R. LASHOF
[27] Lees' immersion theorem and the triangulation of manifolds, (to appear).

R. LASHOF AND M. ROTHENBERG
[28] Microbundles and smoothing. Topology 3 (1965), 357–388.

J. LEES
[29] Immersions and surgeries of topological manifolds, (to appear).

J. W. MILNOR
[30] Microbundles: I. Topology 3, Supp. 1 (1964), 53–80.

C. MORLET
[31] Ph.D. thesis Paris (1968).

C. P. ROURKE
[32] Block bundles and Sullivan theory, (to appear).

C. P. ROURKE AND B. J. SANDERSON
[33, 34, 35] Block bundles: I, II, III. Ann. of Maths. 87 (1968), p.p. 1–28, 255–277, 431–483.

C. P. ROURKE AND B. J. SANDERSON
[36, 37] Δ-sets: I and II, (to appear).

C. P. Rourke and B. J. Sanderson
[38] An embedding without a normal microbundle. Invent. Maths. 3 (1967), 293–299.

C. P. Rourke and B. J. Sanderson
[39] Decompositions and the relative tubular neighbourhood conjecture. Topology, 9 (1970) 225–230.

P. Scott
[40] Ph.D. Thesis, Warwick (1969).

L. Siebenmann
[41] Ph.D. Thesis, Princeton (1966).

J. Stallings
[42] The *PL* structure of Euclidian space. Proc. Cam. Phil. Soc. 58 (1962), 481–488.

C. T. C. Wall
[43] On bundles over a sphere with fibre Euclidean space.

C. T. C. Wall
[44] On *PL* submanifolds of codim 2. Proc. Cam. Phil. Soc. 63 (1967), 5–8.

E. C. Zeeman
[45] Unknotting combinatorial balls. Ann. of Maths. 78 (1963), 501–526.

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