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TWO NEAR-ISOMETRY INVARIANTS OF BANACH SPACES

by

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1. Introduction

In [1, p. 242], Banach defines the quantity $D(X, Y) = \log \inf ||T|| \, ||T^{-1}||$, for topologically isomorphic normed linear spaces X and Y, the infimum being taken over all isomorphisms T mapping X onto Y. If $D(X, Y) = 0$ then X and Y are said to be nearly isometric. An unpublished example due to Pelczyński shows that two Banach spaces can be nearly isometric without being isometric. Thus the properties of Banach spaces that are invariant under near-isometry form a proper subset of the properties invariant under isometry. In this paper we present two numerical-valued functions of Banach spaces related to the metric geometry of the unit ball and show, among other things, that they are invariant under near-isometry.

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2. The numerical functions

Let $S(X)$ denote the unit ball of the normed space X. If K is a compact Hausdorff space, $C(K)$ is the space of all continuous scalar-valued functions on K normed with the supremum norm. $f \in C(K)$ is an extreme point of $||f||S(C(K))$ iff $|f(k)| = ||f||$ for all $k \in K$. It is easily seen that if g is any element of $C(K)$ and f is an extreme point of $||f||S(C(K))$ then $\sup_{|\alpha|=1} ||f+\alpha g|| = ||f||+||g||$. Imitating this phenomenon, we are led to define a sort of measure of the existence of extreme points on the unit ball of a Banach space. We know that if x is extreme on $||x||S(X)$ then for every $y \neq 0$, at least one of $||x+y||$ and $||x-y||$ is greater than $||x||$. Our adaptation of this fact requires that the size of $\sup_{|\alpha|=1} ||x+\alpha y||$ depends on $||y||$.

Definition 1: If X is a Banach space then

$$\xi(X) = \sup\{\gamma : \forall\varepsilon > 0\ \exists x \in X \text{ with } \|x\| \geqq 1, \text{ and } \forall y \in X, \\ \sup_{|\alpha|=1} \|x+\alpha y\| \geqq \|x\|+\gamma\|y\|-\varepsilon\}.$$

From the triangle inequality it follows that $0 \leqq \xi(X) \leqq 1$, and the above discussion shows that $\xi(C(K)) = 1$ for all K.

THEOREM 1: *If $D(X, Y) = 0$, then $\xi(X) = \xi(Y)$*

PROOF: By hypothesis, there is a sequence $\{T_n\}$ of isomorphisms of X onto Y such that $\|x\| \leqq \|T_n x\| \leqq (1+1/n)\|x\|$ for all $x \in X$. Let $\varepsilon > 0$ be given and let $0 \leqq \gamma \leqq \xi(X)$. Then there is an $x \in X$ such that $\|x\| \geqq 1$ and $\sup_{|\alpha|=1} \|x+T_n^{-1}\alpha y\| \geqq \|x\|+\gamma\|T_n^{-1}y\|-\varepsilon/(2+\gamma)$ for all $y \in Y$ and all n. Now, for every n we have

$$\sup_{|\alpha|=1} \|T_n x+\alpha y\| \geqq \|x\|+\gamma\|T_n^{-1}y\|-\varepsilon/(2+\gamma).$$

We can choose N large enough that $\|x\| \geqq \|T_N x\|-\varepsilon/(2+\gamma)$ and $\|T_N^{-1}y\| \geqq \|y\|-\varepsilon/(2+\gamma)$. Then we have $\sup_{|\alpha|=1}\|T_N x+\alpha y\| \geqq \|T_N x\|+\gamma\|y\|-\varepsilon$. Thus $\gamma \leqq \xi(X)$ implies that $\gamma \leqq \xi(Y)$. The argument is symmetric in X and Y, so it follows that $\xi(X) = \xi(Y)$. Q.E.D.

REMARK: We have seen that $\xi(C(K)) = 1$ for all K. It is easily shown that if Q is a non-compact, locally compact Hausdorff space, and if $C_0(Q)$ is the Banach space of all continuous, scalar-valued functions on Q that vanish at infinity, normed by the supremum norm, then $\xi(C_0(Q)) = 0$. In particular $\xi(c) = 1$ and $\xi(c_0) = 0$, providing a new proof that $D(c, c_0) > 0$. The question of whether $D(c, c_0) > 0$ originated with Banach [1, p. 243], and a considerable amount of work has been devoted to it [2, 3, 4]. However, our proof has the merit that it relates the fact that $D(c, c_0) > 0$ to the extreme point structure of the unit balls of these spaces.

DEFINITION 2: If X is a Banach space we define

$$\eta(X) = \sup\{\gamma : \forall\varepsilon > 0, \forall x\|x\| \geqq 1, \exists y\|y\| \geqq \gamma \text{ and} \\ \sup_{|\alpha|=1} \|x+\alpha y\| \leqq \|x\|+\varepsilon\}$$

In the case of real scalars, to say that $\eta(X) > 0$ is to say that for every x on the unit sphere one can find a segment of length arbitrarily close to $2\eta(X)$ that is arbitrarily close to the surface of the unit ball, and such that x is the midpoint of the segment.

THEOREM 2: *If $D(X, Y) = 0$, then $\eta(X) = \eta(Y)$.*

PROOF: By hypothesis there is a sequence $\{T_n\}$ of isomorphisms mapping Y onto X such that $\|y\| \leqq \|T_n y\| \leqq (1+1/n)\|y\|$, for all

$y \in Y$. Suppose first that $\eta(X) > 0$ and $\eta(Y) > 0$ and let $\gamma < \eta(X)$. Then for every $\varepsilon > 0$ and every $x \in X$ such that $||x|| \geqq 1$ there is a w such that $||w|| \geqq \gamma$ and $\sup_{|\alpha|=1} ||x+\alpha w|| \leqq ||x||+\varepsilon$. Let $\varepsilon_0 > 0$ and $y \in Y$ with $||y|| \geqq 1$ be given. Then $||T_n y|| \geqq 1$ for every n, so for each n there is a $w_n \in X$ with $||w_n|| \geqq \gamma$ such that $\sup_{|\alpha|=1} ||T_n y+\alpha w_n|| \leqq ||T_n y|| +\varepsilon_0/2$. Since T_n is norm-increasing, T_n^{-1} is norm-decreasing for every n, so we have $\sup_{|\alpha|=1} ||y+\alpha T_n^{-1} w_n|| \leqq ||T_n y||+\varepsilon_0/2$. There is an N such that if $n \geqq N$ then $||T_n y|| \leqq ||y||+\varepsilon_0/2$. Thus, if $n \geqq N$ we have $\sup_{|\alpha|=1} ||y+\alpha T_n^{-1} w_n|| \leqq ||y||+\varepsilon_0$. We also have $||T_n^{-1} w_n|| \geqq ||w_n|| n/(n+1)$ for all n. Thus we know that if $n \geqq N$ and $\gamma \leqq \eta(X)$, then $\eta(Y) \geqq n\gamma/(n+1)$. But $n/(n+1) \to 1$ as $n \to \infty$, so we conclude that $\eta(Y) \geqq \gamma$. This proves that $\eta(Y) \geqq \eta(X)$ and interchanging X and Y in the argument above proves the reverse inequality.

If both $\eta(X)$ and $\eta(Y)$ are 0 there is nothing to prove. Suppose, without loss of generality, that $\eta(X) > 0$ and $\eta(Y) = 0$. The argument above shows that if $\eta(X) > 0$ then $\eta(Y) > 0$, proving that if either $\eta(X) = 0$ or $\eta(Y) = 0$ and $D(X, Y) = 0$ then both must be 0. Q.E.D.

REMARK: It is easily shown that if one of $\eta(X)$ and $\xi(X)$ is non-zero then the other is 0. It follows then that $\eta(C(K)) = 0$ for any compact Hausdorff space K. A simple computation shows that $\eta(C_0(Q)) = 1$ for Q a locally compact non-compact Hausdorff space. Thus $\eta(c) = 0$ and $\eta(c_0) = 1$ giving another proof that $D(c, c_0) > 0$.

3. The continuity of ξ and η

Banach observed in [1, p. 243] that D defines a pseudometric on the class of all Banach spaces topologically isomorphic to a Banach space X, with isometric spaces identified. Since ξ and η are functions from this pseudometric space to the real line, it is natural to ask about their continuity. In this direction we have the following two results.

EXAMPLE: Let K be an infinite compact Hausdorff space and let $p \in K$ be a point which is not isolated. Let $||\ ||$ denote the supremum norm on $C(K)$ and let $|||\ |||_\alpha$, for $0 < \alpha \leqq 1$ denote the norm on $C(K)$, equivalent to $||\ ||$, that is defined by

$$|||f|||_\alpha = \max \{\alpha||f||, |f(p)|\}.$$

It can be shown that $\xi((C(K), |||\ |||_\alpha)) = 0$ when $\alpha < 1$. When $\alpha = 1$ we have $|||\ |||_1 = ||\ ||$. It can also be shown that $D((C(K), ||\ ||), (C(K), |||\ |||_\alpha)) \leqq \log (1/\alpha)$. Thus, as $\alpha \to 1$, $\xi((C(K), |||\ |||_\alpha)) \nrightarrow 1$.

THEOREM 3: *If* $D(X_n, X) \to 0$ *as* $n \to \infty$ *then* $\eta(X) \geqq \limsup \eta(X_n)$.

PROOF: By hypothesis there exist norm-increasing isomorphisms $T_n : X \to X_n$ which are onto and such that $||T_n|| \leqq 1+1/n$. Let $\varepsilon > 0$ and $x \in X$ such that $||x|| \geqq 1$ be arbitrary but fixed. Then for each n there exists $x_n \in X_n$ such that $||x_n|| \geqq \eta(X_n)-1/n$ and $\sup_{|\alpha|=1} ||T_n x+\alpha x_n|| \leqq ||T_n x||+\varepsilon/2$. T_n^{-1} is norm-decreasing so $\sup_{|\alpha|=1} ||x+\alpha T_n^{-1} x_n|| \leqq ||T_n x||+\varepsilon/2$. There exists an N such that if $n > N$ then $||T_n x|| \leqq ||x||+\varepsilon/2$. Thus, for $n > N$, we have $\sup_{|\alpha|=1} ||x+\alpha T_n^{-1} x_n|| \leqq ||x||+\varepsilon$. From the inequality $||x_n||/||T_n|| \leqq ||T_n^{-1} x_n||$ and the above we obtain $(1/||T_n||)(\eta(X_n)-1/n) \leqq ||T_n^{-1} x_n||$ for $n > N$. Taking limits superior gives us $\eta(X) \geqq \limsup \eta(X_n)$, since $||T_n|| \to 1$ and $1/n \to 0$. Q.E.D.

BIBLIOGRAPHY

S. BANACH
[1] Théorie des Opérations Linéaires, Chelsea, New York, 1955.

M. CAMBERN
[2] A Generalized Banach-Stone Theorem, Proc. Amer. Math. Soc. 18 (1967) 1062–1066.

M. CAMBERN
[3] On Mappings of Sequence Spaces, Studia Math. 30 (1968) 73–77.

R. WHITLEY
[4] The Size of the Unit Sphere, Canad. J. Math. 20 (1968) 450–455.

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