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HOMOTOPY TYPE OF MAPPING TRACKS

by

F. H. Croom

1. Introduction

Let (E, e_0) and (B, b_0) be pointed spaces and $p: (E, e_0) \to (B, b_0)$ a continuous function. If (E, p, B) has the weak covering homotopy property, it follows that the basic fiber $F = p^{-1}$ (b_0) and the mapping track $\Sigma p = \{(e, \alpha) \in E \times B^I : p(e) = \alpha(0) \text{ and } \alpha(1) = b_0\}$ have the same homotopy type, but necessary and sufficient conditions for the existence of such a homotopy equivalence are not known. For the case in which (E, p, B) is a principal fiber structure, this paper gives necessary and sufficient conditions in terms of a lifting function that the fiber structures (F, i, E) and $(\Sigma p, \pi, E)$ be H-isomorphic. Here $i: F \to E$ is the inclusion map and $\pi: \Sigma p \to E$ is the projection on the first component.

2. Preliminaries

DEFINITION. A pair (A, q, C) and (A', q', C) of fiber structures over the same base C have the same homotopy type or are homotopy equivalent means that there is a homotopy equivalence $h: A \to A'$ such that $q'h \sim q$ (homotopic).

DEFINITION. A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

of topological spaces with base points and continuous maps is exact means that

- (1) The composition gf is null-homotopic (i.e. homotopic to the constant map whose only value is the base point of Z); and
- (2) for each space W and continuous map $h: W \to Y$ such that gh is null-homotopic, there is a continuous map $h': W \to X$ such that $fh' \sim h$.

Note. The functions involved in this paper are not assumed to be base point preserving unless specifically stated. All function spaces are assigned the compact-open topology.

DEFINITION. The fiber structure (E, p, B) is principal means that the

basic fiber F is an H-group which operates on E in the following sense: There is a continuous map $\mu: E \times F \to E$ such that the restriction of μ to $F \times F$ is homotopic to the composition of the multiplication on F and the inclusion of F in E.

DEFINITION. A quasi-lifting function for (E, p, B) is a continuous map $\lambda: \Sigma p \to E^I$ with the following properties:

- (1) $\lambda(e, \alpha)$ (0) = e, $\lambda(e, \alpha)$ (1) $\in F$ $(e, \alpha) \in \Sigma p$,
- (2) the map $l: \Omega B \to \Omega(E, F)$ defined by

$$l(\beta) = \lambda(e_0, \beta) \quad \beta \in \Omega B$$

is a homotopy equivalence, and

(3) There is a continuous map $\theta: \Omega E \to \Omega E$ such that the diagram

$$\begin{array}{c} i \\ \Omega E \subseteq \longrightarrow \Omega(E, F) \\ \theta \downarrow & \uparrow l \\ \Omega E \longrightarrow \Omega B \\ \Omega_p \end{array}$$

commutes up to homotopy.

Here ΩB is the space of based loops in B, $\Omega(E,F)$ is the space of paths in E beginning at e_0 and ending in F and Ω_p is the natural map induced by p. If $\lambda: \Sigma p \to E^I$ is a quasi-lifting function, there is a continuous map $\lambda^*: \Sigma p \to F$ defined by

$$\lambda^*(e, \alpha) = \lambda(e, \alpha)(1) \quad (e, \alpha) \in \Sigma p.$$

THEOREM 1. If (E, p, B) has the weak covering homotopy property, then it has a quasi-lifting function.

Proof. Let

$$\rho: \varDelta = \left\{ (e,\,\alpha) \in E \times B^I : p(e) = \,\alpha(0) \right\} \to E^I$$

be a weak lifting function and $G: \Delta \times I \to E$ a homotopy such that

$$G(e, \alpha, 0) = e, G(e, \alpha, 1) = \rho(e, \alpha)(0),$$

 $pG(e, \alpha, t) = p(e)$ $(e, \alpha) \in A, t \in I.$

Let (PE, π_E, E) denote the usual path fibration (PE) is the space of paths in E with initial point e_0 and π_E is defined by evaluation at the terminal point). Then $(PE, p\pi_E, B)$ and (PB, π_B, B) have the weak covering homotopy property and both total spaces are contractible.

Define $\mu: PB \to PE$ by

$$\mu(\beta) = G(e_0, \beta, \cdot) * \rho(e_0, \beta) \qquad \beta \in PB$$

where * denotes the usual operation of juxtaposition of paths. Since μ is a fiber map, it is a fiber homotopy equivalence between (PB, π_B, B) and $(PE, p\pi_E, B)$ [2, Theorem 6.1]. In particular, the restriction of μ to ΩB is a homotopy equivalence between ΩB and $\Omega(E, F)$.

A quasi-lifting function for (E, p, B) is then defined by

$$\lambda(e, \alpha) = G(e, \alpha, \cdot) * \rho(e, \alpha)$$
 $(e, \alpha) \in \Sigma p.$

In this case we take $\theta = id_{\Omega E}$.

3. Homotopy type of Σp

THEOREM 2. If (E, p, B) is a principal fiber structure such that Σp is an H-group, then (F, i, E) and $(\Sigma p, \pi, E)$ are H-isomorphic if and only if there is a quasi-lifting function $\lambda : \Sigma p \to E^I$ such that λ^* is an H-homomorphism.

PROOF. (Sufficiency) The sequence

$$\Omega\Sigma p \overset{\Omega\pi}{\to} \Omega E \overset{\Omega p}{\to} \Omega B \overset{q}{\to} \Sigma p \overset{\pi}{\to} E \overset{p}{\to} B$$

is exact where $q(\beta) = (e_0, \beta)$ for each $\beta \in \Omega B$ [1, Theorem 3]. The existence of a quasi-lifting function implies the exactness of the sequence

$$\Omega F \stackrel{i}{\to} \Omega E \stackrel{i}{\to} \Omega(E,F) \stackrel{m}{\to} F \stackrel{i}{\to} E \stackrel{p}{\to} B$$

where i denotes inclusion maps and $m: \Omega(E, F) \to F$ is the evaluation at the terminal point.

Let $t: \Omega(E, F) \to \Omega B$ be a homotopy inverse for l and define $v: F \to \Sigma p$ by

$$v(x) = (x, c(b_0)), c(b_0)(I) = b_0.$$

Then $i\lambda^* v \sim i$ so that $i(\lambda^* v \cdot j)$ is null-homotopic where \cdot denotes the H-group operation and j denotes inversion. Hence there is a continuous map $s: F \to \Omega(E, F)$ such that

$$ms \cdot id_F \sim \lambda^* v$$
.

Then

$$id_F \sim jms \cdot \lambda^* v \sim j\lambda^* qts \cdot \lambda^* v \sim \lambda^* (jqts \cdot v)$$

so the map $\rho = jqts \cdot v$ is a right homotopy inverse for λ^* .

Now consider $\rho \lambda^*$: $\Sigma p \to \Sigma p$ and observe that

$$0 \sim i\lambda^*(\rho\lambda^*\cdot j) \sim \pi(\rho\lambda^*\cdot j).$$

Then there is a continuous map $\sigma: \Sigma p \to \Omega B$ such that

$$q\sigma \sim \rho \lambda^* \cdot j$$
.

Hence

$$ml\sigma \sim \lambda^* q\sigma \sim \lambda^* (\rho \lambda^* \cdot j) \sim 0$$

so there is a continuous map $\sigma': \Sigma p \to \Omega E$ such that

$$i\sigma' \sim l\sigma$$
.

Let $\theta: \Omega E \to \Omega E$ denote the map specified in the definition of quasilifting function. Since $l\Omega_n\theta \sim i$, then

$$qti \sim qtl\Omega_n\theta \sim q\Omega_n\theta \sim 0.$$

Hence

$$\rho \lambda^* \cdot i \sim a\sigma \sim atl\sigma \sim ati\sigma' \sim 0$$

so that $\rho \lambda^*$ is homotopic to the identity on Σp . Since λ^* is an H-homomorphism and $i\lambda^* \sim \pi$, it follows that $(\Sigma p, \pi, E)$ and (F, i, E) are H-isomorphic.

(Necessity) Suppose now that $h: F \to \Sigma p$ is an H-isomorphism such that $\pi h \sim i$. Since $(\Sigma p, \pi, E)$ has the weak covering homotopy property, there is a continuous map $r = (r_1, r_2): F \to \Sigma p$ such that r is homotopic to h and $r_1 = i$. Let $\delta: \Sigma p \to F$ be a homotopy inverse for r and $R = (R^1, R^2): \Sigma p \times I \to \Sigma p$ a homotopy such that

$$R_0 = id_{\Sigma p}, \quad R_1 = r\delta.$$

Define $\lambda: \Sigma p \to E^I$ and $t: \Omega(E, F) \to \Omega B$ by

$$\lambda(e, \alpha)(s) = R^{1}(e, \alpha, s) \quad (e, \alpha) \in \Sigma p, \quad s \in I$$

$$t(\sigma) = p\sigma * r_{2}\sigma(1) \quad \sigma \in \Omega(E, F).$$

Then t is a homotopy inverse for the induced map $l: \Omega B \to \Omega(E, F)$ and the composite $qti: \Omega E \to \Sigma p$ is null-homotopic. Hence there is a continuous map $\theta: \Omega E \to \Omega E$ such that $\Omega_p \theta \sim ti$. Then

$$l\Omega_{p}\theta \sim lti \sim i$$

so that λ is a quasi-lifting function such that $\lambda^* = \delta$ is an *H*-homomorphism.

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