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ON THE UNIVALENCE OF POLYNOMIALS

by

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Let $f(z)$ be a polynomial of degree n ($n \geq 2$). If the moduli of all the zeros of $f'(z)$ are greater than or equal to $\operatorname{cosec}(\pi/n)$ then by a theorem of S. Kakeya [1] $f(z)$ is univalent in $|z| < 1$. M. Robertson [2] gave a necessary and sufficient condition for $f(z)$ to have the radius of univalence exactly equal to 1. I [3] formed a counter example to show that this result is not sufficient. In connection with the same problem I will now prove another necessary and sufficient condition, Theorem 1, and by this proof will also deduce Kakeya's result. Then I will consider a result given by L. N. Čakalov, which will follow from Theorem 1, and will give some improved results.

THEOREM 1. *Let $f(z)$ be a polynomial of degree n ($n \geq 2$). If the moduli of all the zeros of $f'(z)$ are greater than or equal to $\operatorname{cosec}(\pi/n)$ then the necessary and sufficient condition for $f(z)$ to have the radius of univalence exactly equal to 1 is that all the zeros of $f'(z)$ should be concentrated at the same point on $|z| = \operatorname{cosec}(\pi/n)$.*

The condition is sufficient for all $n \geq 2$. If $n = 2$ this can be seen by the polynomial $z^2 + 2z$, since $\operatorname{cosec}(\pi/n) = 1$ and the derivative vanishes at $z = -1$. If $n > 2$ let us consider $f'(z) = (z - \operatorname{cosec}(\pi/n))^{n-1}$. Then $f(z)$ takes the same value at the points $\alpha = e^{\pi(2-n)i/2n}$, $\beta = e^{\pi(n-2)i/2n}$, because if we put $w = e^{(2\pi i)/n}$, then α, β satisfy the equation

$$\operatorname{cosec} \frac{\pi}{n} = \frac{\alpha - \beta w}{1 - w},$$

which implies that

$$\left(\frac{\alpha - \operatorname{cosec} \frac{\pi}{n}}{\beta - \operatorname{cosec} \frac{\pi}{n}} \right)^n = 1.$$

In order to prove that the condition is also necessary I will use the principle of apolarity of polynomials. Therefore first I will define this principle and state a theorem of Grace about apolarity.

DEFINITION. If the coefficients of two polynomials

$$f(z) = a_0 + C_n^1 a_1 z + C_n^2 a_2 z^2 + \cdots + a_n z^n$$

$$g(z) = b_0 + C_n^1 b_1 z + C_n^2 b_2 z^2 + \cdots + b_n z^n$$

of degree n satisfy the condition

$$a_0 b_n - C_n^1 a_1 b_{n-1} + C_n^2 a_2 b_{n-2} - \cdots + (-1)^n a_n b_0 = 0 \tag{1}$$

then $f(z)$ and $g(z)$ are called apolar polynomials.

Let

$$f(z) = a_0 + C_n^1 a_1 z + C_n^2 a_2 z^2 + \cdots + a_n z^n,$$

where the coefficients satisfy a linear relation

$$a_0 l_n + C_n^1 a_1 l_{n-1} + C_n^2 a_2 l_{n-2} + \cdots + a_n l_0 = 0$$

then

$$g(z) = l_0 - C_n^1 l_1 z + C_n^2 l_2 z^2 - \cdots + (-1)^n l_n z^n$$

is apolar to $f(z)$. If we write the same relation for the particular polynomial

$$F(x) = (x - z)^n = x^n - C_n^1 x^{n-1} z + \cdots,$$

regarding x as a parameter, we find that

$$g(z) = 0.$$

Therefore if the coefficients of a polynomial $f(z)$ satisfy a linear relation then we can obtain a polynomial $g(z)$ apolar to $f(z)$ directly from this relation ². I will use this fact in order to prove Theorem 1.

For the relative location of the zeros of apolar polynomials we have the following theorem of Grace ³.

THEOREM 2. *If two polynomials are apolar then any circular domain ⁴ containing all the zeros of one of these polynomials contains at least one zero of the other.*

By using arguments similar to those used in the proof of Theorem 2 I will prove the following result.

THEOREM 3. *Let $f(z)$ and $g(z)$ be apolar polynomials of degree $n \geq 2$. Let C be the circle $|z| = r$ such that one zero of $f(z)$ is on C and this is*

¹ C_n^p denotes the $(p+1)$ th coefficient of the n 'th power of the binomial, i.e.

$$C_n^p = \frac{n!}{p!(n-p)!}.$$

² See, e.g. [4] p. 19–20.

³ [5], see e.g., [4] p. 16–19.

⁴ By a circular domain we mean the interior or exterior of a circle or half plane.

not a zero of $f'(z)$; and $n-1$ zeros of $f(z)$ lie in the interior of C . If all the zeros of $g(z)$ are not concentrated at the same point on C then there exists at least one zero of $g(z)$ in the interior of C .

PROOF. Let two polynomials

$$\begin{aligned} f(z) &= a_0 + C_n^1 a_1 z + C_n^2 a_2 z^2 + \dots + a_n z^n \\ g(z) &= b_0 + C_n^1 b_1 z + C_n^2 b_2 z^2 + \dots + b_n z^n \end{aligned} \tag{2}$$

of degree n be apolar. Then their coefficients satisfy the condition of apolarity (1). We denote the zeros of $f(z)$ by $\alpha_1, \alpha_2, \dots, \alpha_n$ and the zeros of $g(z)$ by z_1, z_2, \dots, z_n .

Putting

$$S_k = (-1)^k C_n^k \frac{b_{n-k}}{b_n},$$

the relation (1) can be written as

$$a_0 S_0 + a_1 S_1 + \dots + a_n S_n = 0, \tag{3}$$

where

$$\begin{aligned} S_0 &= 1 \\ S_1 &= z_1 + z_2 + \dots + z_n \\ &\dots\dots\dots \\ S_n &= z_1 z_2 \dots z_n \end{aligned}$$

In this way we associate the relation (3) with the equation (2). We will show that, if z_1, z_2, \dots, z_n is a system of solutions of equation (3) and if all z_k 's are not concentrated at the same point on C then at least one z_k lies in the interior of C . We may assume that at least one of the points z_k is exterior to or on C , otherwise there is nothing to prove. Supposing that this point is $z_n = \zeta$ we will show that one of the points z_1, z_2, \dots, z_{n-1} lies in the interior of C . Let us put

$$\begin{aligned} s_0 &= 1 \\ s_1 &= z_1 + z_2 + \dots + z_{n-1} \\ &\dots\dots\dots \\ s_{n-1} &= z_1 z_2 \dots z_{n-1}. \end{aligned}$$

Then

$$\begin{aligned} S_0 &= s_0 \\ S_1 &= s_1 + \zeta s_0, \\ &\dots\dots\dots \\ S_n &= \zeta s_{n-1}. \end{aligned}$$

By substituting these values in (3) then we obtain the relation

$$(a_0 + a_1 \zeta)s_0 + (a_1 + a_2 \zeta)s_1 + \dots + (a_{n-1} + \zeta a_n)s_{n-1} = 0;$$

but this relation is associated with the equation

$$G(z) = (a_0 + a_1 \zeta) + C_{n-1}^1(a_1 + a_2 \zeta)z + \dots + (a_{n-1} + \zeta a_n)z^{n-1} = 0 \quad ^5$$

Now it is sufficient to show that all the roots of this last equation are in the interior of C . Since z_1, z_2, \dots, z_{n-1} are the zeros of a polynomial apolar to $G(z)$, then by Grace's theorem at least one z_k lies in the interior of C . Writing $G(z)$ as

$$G(z) = a_0 + C_{n-1}^1 a_1 z + \dots + a_{n-1} z^{n-1} + \zeta(a_1 + C_{n-1}^1 a_2 z + \dots + a_n z^{n-1}),$$

since

$$f'(z) = n[a_1 + C_{n-1}^1 a_2 z + \dots + a_n z^{n-1}],$$

we have

$$nG(z) = \zeta f'(z) + n[a_0 + C_{n-1}^1 a_1 z + \dots + a_{n-1} z^{n-1}],$$

and subtracting

$$nf(z) = na_0 + nC_{n-1}^1 a_1 z + \dots + na_n z^n$$

we obtain

$$nG(z) = nf(z) + (\zeta - z)f'(z). \quad (4)$$

Division by $f(z)$ gives

$$h(z) = n + (\zeta - z) \frac{f'(z)}{f(z)}.$$

First we will show that if $G(z)$ had a zero, z_0 , outside or on C then we would have $f(z_0) \neq 0$, and so $G(z_0) = 0$ would imply that $h(z_0) = 0$. Then we will show that $h(z)$ cannot have any zero outside or on C which will complete the proof. Now let us suppose that z_0 is outside or on C and $G(z_0) = 0$, $f(z_0) = 0$. Then by equation (4) we have either $f'(z_0) = 0$ or $\zeta = z_0$. If z_0 is on C , since there exists just one zero of $f(z)$ on C which is not a zero of $f'(z)$, then $f'(z_0) \neq 0$. If z_0 is exterior to C , then z_0 cannot be a zero of $f'(z)$ because the circle C encloses all the zeros of $f(z)$ and therefore encloses all the zeros of $f'(z)$ ⁶. The second possibility, $\zeta = z_0$, does not hold either since at the beginning we can choose ζ such that $f(\zeta) \neq 0$. This can be done because the zeros of $g(z)$ are not

⁵ $-a_{n-1}/a_n$ lies in the interior of C because

$$\frac{-C_{n-1}^{n-1} a_{n-1}}{a_n} = \alpha_1 + \alpha_2 + \dots + \alpha_n, \text{ and so } -\frac{a_{n-1}}{a_n} = \frac{\alpha_1 + \alpha_2 + \dots + \alpha_n}{n},$$

and since all the zeros of $f(z)$ do not have the same modulus then $|a_{n-1}/a_n|$, is less than the maximum modulus. Therefore $\zeta \neq -a_{n-1}/a_n$ and so $G(z)$ is a polynomial of degree $n-1$.

⁶ [6], p. 15, Thm. (6.2).

concentrated at the same point, therefore there exists at least one zero of $g(z)$ which lies either in the interior of C , when there is nothing to prove, or lies on or outside C and is not a zero of $f(z)$. So we can choose this zero as ζ .

Now $h(z)$ cannot have a zero exterior to or on C , for suppose that z_0 is a zero exterior to or on C , then

$$\frac{f'(z_0)}{f(z_0)} = \sum_{i=1}^n \frac{1}{z_0 - \alpha_i},$$

and

$$h(z_0) = \sum_{i=1}^n \left(1 + \frac{\zeta - z_0}{z_0 - \alpha_i} \right) = \sum_{i=1}^n \frac{\zeta - \alpha_i}{z_0 - \alpha_i} = 0 \quad (5)$$

The image of the interior of C under the transformation

$$Z = \frac{\zeta - z}{z_0 - z}$$

is a convex domain. Let us denote this domain by Γ . Since ζ is not in the interior of C , $Z = 0$ is exterior to Γ . By (5) the sum of the transforms of α_i 's is 0; so $Z = 0$ is their centre of gravity. But the transform of at least one α_i is in Γ ; thus $Z = 0$ is also in Γ ⁷. This contradiction completes the proof.

Now I will prove the necessary condition of Theorem 1, and by the same proof I will also deduce Kakeya's Theorem.

PROOF. Let $f(z)$ be a polynomial of degree $n \geq 2$ which attains the same value at two distinct points z_1, z_2 in the closed unit circle. The relation

$$f(z_1) = f(z_2)$$

is a linear relation between the coefficients of $f'(z)$. As it is explained in the argument following the definition of apolarity, by writing the same linear relation between the coefficients of

$$F(x) = (x - z)^{n-1}$$

we find that

$$y(z) = \int_{z_1}^{z_2} (x - z)^{n-1} dx = 0, \quad (6)$$

where $y(z)$ and $f'(z)$ are apolar polynomials. Now let Z be the zero of $y(z)$ of maximum modulus. By writing equation (6) as

⁷ If n points are located in or on the boundary of a convex domain and if at least one of them lies in the domain then the centre of gravity of these points also lies in the domain.

$$\left(\frac{z_1 - z}{z_2 - z}\right)^n = 1,$$

we have

$$z = \frac{z_1 - wz_2}{1 - w}$$

where w is an n 'th root of unity different from 1. If we allow z_1, z_2 to vary in the closed unit circle, then we have

$$\begin{aligned} \max \left| \frac{z_1 - wz_2}{1 - w} \right| &= \frac{2}{\min |1 - w|} = \frac{2}{\left| 1 - \cos \frac{2\pi}{n} - i \sin \frac{2\pi}{n} \right|} \\ &= \frac{2}{\sqrt{\left(1 - \cos \frac{2\pi}{n}\right)^2 + \sin^2 \frac{2\pi}{n}}} \\ &= \frac{2}{\sqrt{2 \left(1 - \cos \frac{2\pi}{n}\right)}} = \frac{2}{\sqrt{4 \sin^2 \frac{\pi}{n}}} = \operatorname{cosec} \frac{\pi}{n}. \end{aligned}$$

Thus $|Z| \leq \operatorname{cosec} \pi/n$. Suppose that $n > 2$ and $|Z| = \operatorname{cosec} \pi/n$, then $y(z)$ satisfies the conditions of $f(z)$ in Theorem 3⁸. Therefore if all the zeros of $f'(z)$ are not concentrated at the same point on $|z| = \operatorname{cosec} \pi/n$ then there exists at least one zero of $f'(z)$ in the interior of $|z| = \operatorname{cosec} \pi/n$. Thus if the zeros of $f'(z)$ are not concentrated at the same point on $|z| = \operatorname{cosec} (\pi/n)$, and if $f'(z)$ does not vanish in $|z| < \operatorname{cosec} (\pi/n)$ then $f(z)$ cannot attain the same value at two distinct points in $|z| \leq 1$. Hence by an argument of J. Dieudonné⁹ the radius of univalence of $f(z)$ is greater than 1¹⁰. Thus the necessary condition of Theorem 1 follows.

In the above proof if we allow z_1, z_2 to vary only in the interior of the unit circle then by applying Grace's Theorem we deduce Kakeya's Theorem.

L. N. Čakalov [8, Theorem 2] formed a special type of distribution of the zeros of $f'(z)$ outside the unit disc, for which he showed that $f(z)$

⁸ Max. modulus of the zeros of $y'(z)$ cannot be greater than $\operatorname{cosec} (\pi/n-1)$, and since $n > 2$ then $\operatorname{cosec} (\pi/n-1) < \operatorname{cosec} (\pi/n)$.

⁹ [7], p. 309–310. If $|z| = R$ is the largest circle about the origin in which a polynomial $f(z)$ is univalent then either $f(z)$ takes the same value at two distinct points on $|z| = R$ or $f'(z)$ vanishes on $|z| = R$. Otherwise $f(z)$ is univalent in a larger circle.

¹⁰ If $n = 2$ then $\operatorname{cosec} (\pi/n) = 1$, and if the derivative does not vanish inside or on the unit circle then by Kakeya's Theorem the radius of univalence of $f(z)$ is greater than 1.

is univalent in a larger circle than that given by *Takeya's Theorem*. His result is as follows:

THEOREM 4. *Suppose that m is a non-negative integer less than $(n+1)/2$, and let*

$$R = \sin \frac{\pi}{n+1} : \sin \frac{(n+1-2m)\pi}{(n-m)(2n+2)}.$$

*Let m of the zeros of the polynomial $Q(z) = \prod_{k=1}^m (1 - (z/z_k))$ lie in the annulus $1 \leq |z| \leq R$ and the remaining $n-m$ be situated in the region $|z| > R$. Then the polynomial $P(z) = \int Q(z)dz$ is univalent in $|z| < r_0$ where r_0 is larger than the radius $\sin [\pi/(n+1)]$ given by the theorem of *Takeya*.*

Now this result follows from *Theorem 1* since the zeros of $Q(z)$ are not concentrated at the same point on the unit circle. By using arguments similar to *Čakalov's*, we can obtain the following improved results, *Theorems 5 and 6*.

THEOREM 5. *Suppose that $n > 1$, x is a real number such that $1/n^2 < x < 1/n$ and*

$$R = \sin \frac{x\pi n}{xn+1} : \sin \frac{(1-xn)\pi}{2(n-1)(xn+1)}.$$

Let one of the zeros of the polynomial $Q(z) = \prod_{k=1}^n (1 - (z/z_k))$ lie in the annulus $1 \leq |z| \leq R$ and the remaining $n-1$ be situated in the region $|z| > R$. Then the polynomial $P(z) = \int Q(z)dz$ is univalent in $|z| < r_0$ where

$$r_0 > \sin \frac{x\pi n}{xn+1} > \sin \frac{\pi}{n+1}.^{11}$$

THEOREM 6. *Suppose that $n > 1$, x is a real number such that $x > 1$, k is an integer such that $0 < k < n$ and*

$$R = \sin \frac{x\pi n}{2k(xn+1)} : \sin \frac{\pi}{2(n-k)(xn+1)}.$$

Let k of the zeros of the polynomial $Q(z) = \prod_{k=1}^n (1 - (z/z_k))$ lie in the annulus $1 \leq |z| \leq R$ and the remaining $n-k$ be situated in the region $|z| > R$. Then the polynomial $P(z) = \int Q(z)dz$ is univalent in $|z| < r_0$ where

$$r_0 > \sin \frac{x\pi n}{2k(xn+1)}.^{12}$$

¹¹ If x is near to $1/n$ then R becomes large and $\sin x\pi n/(xn+1)$ is near to 1.

¹² For $k = 1$ and for large x , R becomes large, and $\sin (x\pi n/2k(xn+1))$ is near to 1.

If $k < (xn^2 + xn)/2(xn + 1)$ we have further

$$r_0 > \sin \frac{x\pi n}{2k(xn + 1)} > \sin \frac{\pi}{n + 1}.^{13}$$

I wish to thank Professor F. R. Keogh for suggesting that I should prove whether or not the condition of Theorem 1 is necessary.

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¹³ In [9, Theorem 2], by considering the distribution of the zeros of a polynomial $g(z)$ relative to an annulus, I obtained the best possible results for the minimum radius of univalence and the minimum radius of starlikeness of $f(z) = zg(z)$. These results are valid for every annulus about the origin and for every type of distribution of the zeros relative to an annulus.