COMPOSITIO MATHEMATICA

K. W. ROGGENKAMP

Projective ideals in clean orders

Compositio Mathematica, tome 22, nº 2 (1970), p. 197-201
<http://www.numdam.org/item?id=CM_1970__22_2_197_0>

COMPOSITIO MATHEMATICA, Vol. 22, Fasc. 2, 1970, pag. 197-201
Wolters-Noordhoff Publishing
Printed in the Netherlands

PROJECTIVE IDEALS IN CLEAN ORDERS

by

K. W. Roggenkamp

1. Introduction

In [5] Faddeev has treated very explicitly the theory of invertible ideals over orders in separable algebras. In § 2 we review the connections between invertible ideals and progenerators (cf. [7]). In [4], [6], [9] and [10] 'clean orders' have been treated; i.e. orders for which every special projective left module is a generator. Two orders are said to lie on the 'same level' if they are linked by an invertible ideal. The main results of § 3 are: Orders on the same level are simultaneously clear or not clean. An order is 'left clean' if and only if it is 'right clean'.

2. Preliminaries

NOTATION:

R = Dedekind domain
K = quotient field of R
R_p = localization of R at the prime ideal p in R
A = finite dimensional separable K-algebra
Λ = R-order in A (cf. [1]).

All modules are assumed to be finitely generated. *Homomorphisms will be written opposite to the scalars.*

(1) DEFINITIONS: Let V be a left A-module.

(i) If $\Omega(V) =: \mathrm{Hom}_A(V, V)$, then V is an $(A, \Omega(V))$-bi-module.

(ii) An R-lattice M in V is a finitely generated R-submodule of V, such that $K \otimes_R M = V$; in particular, M is R-torsion free.

(iii) $\Lambda_l(M) = \{x \in A | xM \subset M\}$,
$\Lambda_r(M) = \{x \in \Omega(V) | Mx \subset M\}$ are the left and right orders of M resp.

(2) MORITA EQUIVALENCE: Let V be a faithful left A-module, then V is a progenerator and

$$A \cong \mathrm{Hom}_{\Omega(V)}(V, V),$$

and we have a natural isomorphism (cf. [7]),

$$\mathrm{Hom}_A(V, A) \cong \mathrm{Hom}_{\Omega(V)}(V, \Omega(V)). \tag{2*}$$

We identify both structures and denote this module by V^*. Then V^* is an $(\Omega(V), A)$-bimodule.

(3) Definitions: Let V be a faithful left A-module, M an R-lattice in V and N an R-lattice in V^*. Then we have the two natural isomorphisms (cf. [7])

$$\mu : V^* \otimes_A V \to \Omega(V)$$
$$v_0(v^* \otimes v)^\mu = (v_0)v^* \cdot v, v_0, v \in V, v^* \in V^*$$

and

$$\tau : V \otimes_{\Omega(V)} V^* \to A, v \otimes v^* \to (v)v^*, v \in V, v^* \in V^*.$$

Now we define the products $NM = \mathrm{im}\, \mu|_{N \otimes M}$, $MN = \mathrm{im}\, \tau|_{M \otimes N}$. Then MN is an R-lattice in A and NM is an R-lattice in $\Omega(V)$.

(4) Definitions: An R-lattice M in the faithful left A-module V is called *invertible* if there exists an R-lattice M^{-1} in V^* such that

(i) $\Lambda_r(M) = \Lambda_l(M^{-1})$, $\Lambda_r(M) = \Lambda_r(M^{-1})$,
(ii) $MM^{-1} = \Lambda_l(M)$, $M^{-1}M = \Lambda_r(M)$.

M^{-1} is called the inverse to M; it is uniquely determined by M, if M is invertible.

(5) Lemma: *An R-lattice M in the faithful left A-module V is invertible if and only if M is a progenerator with respect to the category of left $\Lambda_l(M)$-modules.*

Proof. If M is a progenerator, the maps

$$\mu : \mathrm{Hom}_{\Lambda_l(M)}(M, \Lambda_l(M)) \otimes_{\Lambda_l(M)} M \to \Lambda_r(M)$$

and

$$\tau : M \otimes_{\Lambda_r(M)} \mathrm{Hom}_{\Lambda_l(M)}(M, \Lambda_l(M)) \to \Lambda_l(M)$$

(for the definition cf. (3)) are natural isomorphisms. Hence $M^{-1} = \mathrm{Hom}_{\Lambda_l(M)}(M, \Lambda_l(M))$. *Conversely*, let M be invertible. From (4, ii) it follows then, that M is a projective left $\Lambda_l(M)$-module, and a projective right $\Lambda_r(M)$-module. From the definition (3) it follows that M is a progenerator for the category of left $\Lambda_l(M)$-modules and also for the category of left $\Lambda_r(M)$-modules (cf. [7]).

(6) Remark: Let Λ be an R-order in A and let $\alpha \in \mathrm{Hom}_A(A^{(n)}, A^{(n)})$ be regular, then the R-lattice $\Lambda^{(n)}\alpha$ is invertible ($X^{(n)}$ denotes the direct sum of n copies of X). In fact, the inverse is $\mathrm{Hom}_\Lambda(\Lambda^{(n)}\alpha, \Lambda)$ and $\Lambda_r(\Lambda^{(n)}\alpha) = \alpha^{-1}\, \mathrm{Hom}_\Lambda(\Lambda^{(n)}, \Lambda^{(n)})\alpha$.

(7) Remark: If M is an R-lattice in the faithful left A-module V, then M is invertible if and only if $R_p \otimes_R M$ is invertible for every prime ideal p in R. This follows from (5).

(8) Definition: Two R-orders Λ_1 in A and Λ_2 in $\Omega(V)$, where V is a free left A-module with a finite basis are said to lie on the *same level*, if there exists an invertible R-lattice M in V such that $\Lambda_l(M) = \Lambda_1$ and $\Lambda_r(M) = \Lambda_2$. With other words Λ_1 and Λ_2 are Morita equivalent. We remark that every maximal R-order in A and every maximal R-order in $\Omega(V)$ lie on the same level. It is not necessary to define the 'levels' only if V is A-free, it suffices to assume that V is a faithful left A-module of finite type. However, clean orders (cf. § 3) are in general not invariant under Morita equivalences via a Λ-lattice M; only if M is a special projective Λ-module (cf. [8]).

3. Clean orders

(9) Definition: An R-order Λ in A is called a *left clean R-order* (cf. [10]), if every special projective left Λ-module (i.e. a projective module which spans a free left A-module) is a progenerator for the category of left Λ-modules.

(10) Theorem. (Strooker [10]. Lam [6]): *For an R-order Λ in A the following are equivalent*

(i) *Λ is a left clean R-order,*
(ii) *$R_p \otimes_R \Lambda = : \Lambda_p$ is a left clean R_p-order for every prime ideal p in R,*
(iii) *every special projective left Λ-module is locally free.*
(iv) *If P_1, P_2 are projective left Λ-modules such that $KP_1 \cong_A KP_2$, then P_1 and P_2 are locally isomorphic.*
(v) *The Cartan-matrix of $\Lambda/p\Lambda$ is non-singular for every prime ideal p in R.*

(11) Remark: (10, v) shows that group rings of finite groups are clean. Commutative orders are clean.

(12) Lemma: *Let Λ be a left clean R-order in A and M an R-lattice the free A-module V. If $\Lambda_l(M) = \Lambda$, then M is invertible if and only if M is a projective left Λ-module.*

Proof. If M is a projective left Λ-module, then M is special projective; i.e. M is a progenerator for the category of left Λ-modules. From (5) it follows that M is invertible and $\Lambda_l(M) = \Lambda$. The other direction of (12) is obvious.

(13) LEMMA: *Let V be a free left A-module with a finite basis. If Λ is a left clean R-order in A, and if Λ_1 is an R-order in $\Omega(V)$, on the same level as Λ, then Λ_1 is also left clean.*

PROOF. By (8) we have a Morita equivalence between Λ and Λ_1 via a special projective Λ-lattice E, which spans V, and we have to show that 'being clean' is invariant under such Morita equivalences. $\Lambda_1 = \mathrm{End}_\Lambda(E)$, if M is a special projective Λ_1-lattice, say $KM = \Omega(V)^{(n)}$, then $\mathrm{Hom}_{\Lambda_1}(E^*, M)$ is a projective Λ-lattice. We have

$$\begin{aligned}K \otimes_R \mathrm{Hom}_{\Lambda_1}(E^*, M) &\cong \mathrm{Hom}_{\Omega(V)}(V^*, KM)\\ &\cong \mathrm{Hom}_{\Omega(V)}(V^*, \Omega(V)^{(n)}) \cong \mathrm{Hom}_{\Omega(V)}(V^*, \Omega(V))^{(n)} \cong V^{(n)} \cong A^{(n\cdot m)},\end{aligned}$$

where $V \cong A^{(m)}$. Here $E^* = \mathrm{Hom}_\Lambda(E, \Lambda)$ and $V^* = \mathrm{Hom}_A(V, A)$. Thus $\mathrm{Hom}_{\Lambda_1}(E^*, M)$ is a special projective Λ-lattice; whence a progenerator, Λ being clean. Consequently, M is a progenerator, since progenerators are preserved under Morita equivalences. In [8] we have shown that in general, clean orders are not preserved under Morita equivalences.

(14) THEOREM: *An R-order Λ is left clean if and only if is right clean (with the obvious definition of right clean).*

PROOF. Let M be a special projective right Λ-module. We have to show that M is a progenerator for the category of right Λ-modules. Since M is a projective right Λ-module, we have the isomorphism

$$\mu : M \otimes_\Lambda \mathrm{Hom}_\Lambda(M, \Lambda) \to \Lambda_l(M)$$

but M is naturally isomorphic to $\mathrm{Hom}_\Lambda((M, \Lambda), \Lambda)$ and hence $\mathrm{Hom}_\Lambda(M, \Lambda)$ is a projective left Λ-module. Since M was special projective, so is $\mathrm{Hom}_\Lambda(M, \Lambda)$; i.e. $\mathrm{Hom}_\Lambda(M, \Lambda)$ is a progenerator for the category of left Λ-lattices. Using (5) we conclude that $\mathrm{Hom}_\Lambda(M, \Lambda)$ is invertible. But the inverse of $\mathrm{Hom}_\Lambda(M, \Lambda)$ is M, whence M is invertible and thus M is a progenerator for the category of right Λ-lattices.

(15) COROLLARY: *The Cartan matrix for the left $\Lambda/p\Lambda$-modules is non-singular of and only if the Cartan matrix for the right $\Lambda/p\Lambda$-modules is non-singular; here Λ is any R-order in A and p a prime ideal in R.*

(16) LEMMA: *Let Λ be a clean R-order in A. Then the two-sided Λ-modules in A which span A and are left Λ-projective form a group.*

PROOF. Claim $\Lambda_r(M) = \Lambda$, if M is a two-sided Λ-module in A, which spans A and is left Λ-projective. From (10, iii) we conclude that $R_p \otimes_R M \cong (R_p \otimes_R \Lambda)\alpha_p$ for a regular element α_p in A. Then $\Lambda_r(R_p \otimes_R M) = \alpha_p^{-1}(R_p \otimes_R \Lambda)\alpha_p$, since $\Lambda_1(R_p \otimes_R M) = R_p \otimes_R \Lambda)$. Thus $\alpha^{-1}(R_p \otimes_R \Lambda) \alpha_p \supset R_p \otimes_R \Lambda$. But a proper inclusion is impossible (cf. [5]). Hence

$\Lambda_r(R_p \otimes_R M) = R_p \otimes_R \Lambda$ and hence $\Lambda_r(M) = \Lambda$. This proves the claim. If now M, N are Λ-modules with the above properties, then M and N are invertible R-lattices in A (cf. (12)) with $\Lambda_l(M) = \Lambda_l(N) = \Lambda_r(M) = \Lambda_r(N)$. Moreover, MN is invertible; in fact, the product MN is proper and hence $\Lambda_l(M) = \Lambda_l(MN) = \Lambda$ (cf. [3]). Hence $\Lambda_l(N^{-1}M^{-1}) = \Lambda_r(MN) = \Lambda$ and $\Lambda_r(N^{-1}M^{-1}) = \Lambda_l(MN) = \Lambda$. Thus $(MN)(N^{-1}M^{-1}) = \Lambda$ and $(N^{-1}M^{-1})(MN) = \Lambda$; i.e. MN is invertible; similarly one shows that NM is invertible.

(17) REMARK (i): (16) becomes wrong if Λ is not clean. In fact, Faddeev [5] has given a counter example.

(ii) If Λ is a clean order in A, then among the orders in A which lie on the same level via a projective Λ-ideal I in A there are no inclusion relations; this follows immediately from the claim since I is locally free.

REFERENCES

C. W. CURTIS AND I. REINER

[1] 'Representation theory of finite groups and associative algebras'. Interscience, N.Y. 1962.

E. C. DADE, O. TAUSSKY AND H. ZASSENHAUS

[2] 'On the theory of orders, in particular on the semigroup of ideal classes and genera of an order in an algebraic number field.' Math. Ann. 148, (1962), 31–64.

M. DEURING

[3] 'Algebren,' Springer, Berlin (1935).

S. ENDO

[4] 'Completely faithful modules and quasi-Frobenius algebras.' J. Math. Soc. Japan, 19, (1967), 437–456.

D. K. FADDEEV

[5] 'An introduction to the multiplicative theory of integral representation modules'. Trudy Math. Inst. Steklov, 80, (1965), 145–182.

T. Y. LAM

[6] Induction theorems for Grothendieck groups and Whitehead groups of finite groups'. Ann. Scient. Ec. Norm. Sup. 4e serie, t.l. (1968), 91–148.

K. MORITA

[7] 'Duality for modules'. Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 6 (1958), 83–142.

K. W. ROGGENKAMP

[8] 'Das Krull Schmidt Theorem für projective Gitter über lokalen Ringen'. Mitt. Math. Sem. Giessen 80 (1969), 29–50.

[9] 'Projective modules over clean orders'. Comp. Math. 21 (1969), 185–194.

J. R. STROOKER

[10] 'Faithfully projective modules and clean algebras', Ph. D. thesis, Rijksuniversiteit Utrecht, (1965).

(Oblatum 8-IV-1969)

Math. Institut der Justus Liebig Universität
Arndtstrasse 2
63 - Giessen - Germany