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Representations of semi direct products of groups


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REPRESENTATIONS OF SEMI DIRECT PRODUCTS OF GROUPS

by

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Introduction

Let $G_1$ be a locally compact Abelian group, $G_2$ a locally compact group of continuous automorphisms of $G_1$. In this paper we characterise all pairs of unitary representations $\rho$ and $\sigma$ of $G_1$ and $G_2$ respectively in a Hilbert space $\mathcal{H}$, where $\rho$ is cyclic and

$$\sigma(\alpha)\rho(x)\sigma(\alpha^{-1}) = \rho(\alpha[x]), \ \alpha \in G_2, \ x \in G_1. \ \ (\star)$$

A set of necessary and sufficient conditions for a pair $(\rho_1, \sigma_1)$ to be unitarily equivalent to a pair $(\rho_2, \sigma_2)$ is given.

It can be shown that the commutation relations $(\star)$ define a system of imprimitivity for the representation $\sigma$. In [3] § 14. Mackey investigates these representations, from a different point of view from ours, primarily as an application of his theory of induced representations.

I would like to thank the referee for his helpful comments.

1. Preliminaries

Definition 1.1. Let $G$ be a locally compact group. A unitary representation of $G$ is a homomorphism $\Pi : g \to \Pi(g)$ of $G$ into the group of unitary transformations of a Hilbert space $\mathcal{H}(\Pi)$, such that $\Pi$ is continuous in the weak topology for operators. A closed linear manifold $M$ is called an invariant subspace for $\Pi$ if $\Pi(g) \xi \in M$ for all $g \in G$ and all $\xi \in M$. An invariant subspace $M$ is said to be a cyclic subspace for $\Pi$ if there is an element $\xi_0$ in $\mathcal{H}(\Pi)$, such that the smallest invariant subspace for $\Pi$ containing $\xi_0$ is $M$. $\xi_0$ is called a relative cyclic vector for $\Pi$. If $\mathcal{H}(\Pi)$ is a cyclic subspace, then $\Pi$ is said to be cyclic. The intertwining algebra of a representation $\Pi$ is the set

$$R(\Pi, \Pi) = \{T : T\Pi(g) = \Pi(g)T, \ g \in G\},$$

$T$ being bounded, everywhere defined, linear transformations on $\mathcal{H}(\Pi)$.

Definition 1.2. Let $\mathcal{H}$ be a Hilbert space. $A^\ast$ algebra $\mathfrak{A}$ of (bounded,
everywhere defined, linear) transformations of \( \mathcal{H} \) is called a von Neumann algebra, if \( \mathcal{A} \) is closed in the weak topology for operators. A closed linear manifold \( \mathcal{M} \) is called an invariant subspace for \( \mathcal{A} \), if \( A \xi \in \mathcal{M} \) for all \( A \in \mathcal{A} \) and all \( \xi \in \mathcal{M} \). An invariant subspace \( \mathcal{M} \) is said to be a cyclic subspace for \( \mathcal{A} \) if there is an element \( \xi_0 \) in \( \mathcal{H} \), such that the smallest invariant subspace for \( \mathcal{A} \) containing \( \xi_0 \) is \( \mathcal{M} \). If \( \mathcal{H} \) is an invariant subspace for \( \mathcal{A} \), then \( \mathcal{A} \) is said to be cyclic. The commutant of \( \mathcal{A} \) is the set

\[ \mathcal{A}' = \{ T : TA = AT, A \in \mathcal{A} \}, \]

\( T \) being bounded, everywhere defined, linear transformations on \( \mathcal{H} \).

It is easy to prove that a closed linear manifold \( \mathcal{M} \subseteq \mathcal{H}(\Pi) \) (resp. \( \mathcal{M} \subseteq \mathcal{H} \)) is a cyclic subspace for \( \Pi \) (resp. \( \mathcal{A} \)) if and only if there is an element \( \xi_0 \in \mathcal{H}(\Pi) \) (resp. \( \xi_0 \in \mathcal{H} \)) such that the closed linear manifold generated by \( (\Pi(g)\xi_0 : g \in G) \) (resp. \( (A\xi_0 : A \in \mathcal{A}) \)) is \( \mathcal{M} \).

If \( S \) is a set of elements in a Hilbert space the closed linear manifold generated by \( S \) is denoted by \([s : s \in S]\).

Let \( \Pi : g \mapsto \Pi(g) \) be a representation of a locally compact group \( G \). We shall often use the following well-known results

**Lemma 1.1.**

(i) \( R(\Pi, \Pi) \) is a von Neumann algebra;
(ii) \( R(\Pi, \Pi)' \) is the smallest von Neumann algebra containing the operators \( (\Pi(g) : g \in G) \);
(iii) \( \mathcal{M} \) is an invariant subspace for \( \Pi \) (resp. \( R(\Pi, \Pi)' \)) if and only if \( P \), the projection whose range is \( \mathcal{M} \), belongs to \( R(\Pi, \Pi) \).
(iv) A closed linear manifold \( \mathcal{M} \) is a cyclic subspace for \( \Pi \) if and only if \( \mathcal{M} \) is a cyclic subspace for \( R(\Pi, \Pi)' \).

**Definition 1.3.** Let \( X \) be a locally compact space, \( \mu \) a finite regular measure defined on the \( \sigma \)-ring of Borel subsets of \( X \). We denote by \( L(X) \) the set of all continuous functions with compact support; \( C(X) \) the set of all continuous functions on \( X \). If \( f \in L(X) \) we denote by \( M_f \) the operator on \( L^2(X, \mu) \) defined by \( (M_f h)(x) = f(x)h(x) \), where \( h \in L^2(x, \mu) \).

**Lemma 1.2.** Let \( \hat{G} \) be a locally compact Abelian group, \( \hat{G} \) the character group of \( G \) and \( \mu \) a finite regular measure defined on the \( \sigma \)-ring of Borel subsets of \( \hat{G} \). The mapping \( M : x \mapsto M_x \), where \( (M_x f)(\tau) = x(\tau)f(\tau), f \in L^2(\hat{G}, \mu) \), \( x \in G \) is a cyclic representation of \( G \).

**Proof.** It is easy to verify that \( M : x \mapsto M_x \) is a weakly continuous unitary representation of \( G \). We shall show that \( M \) is cyclic.

Let \( e \) be the function on \( \hat{G} \), \( e(\tau) = 1 \). Since \( \mu \) is a finite measure on \( \hat{G} \), \( e \) belongs to \( L^2(\hat{G}, \mu) \) and therefore \( x = M_x e \in L^2(\hat{G}, \mu) \) for all \( x \in G \). Denote by \( F \) the set of all finite linear combination of elements...
of $G_1$. We recall [[4] § 31, cor. 4] that every continuous function on $\hat{G}_1$ can be approximated uniformly on compact sets by members of $F$. If $f, h_1, h_2$ are continuous functions with compact supports and $\varepsilon > 0$, we can find $s \in F$ such that

$$|f(\tau) - s(\tau)| < \frac{\varepsilon}{||h_1|| ||h_2||} \text{ for all } \tau \in k_1 \cap k_2$$

where $K_i$ is the support of $h_i$. Hence

$$||(M_f - M_s)h_1, h_2|| = \left| \int_{\hat{G}_1} (f(\tau) - s(\tau))h_1(\tau)\overline{h_2(\tau)}d\mu(\tau) \right|$$

$$< \frac{\varepsilon}{||h_1|| ||h_2||} ||h_1|| ||h_2|| = \varepsilon.$$

This is true for all $h_1 \in L(\hat{G}_1)$ and $h_2 \in L(\hat{G}_1)$. Since $L(\hat{G}_1)$ is dense in $L^2(\hat{G}_1, \mu)$, we have proved that $(M_f : f \in L(\hat{G}_1))$ belongs to the weakly closed algebra generated by $(M_x : x \in G_1)$. From Lemma 1.1 (ii) we deduce that $(M_f : f \in L(\hat{G}_1)) \subseteq R(\Pi, \Pi)'$ and therefore from the (iv) of Lemma 1.1, we deduce that $f = M_f e \in [M_x e : x \in G_1]$. That is $L(\hat{G}_1) \subseteq [M_x e : x \in G_1]$. We complete the proof by observing that $L(\hat{G}_1)$ is dense in $L^2(G_1, \mu)$.

**Lemma 1.3.** Let $\Pi : x \rightarrow \Pi(x)$ be a cyclic representation of a locally compact Abelian group $G_1$. There is a regular, finite measure $\mu$ on $\hat{G}_1$, and a linear isometry $S : \hat{G}(\Pi) \rightarrow L^2(\hat{G}_1, \mu)$ such that $S\Pi(x)S^{-1} = M_x$, where $M : x \rightarrow M_x$ is the representation of $G_1$ defined in Lemma 1.2.

**Proof.** Let $\xi_0$ be a cyclic element for the cyclic representation $\Pi$, and let $\Phi(x) = (\Pi(x)\xi_0, \xi_0)$. There is a positive functional $P$ on $R(G_1)$, the group algebra of $G_1$, which corresponds to the continuous positive definite function $\Phi$. Since $R(G_1)$ is a commutative Banach algebra, the positive functional $P$ can be represented in the form

$$P(f) = \int_{\Delta} f(\tau)d\mu(\tau).$$

The spectrum $\Delta$ of $R(G_1)$ is homeomorphic to $\hat{G}_1 \cup \{L^1(G_1)\}$ and $\mu(\{L^1(G_1)\}) = 0$. Therefore, the measure $\mu$ may be considered as a measure defined on $\hat{G}_1[[4] § 31, sec. 3].$

The Gelfand isomorphism theorem allows us to regard $P$ as a positive functional on $C(\Delta)$, where $C(\Delta)$ is the set of all continuous functions on $\Delta$. The positive functional $P$ defines a representation of $C(\Delta)$ which is equivalent to the representation $M : f \rightarrow M_f$ on $L^2(\Delta, \mu)$, where

$$(M_f g)(\delta) = f(\delta)g(\delta), g \in L^2(\Delta, \mu).$$
([4]. ch. 4. § 17]. Since $G_1 \subseteq C(\Delta)$, we obtain a representation $M : x \to M_x$ of $G_1$ in
\[ L^2(\Delta, \mu) = L^2(\hat{G}_1, \mu), \]
where
\[ (M_x \gamma)(\tau) = x(\tau) \gamma(\tau). \]
Since the representations $M$ and $\Pi$ of $G_1$ define the same representation of $R(G_1)$, namely the representation defined by the positive functional $P$, the representations $M$ and $\Pi$ are equivalent. [4] § 29, sec. 3.

2. Semi-direct products

Let $G$ be a locally compact group, $G_2$ a locally compact group of automorphisms of $G$ such that the mapping $(g, \alpha) \to \alpha[g]$ of $G \times G_2$ into $G$ is continuous in both variables. The semi-direct product $G \otimes G_2$ is the set of all pairs $(g, \alpha)$, $g \in G$, $\alpha \in G_2$, whose group operation is defined by
\[ (g, \alpha)(h, \beta) = (gh[\alpha], \alpha \beta). \]
$G \otimes G_2$ is a locally compact group in the product topology. The mapping $g \to (g, \varepsilon)$ where $\varepsilon$ is the identity of $G_2$ is an isomorphism between $G$ and a closed normal subgroup of $G \otimes G_2$. The mapping $x \to (e, \alpha)$ where $e$ is the identity element of $G$ is an isomorphism between $G_2$ and a closed subgroup of $G \otimes G_2$. Finally, $(g, \alpha) = (g, \varepsilon)(e, \alpha)$. [2] pp. 6-7, 58-59, [3] § 14]. The proof of the following lemma is routine.

**Lemma 2.1.** Let $\rho : g \to \rho(g)$ and $\sigma : \alpha \to \sigma(\alpha)$ be representations of $G$ and $G_2$ respectively in a Hilbert space $\mathcal{H}$. The mapping $\Pi : (g, \alpha) \to \Pi(g, \alpha)$, where $\Pi(g, \alpha) = \rho(g)\sigma(\alpha)$ is a representation of $G \otimes G_2$ if and only if
\[ \sigma(\alpha)\rho(g)\sigma(\alpha^{-1}) = \rho(\alpha[g]). \]

In the following pages let $G_1$ be a locally compact Abelian group, $\hat{G}_1$ the character group of $G_1$, $G_2$ a locally compact group of continuous automorphisms of $G_1$ such that the mapping $(x, \alpha) \to \alpha[x]$ of $G_1 \times G_2$ to $G_1$ is continuous in both variables. The group $G_2$ acts as a group of automorphisms of $\hat{G}_1$, if we define $[\tau]x$ by the equation $([\tau]x)(x) = \tau(\alpha[x]), x \in G_1$. [2] 26.9.

**Definition 2.1.** Let $\mu$ be a finite Borel measure defined on $\hat{G}_1$, and for each $\alpha \in G_2$ let $\mu_\alpha$ be the measure on $\hat{G}_1$ defined by $\mu_\alpha(B) = \mu([B]\alpha)$. The measure $\mu$ is said to be $G_2$-quasi invariant if $\mu_\alpha$ is absolutely continuous with respect to $\mu$ for all $\alpha \in G_2$.

**Lemma 2.2.** Let $\mu$ be a $G_2$-quasi invariant measure on $\hat{G}_1$. The mapping
where

$\Pi : (x, \alpha) \rightarrow \Pi(x, \alpha) = \Pi(x, \varepsilon)\Pi(e, \alpha)$,

and

$$(\Pi(x, \varepsilon)f)(\tau) = x(\tau)f(\tau)$$

is a representation of $G_1 \otimes G_2$ in $L^2(\hat{G}, \mu)$.

As the proof consists of a routine verification of the condition given in lemma 2.1, we omit the proof.

**Theorem 2.1.** Let $\Pi : (x, \alpha) \rightarrow \Pi(x, \alpha)$ be a representation of $G_1 \otimes G_2$ in a Hilbert space $\mathcal{H}(\Pi)$ such that the representation $\Pi(x, \varepsilon)$ of $G_1$ in $\mathcal{H}(\Pi)$ is cyclic. There is a $G_2$-quasi invariant measure $\mu$ on $\hat{G}_2$ and a linear isometry $S$ from $\mathcal{H}(\Pi)$ onto $L^2(\hat{G}_1, \mu)$ such that

$$S\Pi(x, \varepsilon)S^{-1}f(\tau) = x(\tau)f(\tau)$$

where $a(\tau, \alpha)$ is a Borel function on $\hat{G}_1 \times G_2$ with the following properties:

i $|a(\tau, \alpha)| = 1$ almost everywhere, and

ii $a(\tau, \alpha_1 \alpha_2) = a(\tau, \alpha_1) a([\tau]\alpha_2), \text{ a.e.}$

**Proof.** Let $\rho(x) = \Pi(x, \varepsilon)$ and $\sigma(\alpha) = \Pi(e, \alpha)$. Since $\rho$ is a cyclic representation of $G_1$ in $\mathcal{H}(\Pi)$, it follows from Lemma 1.2 that there is a finite Borel measure $\mu$ on $\hat{G}_1$ and a linear isometry $S$ from $\mathcal{H}(\Pi)$ onto $L^2(\hat{G}_1, \mu)$ such that $S\rho(x)S^{-1}f(\tau) = x(\tau)f(\tau)$. The well-known Stone-Naimark-Ambrose-Godement theorem asserts that there is a projection valued measure $P : B \rightarrow P_B$ on the Borel subsets of $\hat{G}_1$ to the projections in the intertwining algebra $R(\rho, \rho)'$ such that [[4] § 31. Th. 6]

$$(\rho(x)\xi, \eta) = \int_{\hat{G}_1} x(\tau)d(P_\tau \xi, \eta) \quad (1)$$

for every pair of elements $\xi$ and $\eta$ in $\mathcal{H}(\Pi)$. Moreover, if $\xi_0$ is a cyclic element for the representation $\rho$ then the measure $\mu$ is equivalent to the measure $\nu$ where $\nu(B) = ||P_B \xi_0||^2$. Now

$$(\sigma(\alpha)\rho(x)\sigma(\alpha^{-1})\xi, \eta) = \rho(\alpha[x] \xi, \eta). \quad (2)$$

From (1) we have
(σ(x)ρ(x)σ(α⁻¹)ξ, η) = (ρ(x)σ(α⁻¹)ξ, σ(α⁻¹)η)
= \int_{G_1} x(τ)d(P_τσ(α⁻¹)ξ, σ(α⁻¹)η)
= \int_{G_1} x(τ)d(σ(α)P_τσ(α⁻¹)ξ, η) \tag{3}

Also,

(ρ(α[x])ξ, η) = \int_{G_1} α[x](τ)d(P_τξ, η)
= \int_{G_1} x(τ)d(P_τξ, η) = \int_{G_1} x(τ)d(P_{[\alpha]^{-1}}ξ, η) \tag{4}

It follows from (2), (3) and (4) that

σ(α)P_Bσ(α⁻¹) = P_{[\alpha]^{-1}}. \tag{5}

Now μ(B) = 0 implies ν(B) = 0 and consequently P_Bξ_0 = 0. Since P_B ∈ R(ρ, ρ), the equation 0 = TP_Bξ = P_BTξ = T ∈ R(ρ, ρ)' implies P_BE = 0 where E is the projection on the closed linear manifold generated by (Tξ : T ∈ R(ρ, ρ)')'. However, E = I because ξ_0 is a cyclic element for ρ. Therefore P_B = 0. Thus μ(B) = 0 implies P_B = 0, and from (5) it follows that P_{[\alpha]^{-1}} = 0. That is, μ(B) = 0 implies ν([B]α⁻¹) = 0. Since μ and ν are equivalent, ν([B]α⁻¹) = 0, implies μ([B]α⁻¹) = 0.

Hence μ_{α^{-1}} is absolutely continuous with respect to μ. Since α ∈ G_2 is arbitrary, we have shown that μ is G_2 quasi invariant.

Let

σ_0(α)f(τ) = \sqrt{dμ_α(τ)f([τ]α)}, f ∈ L^2(\hat{G}_1, μ)

and

σ_1(α) = Sσ(α)S^{-1}σ_0(α⁻¹)

where S is the linear isometry S(\hat{\alpha}) → L^2(\hat{G}_1, μ) introduced in the first paragraph of this proof. It is clear that σ_1(α) is a unitary transformation. Now, from the relation σ_0(α⁻¹)M_x = M_{[α]^{-1}x}σ_0(α⁻¹), we have

σ_1(α)Sρ(x)S^{-1}(τ) = Sσ(α)S^{-1}σ_0(α⁻¹)M_xf(τ)
= Sσ(α)S^{-1}M_{[α]^{-1}x}σ_0(α⁻¹)f(τ)
= Sσ(α)xσ_0(α⁻¹)[x]S^{-1}σ_0(α⁻¹)f(τ)
= Sσ(α)xσ_0(α⁻¹)[x]S^{-1}σ_0(α⁻¹)f(τ)
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= Sρ(α)S^{-1}σ_0(α⁻¹)f(τ)
= Sρ(α)S^{-1}σ_0(α⁻¹)f(τ)
This shows that \( \sigma_1(\alpha) \) commutes with \( S_0(\alpha)S^{-1} = M_\alpha \) and consequently \( \sigma_1(\alpha) \) commutes with the von Neumann algebra generated by \( M_\alpha \). It is known [5 cor. 1.1] that a commutative von Neumann algebra with a cyclic vector is maximal Abelian. Therefore \( \sigma_1(\alpha) \) belongs to the von Neumann algebra generated by \( \{ M_\alpha : \alpha \in G_1 \} \) which is the algebra of multiplication by essentially bounded measurable functions on \( (G_1, \mu) \). Hence \( \sigma_1(\alpha)f(\tau) = a(\tau, \alpha)f(\tau) \) where \( a(\tau, \alpha) \) is, for each \( \alpha \) a measurable essentially bounded function of modulus 1. We introduce the operator \( M_\alpha \) in \( L^2(G_1, \mu) \) where \( (M_\alpha f)(\tau) = a(\tau)f(\tau) \).

From the equation \( S_0(\alpha)S^{-1} = M_\alpha \sigma_0(\alpha) S_0 \), we obtain \( S_0(\alpha)S^{-1} = M_\alpha \sigma_0(\alpha) \); that is
\[
S_0(\alpha)S^{-1}f(\tau) = a(\tau, \alpha) \sqrt{\frac{d\mu_\alpha}{d\mu}}(\tau)f([\tau] \alpha).
\]

Finally,
\[
S_0(\alpha_1 \alpha_2)S^{-1}f(\tau) = a(\tau, \alpha_1 \alpha_2) \sqrt{\frac{d\mu_{\alpha_1 \alpha_2}}{d\mu}}(\tau)f([\tau] \alpha_1 \alpha_2)
\]
\[
= a(\tau, \alpha_1) \sqrt{\frac{d\mu_{\alpha_1}}{d\mu}}(\tau)f([\tau] \alpha_1, \alpha_2)
\]
\[
\sqrt{\frac{d\mu_{\alpha_2}}{d\mu}}([\tau] \alpha_1)f([\tau] \alpha_1 \alpha_2)
\]
\[
= a(\tau, \alpha_1)a([\tau] \alpha_1, \alpha_2) \sqrt{\frac{d\mu_{\alpha_1 \alpha_2}}{d\mu_\alpha}}(\tau)
\]
\[
\sqrt{\frac{d\mu_{\alpha_2}}{d\mu}}(\tau)f([\tau] \alpha_1 \alpha_2)
\]
\[
= a(\tau, \alpha_1)a([\tau] \alpha_1, \alpha_2) \sqrt{\frac{d\mu_{\alpha_1 \alpha_2}}{d\mu}}(\tau)f([\tau] \alpha_1 \alpha_2).
\]

Since \( S_0(\alpha_1 \alpha_2)S^{-1} = S_0(\alpha_1)S^{-1} \sigma_0(\alpha_2)S^{-1} \) we have
\[
a(\tau, \alpha_1 \alpha_2) = a([\tau] \alpha_1, \alpha_2)\alpha(\tau, \alpha_1), \text{ a.e.}
\]

This completes the proof of the theorem.

**Definition 2.2.** A Borel measure \( \mu \) on \( G_1 \) is said to be \( G_2 \)-ergodic if
1. \( \mu \) is \( G_2 \)-quasi invariant, and
2. the $G_2$-quasi-invariant non zero measures on $\hat{G}_2$ which are absolutely continuous with respect to $\mu$ are equivalent to $\mu$.

**Theorem 2.** Let $G_1$ and $G_2$ be as in the paragraph preceding Definition 2.1. Let $\Pi : (x, \alpha) \rightarrow \Pi(x, \alpha) = \rho(x)\sigma(x)$ be a representation of $G_1 \otimes G_2$. If the measure $\mu$ defined by the cyclic representation $\rho$ is $G_2$ ergodic, then $\Pi$ is irreducible.

**Proof.** Suppose a closed linear manifold $\mathcal{M}$ of $\hat{S}(\Pi)$ is invariant for $\Pi$. Then clearly $\mathcal{M}$ is invariant for $\rho$ and $\sigma$. Let $E$ be the projection whose range is $\mathcal{M}$. $E$ belongs to $R(\rho, \rho)$. The representation $\rho$ being cyclic, the von Neumann algebra $R(\rho, \rho)'$, generated by the operators $\rho(x) : x \in G_1$, is a commutative von Neumann algebra with a cyclic element. Consequently [5], cor. 1.1. $R(\rho, \rho)'$ is maximal Abelian. Therefore $R(\rho, \rho)' = R(\rho, \rho)$. Since every projection of $R(\rho, \rho)'$ is of the form $P_B$, where $P : B \rightarrow P_B$ is the projection valued measure defined by $\rho$, there is a Borel set $B_0$ of $\hat{G}_1$ such that $E = P_{B_0}$.

Let $\mu_0(B) = \mu(B_0 \cap B)$. Clearly $\mu_0$ is absolutely continuous with respect to $\mu$. We shall show that $\mu_0$ is $G_2$-quasi invariant. The measure $\mu$ is equivalent to the measure $\nu$ where $\nu(B) = \sup_{\pi} \nu(B \pi)$. We may for the purpose of this proof assume, without loss of generality, that $\nu(B) = \sup_{\pi} \nu(B \pi)$. From equation (5) in the proof of Theorem 2.1 we have

$$\sigma(\pi) P_{B_0 \cap B} \sigma(\pi^{-1}) = P_{[B_0 \cap B] \pi^{-1}}.$$  

However,

$$\sigma(\pi) P_{B_0 \cap B} \sigma(\pi^{-1}) = \sigma(\pi) P_{B_0} P_B \sigma(\pi^{-1}) = \sigma(\pi) P_{B_0} \sigma(\pi)^{-1} \sigma(\pi) P_B \sigma(\pi^{-1}).$$

Now suppose $\mu_0(B) = 0$. Then $\mu(B_0 \cap B) = 0$, and by the $G_2$-quasi invariance of $\mu$, it follows that $\mu([B_0 \cap B] \pi^{-1}) = 0$. Consequently,

$$0 = ||P_{[B_0 \cap B] \pi^{-1}} \xi_0||^2$$
$$= ||P_{B_0 \cap [B] \pi^{-1}} \xi_0||^2$$
$$= \mu(B_0 \cap [B] \pi^{-1}) = \mu_0([B] \pi^{-1}).$$

Since $\pi$ in $G_2$ is arbitrary, we have shown that $\mu_0$ is $G_2$-quasi invariant. The measure $\mu$ is $G_2$-ergodic. Therefore either $\mu_0$ is equivalent to $\mu$ or $\mu_0$ is the zero measure. That is either $B_0 = \hat{G}_1$ or $B_0 = \phi$. Consequently, $\mathcal{M} = \mathcal{S}$ or $\mathcal{M} = \{0\}$.

This completes the proof.

**Definition 2.3.** Let $\Pi_i : (x, \alpha) \rightarrow \Pi_i(x, \alpha) = \rho_i(x)\sigma_i(\alpha)$ be representations of $G_1 \ast G_2$ in $\hat{S}(\Pi_i)$, $i = 1, 2$. $\Pi_i$ is said to be equivalent to $\Pi_2$ if there is a linear isometry $S : \hat{S}(\Pi_1) \rightarrow \hat{S}(\Pi_2)$ such that
THEOREM 2.3. Let $n_i : (x, \alpha) \rightarrow \Pi_i(x, \alpha)$ be representations of $G_1 \otimes G_2$ on $S (\omega_i)$ where $\omega_i$ are cyclic. A set of necessary and sufficient condition that $\omega_1$ is equivalent to $\omega_2$ is

1. $\mu^1_1$ is equivalent to $\mu^2_1$ where $\mu^1_1$ is the measure on $\hat{G}_i$ defined by $\Pi_i (o, \omega) \ i = 1, 2$; and
2. there exists a Borel function $b$ on $\hat{G}_i$ with the properties
   2.1. $|b(\tau)| = 1$ almost everywhere, and
   2.2. $a^i_2(\tau, \alpha) = b(\tau) a^i_1(\tau, \alpha) b^{-1}(\tau)\alpha$ where $a^i_1(\tau, \alpha)$ is the function associated with $\Pi_i(e, \omega) : \omega_i \rightarrow \Pi_i(e, \alpha)$ in theorem 2.1.

PROOF. It is evident from Theorem 2.1 that, $\Pi_1$ is equivalent to $\Pi_2$ if and only if the following is true: (*) there is a linear isometry $S : L^2 (\hat{G}_1, \mu^1) \rightarrow L^2 (\hat{G}_1, \mu^2)$ such that $S \rho_1(x) = \rho_2(x) S$ where

$$\rho_i(x)f(\tau) = x(\tau)f(\tau), \ f \in L^2 (\hat{G}_1, \mu^i)$$

and $S \sigma_1(\alpha) = \sigma_2(\alpha) S$, where

$$\sigma_i(\alpha)(\tau) = a_i(\tau, \alpha) \sqrt{\frac{d\mu_i^1}{d\mu_i^2}}(\tau)f([\tau]|\alpha), \ i = 1, 2.$$

Assume that the conditions (*) are satisfied. We recall that $L(\hat{G}_1)$, the set of all continuous functions with compact support, is dense in $L^p(G_1, \mu^1)$ where $p = 1, 2$ and $i = 1, 2$. In the course of the proof of lemma 1.2 we saw that the operators $\rho_i(g)$ where $(\rho_i(g)f)(\tau) = g(\tau)f(\tau), \ g \in L(\hat{G}_1)$ and $f \in L^2 (\hat{G}_1, \mu^i)$ belong to $R(\rho_1, \rho_1)'$. It is easily verified that $S \rho_1(x) S^{-1} = \rho_2(x)$ implies $S \rho_1(g) S^{-1} = \rho_2(g)$ for all $g \in L(\hat{G}_1)$.

Since $S \rho_1(x) S^{-1} = \rho_2(x)$ for all $x$ in $G_1$, the commutative von Neumann algebra $R(\rho_1, \rho_1)'$ generated by $\rho_1(x) : x \in G$ is unitarily equivalent to the von Neumann algebra $R(\rho_2, \rho_2)'$ generated by $\rho_2(x) : x \in G_1$. Since $\rho_i$ are cyclic representations, the commutative von Neumann algebras $R(\rho_1, \rho_1)'$ are cyclic. A commutative von Neumann algebra with a cyclic vector is maximal Abelian ([5] corollary 1.1) and is unitarily equivalent to a multiplication algebra ([5] Lemma 1.2). Consequently, the multiplication algebra on $L^2 (\hat{G}_1, \mu^1)$ is unitarily equivalent to the multiplication algebra on $L^2 (\hat{G}_1, \mu^2)$ and therefore ([6] Theorem 4.1) $\mu^1$ is equivalent to $\mu^2$.

The function $e$, where $e(\tau) = 1$ for all $\tau \in \hat{G}_1$, belongs to $L^2 (\hat{G}_1, \mu^1)$. Let $Se = c \in L^2 (\hat{G}_1, \mu^2)$. We shall show that $c$ is an essentially bounded function.

$S \rho_1(x) S^{-1} = \rho_2(x), \ S \sigma_1(\alpha) S^{-1} = \sigma_2(\alpha)$.
Let
\[ C(g) = \int_{\mathcal{G}_1} |c(\tau)|^2 g(\tau) d\mu^2(\tau) \]
\[ |C(g)| = (gc, c) = (\rho_1(g)c, c) = (\rho_2(g)Se, Se) \]
\[ = (S^{-1}\rho_2(g)Se, e) = (\rho_2(g)e, e) \]
\[ = \int_{\mathcal{G}_1} g(\tau) d\mu^1(\tau). \]  
\[ (ii) \]

Hence \[ |C(g)| \leq \|g\|_1 \] (the \( L^1 \)-norm of \( g \in L^1(\mathcal{G}_1, \mu^1) \)).

That is, \( C(g) \) is bounded on a dense linear subset \( L^1(\mathcal{G}_1) \) of \( L^1(\mathcal{G}_1, \mu^1) \), and can therefore be extended to \( L^1(\mathcal{G}_1, \mu^1) \). Hence \( C \in L^\infty(\mathcal{G}_1, \mu^1) \), and therefore \( \sigma \) is essentially bounded with respect to \( \mu^1 \). Since \( \mu^1 \) and \( \mu^2 \) are equivalent it follows that \( c \) is essentially bounded with respect to \( \mu^2 \).

Since the function \( c \) is essentially bounded the equation (i) can be written in the form \( Sg = M_\sigma g \) where \( M_\sigma \) is the operation of multiplying by \( c \). Since \( M_\sigma \) is a bounded operator and \( L(\mathcal{G}_1) \) is dense in \( L^2(\mathcal{G}_1, \mu) \), the equation \( Sg = M_\sigma g \) holds for all \( g \) in \( L^2(\mathcal{G}_1, \mu^1) \). It follows from the equivalence of \( \mu^1 \) and \( \mu^2 \) and the equation (ii) that
\[ \int_{\mathcal{G}_1} g(\tau)|c(\tau)|^2 d\mu^2(\tau) = \int_{\mathcal{G}_1} g(\tau) d\mu^1(\tau) \]
\[ = \int_{\mathcal{G}_1} g(\tau) \frac{d\mu^1}{d\mu^2}(\tau) d\mu^2(\tau). \]

Hence \[ |c(\tau)|^2 = \frac{d\mu^1}{d\mu^2}(\tau) \]
a virtually everywhere, and
\[ c(\tau) = b(\tau) \sqrt{\frac{d\mu^1}{d\mu^2}(\tau)} \]
where \[ |b(\tau)| = 1 \]
a virtually everywhere.

Now,
\[ S\sigma_1(\alpha)(\tau) = M_\sigma a_1(\tau, \alpha) \sqrt{\frac{d\mu^1}{d\mu^2}(\tau)} g([\tau]\alpha) \]
\[ = b(\tau) \sqrt{\frac{d\mu^1}{d\mu^2}(\tau)} a_1(\tau, \alpha) \sqrt{\frac{d\mu^1}{d\mu^2}(\tau)} g([\tau]\alpha) \]
\[ = b(\tau) x_1(\tau, \alpha) \sqrt{\frac{d\mu^1}{d\mu^2}(\tau)} g([\tau]\alpha). \]
Hence the equation $S\alpha_1(\alpha)g = \sigma_2(\alpha)Sg$ yields

$$b(\tau)\alpha_1(\tau, \alpha) = \alpha_2(\tau, \alpha)b([\tau]_\alpha), \text{ a.e.}$$

The converse is easy to verify and we omit the details. This completes the proof.

The condition 2 of the last theorem can be reformulated in terms of a one dimensional cohomology group. To this end we observe first that $G_2$ as a group of automorphisms of $L^\infty(\hat{G}_1, \mu) : \alpha(g)(\tau) = g([\tau]_\alpha)$. Furthermore, the function $\alpha(\tau, \alpha)$ of Theorem 2.1 defines a mapping $\tilde{\alpha} : G_2 \to L^\infty(\hat{G}_1, \mu)$ where $(\tilde{\alpha}(\alpha))(\cdot) = \alpha(\cdot, \alpha)$. From ii of theorem 2.1 we see that $\tilde{\alpha}$ is a crossed homomorphism. It is evident that $(b(\alpha))(\cdot) = b(\cdot)b^{-1}([\cdot]_\alpha)$, where $b \in L^\infty$, is a principal crossed homomorphism. In view of these observations the condition 2 of Theorem 2.3 states $\alpha_1$ and $\alpha_2$ define the same element of the one dimensional cohomology group $H^1(G_2, L^\infty)$.

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