COMPOSITIO MATHEMATICA

CARL E. GORDON

Comparisons between some generalizations of recursion theory

Compositio Mathematica, tome 22, nº 3 (1970), p. 333-346
<http://www.numdam.org/item?id=CM_1970__22_3_333_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 3, 1970, pag. 333-346
Wolters-Noordhoff Publishing
Printed in the Netherlands

COMPARISONS BETWEEN SOME GENERALIZATIONS OF RECURSION THEORY [1]

by

Carl E. Gordon

0. Introduction

There has been much work done to generalize the motions of 'recursive' and 'recursively enumerable' so that given an arbitrary structure $\mathfrak{A}$ with field A one can make use of a class of relations on A which is somehow analogous, e.g., to the class of recursive relations on the natural numbers.

We concern ourselves here with two of these generalizations, one of which ([RM]) approaches recursiveness from the point of view of definability and the other of which ([YNM]) from the point of view of computability. The main result of this paper is that the two approaches yield the same class of 'recursive' relations.

To do any kind of computation or recursion theory one must work within a rich enough structure so that information can be coded and stored. Clearly very little recursion theory can be done *within* a completely arbitrary structure $\mathfrak{A}$.

Montague's approach ([RM]) is to extend $\mathfrak{A}$ as follows: Let κ be a cardinal. Define:

$$U^{0,\kappa} = A,$$
$$U^{n+1,\kappa} = \{x \subset U^{n,\kappa} : \text{cardinality}(x) < \kappa\}.$$

Consider a language with relation symbols for the relations of $\mathfrak{A}$ and the membership symbol ε and variables of type n to range over $U^{n,\kappa}$. Roughly speaking, a relation is 'κ-recursively enumerable' if it is definable by a formula of this language having no unrestricted universal quantifiers. It is 'κ-recursive' if both it and its complement are 'κ-recursively enumerable'. For our purposes we only consider the case when $\kappa = \aleph_0$. Our 'Σ^t definable' will mean' $\aleph_0$-recursively enumerable'

Moschovakis' approach ([YNM]) is to extend $\mathfrak{A}$ by adding a distinquished element 0 and by closing $A \cup \{0\}$ under the operation of forming

[1] This is a modified and strengthened version of Part II of the author's dissertation ([G]).

ordered pairs. In this extended structure, A^*, one can define the natural numbers and the finite sequences of members of A^*. The class of 'primitive computable' functions (which is the analog of the class of primitive recursive functions on the natural numbers) is defined in a natural way with the ordinary recursion schema being replaced by a schema that allows definitions by recursion over the pairing relation. The definition of the class of 'search computable' functions (which is the analog of the class of recursive functions) as given in [YNM] is good enough to give a theory of functionals 'computable in' given functions or functionals. However, since we concern ourselves here only with first-order relations on a first-order structure, we can bypass the full definition and use the Normal Form Theorem (which is applicable in this case) so that a 'σ_1^0' ('recursively enumerable') relation is one of the form $\exists y R(x_1, \cdots, x_n, y)$ with R primitive computable and a 'search computable' relation is one which is 'σ_1^0' and which has a 'σ_1^0' complement.

Each of these generalizations is good in the sense that much of the theory of recursive relations and much of the theory of the arithmetical hierarchy goes through, including Post's Theorem. Moreover the theory of search computable functions yields a good analog of the class of hyperarithmetic sets, including the hierarchy theorems. Each of the generalizations can be specialized to the case when the given structure is the set of natural numbers, in which case both the $\aleph_0$-recursive and the search computable relations are just the ordinary recursive relations. Furthermore the search computable functions on a recursively regular ordinal α have been shown in [G] to be the α-recursive functions in the sense of Kripke ([K]) and the search computable relations on an admissible set A have been shown in [G] to be the A-recursive relations in the sense of Platek ([P]).

Our metatheory is a set theory with a (unique) empty set 0, and individuals (urelements) which are not sets.

Throughout this paper $\mathfrak{A} = \langle A, R_1, \cdots, R_l \rangle$ will be a fixed structure with A an arbitrary set of urelements and each R_i an n_i place relation on A.

1. Definitions and easy lemmas without proofs

(1.1) Definition. $U = \bigcup_n U^n$, where U^n is defined inductively by:

$$U^0 = A,$$
$$U^{n+1} = \{x | x \text{ is a finite subset of } U^n\}.$$

The elements of U^n are called objects of type n.

(1.2) DEFINITION. $\mathfrak{A}^t = \langle U_n, \in/U, R_1, \cdots, R_l, {}^\sim R_1, \cdots, {}^\sim R_l \rangle_{n<\omega}$, where ${}^\sim R_i$ is the complement, relative to A, of R_i.

(1.3) DEFINITION. $HF(A) = \bigcup_n R^n$, where R^n is defined inductively by:

$$R^0 = A,$$
$$R^{n+1} = R^n \cup \{x | x \text{ is a finite subset of } R^n\}.$$

(1.4) DEFINITION. $HF(\mathfrak{A}) = \langle HF(A), A, \in/HF(A), R_1, \cdots, R_l, {}^\sim R_1, \cdots, {}^\sim R_l \rangle$.

(1.5) DEFINITION. The language Σ^t (for the structure $\mathfrak{A}^t$) has the following symbols:

(a) For each natural number n, a countable sequence $\mathrm{v}_{0,n}, \mathrm{v}_{1,n}, \cdots$ of variables of type n.

(b) Relation symbols $\mathrm{R}_1, \cdots, \mathrm{R}_l, {}^\sim\mathrm{R}_1, \cdots, {}^\sim\mathrm{R}_l$.

(c) The symbols $\wedge$, $\vee$, $\forall$, $\exists$, ε, (,) and , .

The formulas of Σ^t are defined inductively by:

(d) For $i = 1, \cdots, l$, if $x_1, \cdots, x_{n_i}$ are type 0 variables then $\mathrm{R}_i(x_1, \cdots, x_{n_i})$ and ${}^\sim\mathrm{R}_i(x_1, \cdots, x_{n_i})$ are formulas.

(e) If ϕ and ψ are formulas then $(\phi \wedge \psi)$ and $(\phi \vee \psi)$ are formulas.

(f) If ϕ is a formula, x is a variable of type n and y is a variable of type $n+1$ then $\exists x \,\varepsilon\, y \phi$, $\forall x \,\varepsilon\, y \phi$ and $\exists x \phi$ are formulas.

(Notice that $x \,\varepsilon\, y$ is not a formula of Σ^t).

The interpretation of Σ^t in $\mathfrak{A}^t$ is the obvious one with variables of type n ranging over objects of type n.

The relations on A which are Σ^t definable in $\mathfrak{A}^t$ are those which are considered in [RM] as analogs of the recursively enumerable relations.

(1.6) DEFINITION. The language Σ (for the structure $HF(\mathfrak{A})$) has all the symbols of Σ^t, except that it only has variables of one type, and in addition has the symbols A and $\neg$A.

The formulas of Σ are defined inductively by:

(a) If x is variable then $\mathrm{A}(x)$ and $\neg\mathrm{A}(x)$ are formulas.

(b) For $i = 1, \cdots, l$, if $x_1, \cdots, x_{n_i}$ are variables then $\mathrm{R}_i(x_1, \cdots, x_{n_i})$ and ${}^\sim\mathrm{R}_i(x_1, \cdots, x_{n_i})$ are formulas.

(c) If ϕ and ψ are formulas then $(\phi \wedge \psi)$ and $(\phi \vee \psi)$ are formulas.

(d) If ϕ is a formula and x and y are variables then $\exists x \,\varepsilon\, y \phi$, $\forall x \,\varepsilon\, y \phi$ and $\exists x \phi$ are formulas.

The interpretation of Σ in $HF(\mathfrak{A})$ is the obvious one with $\mathrm{A}(x)$ meaning $x \in A$ and $\neg\mathrm{A}(x)$ meaning $x \in HF(A) - A$.

(1.7) DEFINITIONS. If a relation R is Σ^t definable in $\mathfrak{A}^t$ we call R a *Σ^t-relation.* If R is Σ definable in $HF(\mathfrak{A})$ we call R a *Σ-relation.* If R is definable in $HF(\mathfrak{A})$ by a formula of Σ having no unrestricted quantifiers, i.e., no subformula of the form $\exists x\phi$, then R is a Δ_0*-relation.*

(1.8) LEMMA. *For each* n, '$x \in U^n$' *is a* Δ_0*-relation.*

(1.9) LEMMA. *Every* Σ^t*-relation is a* Σ*-relation.*

(1.10) LEMMA. *Every* Σ*-relation is of the form* $\exists y S(u_1, \cdots, u_k, y)$, *for some* Δ_0*-relation* S.

2. Primitive computability and σ_1^0 relations

(2.1) DEFINITION.

(a) $A^0 = A \cup \{0\}$.

(b) $A^* =$ the closure of A^0 under the pairing function

$$(x, y) = \{\{x, y\}, \{y\}\}.$$

(c) For $s, t \in A^*$, $\pi(s, t) = s$ and $\delta(s, t) = t$; for $x \in A$, $\pi x = \delta x = (0, 0)$ and $\pi 0 = \delta 0 = 0$.

(d) The natural numbers $0, 1, 2, \cdots$ are identified with $0, (0, 0), ((0, 0), 0), \cdots$ so that, in particular, $n+1 = (n, 0)$ and the set ω of natural numbers is a subset of A^*.

(e) The sequence $\langle x_1, \cdots, x_n\rangle$ of elements of A^* is identified with the element $(n, (x_1, \cdots, (x_n, 0) \cdots))$ of A^*.

(f) If $x = \langle x_1, \cdots, x_n\rangle$ then $lh(x) = n$ and, for $1 \leqq i \leqq n$, $(x)_i = x_i$.

Type conventions: (a) Lower case Roman Italics, $f, g, \cdots, y, z$, will usually stand for members of A^*, i, j, k, l, m and n will stand for elements of ω. (b) Bold face indicates sequences, in particular $\boldsymbol{u} = u_1, \cdots, u_k$, $\boldsymbol{x} = x_1, \cdots, x_n$ and $\boldsymbol{t}_i = t_1, \cdots, t_{n_i}$ (where n_i is the number of arguments taken by R_i). If, for example, $k = 0$ then $\boldsymbol{u}$ represents an empty sequence. (c) W will stand for a subset of A^*.

Let $\chi_1, \cdots, \chi_l$ be the representing functions of $R_1, \cdots, R_l$ respectively.

Our next project is to define a relation '$\{f\}_{\mathrm{pr}}(\boldsymbol{u}) =_W z$'. The definition can be got from the inductive definition of '$\{f\}_{\mathrm{pr}}(\boldsymbol{u}) = z$' implicit in [YNM] by omitting clause C1 and by relativizing the definition to W.

(2.2) '$\{f\}_{\mathrm{pr}}(\boldsymbol{u}) =_W z$' is defined inductively by:

C0$_i$ ($i = 1, \cdots, l$). If $f = \langle 0, n_i+n, i\rangle$ for some $n \in \omega$ and if $\{\boldsymbol{t}_i, \boldsymbol{x}\} \subseteq W$ and $\chi_i(\boldsymbol{t}_i) = z$ then $\{f\}_{\mathrm{pr}}(\boldsymbol{t}_i, \boldsymbol{x}) =_W z$.

C2. If $f = \langle 2, n+1\rangle$ and $\{\boldsymbol{x}, z\} \subseteq W$ then $\{f\}_{\mathrm{pr}}(z, \boldsymbol{x}) =_W z$.

C3. If $f = \langle 3, n+2\rangle$ and $\{s, t, \boldsymbol{x}, (s, t)\} \subseteq W$ then $\{f\}_{\mathrm{pr}}(s, t, \boldsymbol{x}) =_W (s, t)$.

C4$_0$. If $f = \langle 4, n+1, 0\rangle$ and $\{y, \boldsymbol{x}, \pi y\} \subseteq W$ then $\{f\}_{\text{pr}}(y, \boldsymbol{x}) =_W \pi y$.

C4$_1$. If $f = \langle 4, n+1, 1\rangle$ and $\{y, \boldsymbol{x}, \delta y\} \subseteq W$ then $\{f\}_{\text{pr}}(y, x) =_W \delta y$.

C5. If $f = \langle 5, n, g, h\rangle$, $\{g\}_{\text{pr}}(y, \boldsymbol{x}) =_W z$ and $\{h\}_{\text{pr}}(\boldsymbol{x}) =_W y$ then $\{f\}_{\text{pr}}(\boldsymbol{x}) =_W z$.

C6. (a) If $f = \langle 6, n+1, g, h\rangle$, $h \in PRI^0$ (see 2.3), $(h)_2 = n+4$, $y \in A^0$ and $\{g\}_{\text{pr}}(y, \boldsymbol{x}) =_W z$ then $\{f\}_{\text{pr}}(y, \boldsymbol{x}) =_W z$.

(b) If $f = \langle 6, n+1, g, h\rangle, y = (s, t), \{f\}_{pr}(s, \boldsymbol{x}) =_W u, \{f\}_{pr}(t, \boldsymbol{x}) =_W v$ and $\{h\}_{\text{pr}}(u, v, s, t, \boldsymbol{x}) = {}_W z$ then $\{f\}_{\text{pr}}(y, \boldsymbol{x}) =_W z$.

C7. If $f = \langle 7, n, j, g\rangle, j < n$ and $\{g\}_{\text{pr}}(x_{j+1}, x_1, \cdots, x_j, x_{j+2}, \cdots, x_n) =_W z$ then $\{f\}_{\text{pr}}(\boldsymbol{x}) =_W z$.

(2.3) Definition (See p. 432 of [YNM] and § 5 of this paper). The set PRI^0 is defined inductively by:

C0–C4. For all n and i such that $1 \leqq i \leqq l$; $\langle 0, n_i+n, i\rangle$, $\langle 2, n+1\rangle$, $\langle 3, n+2\rangle$, $\langle 4, n+1, 0\rangle$ and $\langle 4, n+1, 1\rangle$ are elements of PRI^0.

C5. If g and h are in PRI^0, $(g)_2 = n+1$ and $(h)_2 = n$ then $\langle 5, n, g, h\rangle \in PRI^0$.

C6. If g and h are in PRI^0, $(g)_2 = n+1$ and $(h)_2 = n+4$ then $\langle 6, n+1, g, h\rangle \in PRI^0$.

C7. If $g \in PRI^0$, $(g)_2 = n$ and $j < n$ then $\langle 7, n, j, g\rangle \in PRI^0$.

(2.4) We write $\{f\}_{\text{pr}}(\boldsymbol{u}) =^* z$ for $\{f\}_{\text{pr}}(\boldsymbol{u}) =_{A^*} z$.

(2.5) Definition[2]. (a) A function ψ on A^* is *absolutely primitive computable* (with respect to $\mathfrak{A}$) if, for some $f \in PRI^0$ and for all $\boldsymbol{u} \in A^*$,

$$\{f\}_{\text{pr}}(\boldsymbol{u}) =^* \psi(\boldsymbol{u}).$$

(b) A relation R on A^* is *absolutely primitive computable* if its representing function is and (c) R is σ_1^0 if there is some absolutely primitive computable relation S such that, for all $\boldsymbol{u} \in A^*$,

$$R(\boldsymbol{u}) \Leftrightarrow \exists y S(\boldsymbol{u}, y)$$

The σ_1^0 relations on A are the relations which are considered in [YNM] as analogs of the recursively enumerable relations.

Before proceeding with the proof of Theorem 1, we list some facts about the primitive computable and σ_1^0 relations which can be found in [YNM].

(2.6) (a) The relations $R_1, \cdots, R_l$, '$x \in A$' and '$x = 0$' are absolutely primitive computable.

[2] The definition of 'absolutely primitive computable' given here is different than but equivalent to the definition in [YNM]. See § 5.

(b) The absolutely primitive computable relations are closed under Boolean combinations and substitution by absolutely primitive computable functions.

(c) The absolutely primitive computable relations are closed under definitions by course-of-values induction (see Lemma 8 p. 438 of [YNM]).

(d) If S is absolutely primitive computable and
$R(\mathrm{i}, \mathbf{x}) \Leftrightarrow [i \in \omega \;\&\; \exists j < iS(j, \mathbf{x})]$ or
$R(i, \mathbf{x}) \Leftrightarrow [i \in \omega \;\&\; \forall j < iS(j, \mathbf{x})]$ then R is primitive computable.

(e) If S is σ_1^0 and, for all $\mathbf{x}$, $R(\mathbf{x}) \Leftrightarrow \exists y S(\mathbf{x}, y)$ then R is σ_1^0.

(f) The relations 'x is a sequence' and '$x \in \omega$' are absolutely primitive computable.

(g) The functions

$$lh(x) = \begin{cases} \text{the length of } x \text{ if } x \text{ is a sequence,} \\ 0 \text{ otherwise} \end{cases}$$

and

$$(x)_{j+1} = \begin{cases} x_{j+1} \text{ if } x = \langle x_1, \cdots, x_n \rangle, j \in \omega \text{ and } j < n, \\ 0 \text{ otherwise} \end{cases}$$

are absolutely primitive computable.

3.

We encode the elements of $HF(A)$ in A^*. The *decoding* function τ is a many-one function from a subset of A^* onto $HF(A)$. It is defined inductively by:

$$\tau x = x \text{ if } x \in A^0, \qquad \tau\langle x_1, \cdots, x_n \rangle = \{\tau x_1, \cdots, \tau x_n\} \text{ if } n \neq 0. \tag{3.1}$$

It is easy to show that τ is well defined, i.e., single valued, and is *onto* $HF(A)$.

Now associate with each relation R on $HF(A)$ the relation R^* on A^* defined by:

$$R^*(u_1, \cdots, u_k) \Leftrightarrow R(\tau u_1, \cdots, \tau u_k). \tag{3.2}$$

(3.3) Lemma. *The relation '$x \in domain(\tau)$' is an absolutely primitive computable relation on x.*

The proof is a direct application of (2.6).

(3.4) Lemma. *If R is a Δ_0 relation on $HF(A)$ then R^* is absolutely primitive computable.*

The proof is by induction on a Δ_0 definition of R. If R is one of R_i, ${}^{\sim}R_i$ or A then, since τ is the identity function on A, R^* is R and is absolutely primitive computable. If R is defined by $\neg \mathrm{A}(x)$ then $R^*(x) \Leftrightarrow$

$x \in \text{domain}(\tau)$ & $x \notin A$ so, by (3.3) and (2.6)–(b), R^* is absolutely primitive computable. If R is defined by a conjunction or disjunction then, by the induction hypothesis and (2.6)–(b), R^* is absolutely primitive computable. If R is of the form $\exists x \in yS(x, \boldsymbol{u})$ then from the fact that τ is onto $HF(A)$ it is easily seen that, for all $y, \boldsymbol{u} \in A^*$,

$$R^*(y, \boldsymbol{u}) \Leftrightarrow [y \in \text{domain}(\tau) \;\&\; \exists i < lh(y)\, S^*((y)_{i+1}, \boldsymbol{u})]$$

Now the relation $S^*((y)_{i+1}, \boldsymbol{u})$ is got from S^* by substitution of the absolutely primitive computable function $(y)_{i+1}$, the relation $\exists i < j\, S^*((y)_{i+1}, \boldsymbol{u})$ is got from that relation by quantification of the form of (2.6)–(e) and R^* is got from that relation by conjunction with the absolutely primitive computable relation '$x \in \text{domain}(\tau)$' and by substitution of the absolutely primitive computable function $lh(x)$. Therefore R^* is absolutely primitive computable. Similarly, if R is of the form $\forall x \in y\; S(x, \boldsymbol{u})$ then R^* is absolutely primitive computable.

MAIN LEMMA TO THEOREM 1. *Every relation R on A which is a Σ-relation is a σ_1^0-relation.*

PROOF. By (1.10), R is of the form $\exists y S(u_1, \cdots, u_k, y)$, where S is a Δ_0-relation. From the fact that τ is onto $HF(A)$, for all $\boldsymbol{u}$,

$$R(\boldsymbol{u}) \Leftrightarrow \exists y S(\boldsymbol{u}, \tau y).$$

Since $R(\boldsymbol{u})$ holds only for $\boldsymbol{u} \in A$, in which case $\tau u_i = u_i$ $(i = 1, \cdots, k)$,

$$R(\boldsymbol{u}) \Leftrightarrow \exists y S^*(\boldsymbol{u}, y).$$

Hence, by (3.4), R is a σ_1^0-relation.

THEOREM 1. *Every relation on A which is a Σ^t-relation is a σ_1^0-relation.* This is an immediate consequence of the main lemma above and (1.9).

REMARK. Theorem 1 is half of our main result, since it is an immediate corollary that every relation R on A which is 'recursive' in the sense of [RM] is 'recursive' in the sense of[YNM].

4.

We now set out to prove the converse to Theorem 1 (with a certain restriction).

(4.1) LEMMA. *If $W \subseteq W'$ and $\{f\}_{\text{pr}}(\boldsymbol{u}) =_W z$ then $\{f\}_{\text{pr}}(\boldsymbol{u}) =_{W'} z$.* The proof is by an easy induction over the definition of '$\{f\}_{\text{pr}}(\boldsymbol{u}) =_W z$'.

(4.2) LEMMA. *If $\{f\}_{\text{pr}}(\boldsymbol{u}) =_W z$ then there is a finite subset W' of W such that $\{f\}_{\text{pr}}(\boldsymbol{u}) =_{W'} z$.*

The proof is by an easy induction over the definition of '$\{f\}_{\mathrm{pr}}(\boldsymbol{u}) =_W z$'. If, for example, $\{f\}_{\mathrm{pr}}(\boldsymbol{x}) =_W z$ holds by clause C5 of the definition then $f = \langle 5, n, g, h\rangle$ and there is a y and, by the induction hypothesis, there are finite subsets W_1 and W_2 of W such that $\{g\}_{\mathrm{pr}}(y, \boldsymbol{x}) =_{W_1} z$ and $\{h\}_{\mathrm{pr}}(\boldsymbol{x}) =_{W_2} y$. Letting $W' = W_1 \cup W_2$ we have, by Lemma (4.1) and by clause C5, that $\{f\}_{\mathrm{pr}}(\boldsymbol{x}) =_{W'} z$.

Assume for the remainder of this paper that both the equality and inequality relations on A are Σ^t definable in $\mathfrak{A}^t$.

(4.3) Definitions.

(a) $x \in_1 y \equiv x \in y$, $x \in_{n+1} y \equiv \exists z[x \in_n z \;\&\; z \in y]$.

(b) $\{x\}_1 = \{x\}$, $\{x\}_{n+1} = \{\{x\}\}_n$.

(c) $A^n = \{\{x\}_n \,|\, x \in A\}$.

(d) $(x_1, x_2) = \{\{x_1, x_2\}, \{x_2\}\}$,
$(x_1, \cdots x_{n+2}) = ((x_1, \cdots, x_{n+1}), \{x_{n+2}\}_{2n})$.

(4.4) Remarks. (a) If $x_1, \cdots, x_{n+1}$ are elements of U^k then $(x_1, \cdots, x_{n+1}) \in U^{k+2n}$. (b) In view of the preceeding we may (and do) identify finite $n+1$ ary relations on U^k with certain elements of U^{k+2n+1}.

(4.5) Definition. The property $P^k(p)$ holds for $p \in U^{k+5}$ if p is a 3-place relation on U^k and there is some $q \in U^{k+3}$ which is a 2-place relation on U^k such that

(a) p is a one-one function on a subset of $U^k \times U^k$, i.e., if $(u, v, w) \in p$ and $(u', v', w') \in p$ then $(u, v) = (u', v') \Leftrightarrow w = w'$.

(b) If $(u, v, w) \in p$ then $w \neq 0$ and $w \notin A^k$.

(c) If $(u, v, w) \in p$, $x \in \{u, v\}$, $x \neq 0$ and $x \notin A^k$ then $\exists u' \exists v' [(u', v', x) \in p]$.

(d) If $(u, v, w) \in p$ then $(u, w) \in q$ and $(v, w) \in q$.

(e) If $(x, y) \in q$ and $(y, z) \in q$ then $(x, z) \in q$.

(f) If $(x, y) \in q$ then $x \neq y$.

One should think of $P^k(p)$ as meaning that p is a pairing relation on a finite part of U^k, $(u, v, w) \in p$ as meaning that w represents the pair (u, v) and $(x, y) \in q$ as meaning that x 'preceeds' y in the pairing structure determined by p.

(4.6) Definition. For each p such that $P^k(p)$ holds, define a function ρ_p from a subset A^* into U^k inductively by:

(a) $\rho_p 0 = 0$,

(b) $\rho_p x = \{x\}_k$, if $x \in A$ and

(c) If $\rho_p x$ and $\rho_p y$ are defined and $(\rho_p x, \rho_p y, w) \in p$ then $\rho_p(x, y) = w$.

(4.7) Lemma. *The following are Σ^t-relations and so are their complements relative to the appropriate domains*: (a) $x \in_n y$, *restricted to* $U^k \times U^{k+n}$, (b) $x = y$, *restricted to* $U^k \times U^k$, (c) $x = 0$, *restricted to* U^k,

(d) $x = \{y\}_n$, *restricted to* $U^{k+n} \times U^k$, (e) $x \in A^n$ *restricted to* U^n *and* (f) $x = (x_1, \cdots, x_{n+1})$ *restricted to* $U^{k+2n} \times U^k \times \cdots \times U^k$.

To prove (4.7) one merely writes out the various definitions in the obvious way and checks that the definitions are in Σ^t form.

(4.8) Lemma. *If* $P^k(p)$ *holds then* ρ_p *is well defined and one-one.*

Proof. It is easy to see, from (4.5)–(b), that if $\rho_p x = y$ then exactly one of the following must hold: (i) $x = y = 0$, (ii) $x \in A$ and $y = \{x\}_k$ or (iii) $\exists s, t, u, v\ [x = (s, t)\ \&\ (u, v, y) \in p\ \&\ \rho_p s = u\ \&\ \rho_p t = v]$. To show that ρ_p is well defined, assume $\rho_p x = y_1$ and $\rho_p x = y_2$ and show by induction on $x \in A^*$ that $y_1 = y_2$. If $x \in A^0$ then either case (i) or case (ii) holds so $\rho_p x$ is uniquely determined and $y_1 = y_2$. If $x = (s, t)$ then case (iii) holds so $\exists u_1, u_2, v_1, v_2\ [(u_1, v_1, y_1) \in p\ \&\ (u_2, v_2, y_2) \in p\ \&\ \rho_p s = u_1\ \&\ \rho_p s = u_2\ \&\ \rho_p t = v_1\ \&\ \rho_p t = v_2]$. By the induction hypothesis applied to s and t, $u_1 = u_2$ and $v_1 = v_2$ so, by (4.5)–(a), $y_1 = y_2$. Let q be an element of U^{k+3} satisfying (4.5)–(d), (e) and (f). As a relation on U^k, q is a finite, and hence well-founded, partial ordering. To show that ρ_p is one-one assume $\rho_p x_1 = y$ and $\rho_p x_2 = y$. If one of cases (i) or (ii) holds then clearly $x_1 = x_2$. If case (iii) holds, we show that $x_1 = x_2$ by q-induction on y. Assume $x_1 = (s_1, t_1)$, $x_2 = (s_2, t_2)$, $(u_1, v_1, y) \in p$, $(u_2, v_2, y) \in p$, $\rho_p s_1 = u_1$, $\rho_p s_2 = u_2$, $\rho_p t_1 = v_1$ and $\rho_p t_2 = v_2$. By (4.5)–(a), $u_1 = u_2$ and $v_1 = v_2$ and by (4.5)–(d), $(u_1, y) \in q$ and $(v_1, y) \in q$. Applying the q-induction hypothesis to u_1 and v_1, we get $s_1 = s_2$ and $t_1 = t_2$. Therefore, $x_1 = x_2$.

(4.9) Lemma. (a) $A^0 \subseteq$ *domain*(ρ_p). (b) *If* $(s, t) \in$ *domain*(ρ_p) *then* $\{s, t\} \subseteq$ *domain*(ρ_p). (c) *If* $(u, v, w) \in p$ *then* $\{u, v, w,\} \subseteq$ *range*(ρ_p).

Proof. (a) is immediate from the definition of ρ_p, (b) follows from the proof of (4.8). (c) can be easily proved by q-induction on w as follows: if $(u, v, w) \in p$ then $(u, w) \in q$ and $(v, w) \in q$. Let x be one of u, v. Either $x = 0$ or $x \in A^k$, in which case $x \in \text{range}(\rho_p)$ or, by (4.5)–(c), $\exists u' \exists v' [(u', v', x) \in p]$. Applying the q-induction hypothesis to x we get that $x \in \text{range}\ (\rho_p)$. Hence, in any case, $\{u, v\} \subseteq \text{range}(\rho_p)$ so, by the definition of ρ_p, w is also an element of range(ρ_p).

(4.10) Lemma. *The relations* $P^k(p)$ *and* $[P^k(p)\ \&\ x \in$ *range*$(\rho_p)]$ *are* Σ^t *definable in* $\mathfrak{A}^t$.

Proof. Simply write out the definition of $P^k(p)$ and observe that it is in Σ^t form. For p such that $P^k(p)$ holds, $x \in \text{range}(\rho_p) \Leftrightarrow [x = 0$ or $x \in A^k$ or $\exists u\, \exists v[(u, v, x) \in p]]$.

(4.11) DEFINITION. For each n, k and each $f \in PRI^0$ with $(f)_2 = k$, let R^n_f be the relation defined by:

$$R^n_f(p, \boldsymbol{u}, z) \equiv P^n(p) \;\&\; \{\boldsymbol{u}, z\} \subseteq \text{range}(\rho_p) \\ \&\; \{f\}_{\text{pr}}(\rho_p^{-1} u_1, \cdots, \rho_p^{-1} u_k) =_{D(p)} \rho_p z,$$

where $D(p) = \text{domain}(\rho_p)$.

(4.12) LEMMA. *R^n_f is a Σ^t-relation.*

The proof is by induction on $f \in PRI^0$. We take three sample cases and indicate how to express the relation in Σ^t form. The remaining cases are left for the reader.

Case CO_ι, $f = \langle 0, n_i + n, i\rangle$. $R^k_f(p, \boldsymbol{t}_i, \boldsymbol{x}, z) \Leftrightarrow P^k(p)$ & $\{\boldsymbol{t}_i, \boldsymbol{x}, z\} \subseteq \text{range}(\rho_p)$ & $[\exists\, t'_1 \in_k t_1 \cdots \exists\, t'_{n_i} \in_k t_{n_i} [t_1 = \{t'_1\}_k \;\&\; \cdots \;\&\; t_{n_i} = \{t'_{n_i}\}_k$ & $[[R_i(t'_1, \cdots, t'_{n_i})$ & $z = 0]$ or $[{}^{\sim}R_i(t'_1, \cdots, t'_{n_i})$ & $(0, 0, z) \in p]]]$ or $[\forall\, t'_1 \in_k t_1 \cdots \forall\, t'_{n_i} \in_k t_{n_i} [t_1 \neq \{t'_1\}_k$ or $\cdots$ or $t_{n_i} \neq \{t'_{n_i}\}_k]$ & $(0, 0, z) \in p]]$.

Case C3, $f = \langle 3, n+2\rangle$. $R^k_f(p, s, t, \boldsymbol{x}, z) \Leftrightarrow P^k(p)$ & $\{s, t, \boldsymbol{x}, z\} \subseteq \text{range}(\rho_p)$ & $(s, t, z) \in p$.

Case C6, $f = \langle 6, n+1, g, h\rangle$. $R^k_f(p, y, \boldsymbol{x}, z) \Leftrightarrow \exists\, r[r$ is a binary relation on U^k & $(y, z) \in r$ & $\forall\, w \in r\, \exists y' \in_2 w\, \exists z' \in_2 w[w = (y', z')$ & $[[y' \in A^k \cup \{0\}$ & $R^k_g(p, y', \boldsymbol{x}, z')]$ or $\exists s\, \exists t\, \exists u\, \exists v\, [(s, t, y') \in p$ & $(s, u) \in r$ & $(t, v) \in r$ & $R^k_h(p, u, v, s, t, \boldsymbol{x}, z')]]]]$.

(4.13) LEMMA. *Let W be a finite subset of A^*,* (a) *if A is infinite then there is a p such that $P^1(p)$ and $W \subseteq domain(\rho_p)$ and* (b) *if A is finite there is an n and a p such that $P^n(p)$ and $W \subseteq domain(\rho_p)$.*

PROOF. Let $W^\square$ be the closure of W under π and δ and let $D = W^\square \cup A^0$. If A is infinite then A, A^1, D and U^1 all have the same cardinality so there is a one-one function γ from D into U^1 such that (i) $\gamma^0 = 0$ and (ii) $\gamma x = \{x\}$ if $x \in A$. Let $p = \{(\gamma u, \gamma v, \gamma(u, v)) \mid (u, v) \in W^\square\}$. It is now easy to show that $P^1(p)$ holds and that $W \subseteq D = \text{domain}(\rho_p)$.

If A is finite then $\{\text{cardinality}(U^n) : n = 0, 1, \cdots\}$ is an increasing sequence while, for all n, $\text{cardinality}(A^n) = \text{cardinality}(A)$. So, for n sufficiently large, there is a one-one function γ from the finite set D into U^n such that (i) $\gamma 0 = 0$ and (ii) $\gamma x = \{x\}_n$, for $x \in A$. Let $p = \{(\gamma u, \gamma v, \gamma(u, v) \mid (u, v) \in W^\square\}$. It can now be shown that $P^n(p)$ holds and that $W \subseteq D = \text{domain}(\rho_p)$.

(4.14) LEMMA. *If A is infinite and R is a σ^0_1 relation on A then, for some $f \in PRI^0$,*

$$R(\boldsymbol{u}) \Leftrightarrow \exists\, y\, \exists\, p R^1_f\,(p, \{u_1\}, \cdots, \{u_k\}, y, 0).$$

PROOF. Pick $f \in PRI^0$ such that, for all $\boldsymbol{u}$,

$$R(\boldsymbol{u}) \Leftrightarrow \exists\, y[\{f\}_{\text{pr}}(\boldsymbol{u}, y) =^* 0].$$

Suppose that $R(\boldsymbol{u})$ then, by (4.2) $\exists\, y \,\exists\, W[W$ is finite and $\{f\}_{\text{pr}}(\boldsymbol{u}, y) =_W 0]$. Now, by (4.13) and (4.1), $\exists y\, \exists p[P^1(p)$ and $\{f\}_{\text{pr}}(\boldsymbol{u}, y) =_{\text{domain}(\rho_p)} 0]$. Hence $\exists y\, \exists p R^1_f(p, \rho_p \boldsymbol{u}, \rho_p y, \rho_p 0)$. Since by assumption R is a relation on A and $R(\boldsymbol{u})$ holds we have $\boldsymbol{u} \in A$ so $\rho_p \boldsymbol{u} = \{u_1\}, \cdots, \{u_k\}$. Therefore $\exists y\, \exists p R^1_f(p, \{u_1\}, \cdots, \{u_k\}, y, 0)$. That $R^1_f(p, \{u_1\}, \cdots, \{u_k\}, y, 0)$ implies $R(\boldsymbol{u})$ is immediate from the definitions.

(4.15) Lemma. *If A is finite and R is a σ^0_1-relation on A then, for some $f \in PRI^0$ and some n,*

$$R(\boldsymbol{u}) \Leftrightarrow \exists y\, \exists p R^n_f(p, \{u_1\}_n, \cdots, \{u_k\}_n, y, 0).$$

Proof. Pick $f \in PRI^0$ such that, for all $\boldsymbol{u}$,

$$R(\boldsymbol{u}) \Leftrightarrow \exists y[\{f\}_{\text{pr}}(\boldsymbol{u}, y) =^* 0].$$

By (4.1) and (4.2),

$$R(\boldsymbol{u}) \Leftrightarrow \exists y\, \exists W\, [W \text{ is finite and } \{f\}_{\text{pr}}(\boldsymbol{u}, y) =_W 0].$$

Now R is a finite relation so there is a finite class $\mathscr{W}$ of finite subsets of A^* such that

$$R(\boldsymbol{u}) \Leftrightarrow \exists y\, \exists W \in \mathscr{W}[\{f\}_{\text{pr}}(\boldsymbol{u}, y) =_W 0].$$

Let $X = \cup\, \mathscr{W}$, then X is finite so, by (4.13), there is an n and a p such that $P^n(p)$ and $X \subseteq \text{domain}(\rho_p)$. Assume that $R(\boldsymbol{u})$. Then, for some y and some $W \in \mathscr{W}$, $\{f\}_{\text{pr}}(\boldsymbol{u}, y) =_W 0$. Now $W \subseteq X \subseteq \text{domain}(\rho_p)$ so $\{f\}_{\text{pr}}(\boldsymbol{u}, y) =_{\text{domain}(\rho_p)} 0$, hence $R^n_f(p, \rho_p \boldsymbol{u}, \rho_p y, \rho_p 0)$. By our assumptions on R and $\boldsymbol{u}$, $\boldsymbol{u} \in A$ so $\rho \boldsymbol{u} = \{u_1\}_n, \ldots, \{u_k\}_n$, therefore $\exists y\, \exists p R^n_f(p, \{u_1\}_n, \cdots, \{u_k\}_n, y, 0)$. That $R^n_f(p, \{u_1\}_n, \cdots, \{u_k\}_n, y, 0)$ implies $R(\boldsymbol{u})$ follows directly from the definitions.

From (4.7), (4.12). (4.14) and (4.15) we have

Theorem 2. *If the equality relation on A and its complement relative to A are Σ^t-relations then every σ^0_1-relation on A is a Σ^t-relation.*

This completes the proof of our main result since (in the case that equality is 'recursive') it is an immediate corollary that every relation which is 'recursive' in the sense of [YNM] is 'recursive' in the sense of [RM]. The problem of strengthening Theorem 2 by removing the requirement that equality be 'recursive' remains open.

5. Reconciliation of the definitions given in [YNM] and the definitions of this paper, computability from parameters.

(5.1) Definition ([YNM], p 432). The set *PRI* is defined inductively. The definition can be obtained from the definition of PRI^0 by (a) replacing PRI^0 by PRI throughout and (b) adjoining the additional clause:

C1. For all $z \in A^*$ and for all $n \in \omega$, $\langle 1, n, z\rangle \in PRI$

(5.2) DEFINITION. The relation $\{f\}_{\text{pr}}(u_1, \cdots, u_k) = z$ is defined inductively. The definition can be obtained from the definition of $\{f\}_{\text{pr}}(u_1, \cdots, u_k) =_{A^*} z$ (2.2) by (a) omitting the subscript 'A^*' throughout, (b) replacing 'PRI^0' by 'PRI' in clause C6 and (c) adjoining the clause:

C1. If $f = \langle 1, n, z\rangle$ then $\{f\}_{\text{pr}}(\boldsymbol{x}) = z$.

(5.3) DEFINITION ([YNM], p. 429). For each subset W of A^* let $W^* =$ the closure of $W \cup \{0\}$ under π, δ and $\lambda xy(x, y)$.

(5.4) DEFINITION ([YNM]). (a) If $W \subseteq A^*$, and ψ is a k-place function on A^* then ψ is *primitive computable from* W if there is an $f \in PRI \cap W^*$ such that, for all $\boldsymbol{u}$,

$$\{f\}_{\text{pr}}(\boldsymbol{u}) = \psi(\boldsymbol{u}).$$

(b) A relation R on A^* is *primitive computable from* W if its representing function is and (c) R is a $\sigma_1^0(W)$*-relation* if there is a relation S, which is primitive computable from W, such that for all $\boldsymbol{u} \in A^*$,

$$R(\boldsymbol{u}) \Leftrightarrow \exists y S(\boldsymbol{u}, y).$$

The definition of 'ψ is primitive computable from W' is such that constant functions may be used in the definition of ψ but only for parameters (constants) from W^*. An alternative but equivalent definition would only allow parameters from $W^* \cap A$. These definitions are equivalent since π, δ, $\lambda xy(x, y)$ and the constantly 0 functions are absolutely primitive computable. In [YNM], a function is called absolutely primitive computable if it is primitive computable from 0. By the preceeding remarks this can be seen to be equivalent to the definition given here.

(5.5) LEMMA. *A k-place function* ψ *is primitive computable from a subset* W *of* A^* *if and only if there is a finite subset* $\{c_1, \cdots, c_n\}$ *of* $W^* \cap A$ *and an absolutely primitive computable,* $k+n$ *place function* ϕ *such that, for all* $\boldsymbol{u} \in A^*$,

$$\psi(\boldsymbol{u}) = \phi(\boldsymbol{u}, \boldsymbol{c}).$$

PROOF. The implication from right to left is immediate, since ψ is obtained from ϕ and the constant functions $c_1, \cdots, c_n$ by substitution. The implication from left to right is proved by induction on a primitive computable index f for ψ such that $f \in PRI \cap W^*$, If $f \in PRI$ by one of clauses C0, C2, C3, or C4 then ψ is already absolutely primitive computable. If $f \in PRI$ by clause C5, then $f = \langle 5, n, g, h\rangle$ and g and h are necessarily in W^* since f is. By the induction hypothesis there are $c_1, \cdots, c_m, c_{m+1}, \cdots, c_p$ in $W^* \cap A$ and absolutely primitive computable ϕ_1 and ϕ_2 such that, for all y, $\boldsymbol{x}$,

$$\{g\}_{\mathrm{pr}}(y, \boldsymbol{x}) = \phi_1(y, \boldsymbol{x}, c_1 \cdots c_m) \quad \text{and}$$
$$\{h\}_{\mathrm{pr}}(\boldsymbol{x}) = \phi_2(\boldsymbol{x}, c_{m+1}, \cdots, c_p).$$

Let $\phi(\boldsymbol{x}, c_1, \cdots, c_p) = \phi_1(\phi_2(\boldsymbol{x}, c_{m+1}, \cdots, c_p), \boldsymbol{x}, c_1, \cdots, c_m)$ then ϕ is absolutely primitive computable and $\psi(\boldsymbol{x}) = \phi(\boldsymbol{x}, c_1, \cdots, c_p)$. Clauses C6 and C7 are handled similarly to C5. The remaining clause is C1. It is clearly sufficient to show that for each $z \in W^*$ the constant function

$$\psi_z(\boldsymbol{u}) = z$$

satisfies the lemma. If $z = 0$ then ψ_z itself is absolutely primitive computable. If $z \in A$, let ϕ be the function $\phi(\boldsymbol{u}, z) = z$. Then ϕ is absolutely primitive computable and $\psi_z(\boldsymbol{u}) = \phi(\boldsymbol{u}, z)$. If $z = (s, t)$ and if there are $c_1, \cdots, c_p \in W^* \cap A$ and absolutely primitive computable functions ϕ_1 and ϕ_2 such that, for all $\boldsymbol{u}$, $\phi_1(\boldsymbol{u}, c_1, \cdots, c_m) = s$ and $\phi_2(\boldsymbol{u}, c_{m+1}, \cdots, c_p) = t$ then let ϕ be the function such that, for all $\boldsymbol{u}, v_1, \cdots, v_p$,

$$\phi(\boldsymbol{u}, v_1, \cdots, v_p) = (\phi_1(\boldsymbol{u}, v_1, \cdots, v_m), \phi_2(\boldsymbol{u}, v_{m+1}, \cdots, v_p)).$$

Now ϕ is absolutely primitive computable and, for all $\boldsymbol{u}$, $\psi_z(\boldsymbol{u}) = \phi(\boldsymbol{u}, c_1, \cdots, c_p)$. Hence we have shown, by induction on $z \in A^*$, that if $z \in W^*$ then ψ_z satisfies the lemma.

For each subset W of A let $\Sigma^t(W)$ be the language Σ^t enriched by adding constants for elements of W. The above lemma gives the following strengthened versions of Theorems 1 and 2. Let W be a subset of A.

THEOREM 1′. *Every relation on A which is definable in $\mathfrak{A}^t$ by a formula of $\Sigma^t(W)$ is a $\sigma_1^0(W)$-relation.*

THEOREM 2′. *If the equality relation on A and its complement relative to A are definable in $\mathfrak{A}^t$ by $\Sigma^t(W)$ formulas, then every $\sigma_1^0(W)$ relation on A is $\Sigma^t(W)$ definable in $\mathfrak{A}^t$.*

REFERENCES

C. GORDON

[G] A comparison of abstract computability theories, Doctoral Dissertation, University of California at Los Angeles (1968).

S. KRIPKE

[K] Transfinite recursions on admissible ordinals. I, II (abstracts), J. Symb. Logic, 29 (1964), 162.

R. MONTAGUE

[RM] Recursion theory as a branch of model theory, Proc. of the third international congr. on Logic, Methodology and the Philos. of Sc., 1967, Amsterdam, 1968, 63–86.

Y. N. Moschovakis

[YNM] Abstract first order computability. I, II, Trans. Amer. Math. Soc. 138 (1969), 427–464.

R. Platek

[P] Foundations of recursion theory, Doctoral Dissertation and supplement. Stanford University (1966).

(Oblatum 19–I–70)

Mathematisch Instituut der Rijksuniversiteit
Utrecht