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RIESZ TRACE CLASS OPERATORS

by

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1. Introduction

When, in certain situations, operators on functions defined on topological spaces are considered, an interesting project, formulated by James Eells, is to investigate whether compactness of the topological space can be replaced by compactness of the operator: that is one should try to 'put the compactness into the operator'. Leray in his paper [9] defines the trace of a linear operator on a possibly infinite dimensional space E when, by factoring out the subspace K on which the operator is nilpotent, the quotient space E/K is finite dimensional. However, as indicated in the example below, this approach neglects the topology because the space K may not in general be closed. The object of this paper is, like [9] to define the trace of a large class of operators on Hilbert space, including the nilpotent operators. This is done by considering operators of the form $C+Q$, where C is an operator of trace class on H and Q is quasi-nilpotent. That is, 'we put the nilpotency into the operator'.

The author is much indebted to Professor Eells for suggesting that the notion of trace be extended, and for several stimulating conversations.

2. Examples

Leray [9] considers an endomorphism T of a linear space E and puts $K = \bigcup_{p=1}^{\infty} \{x : T^p x = 0\}$. Clearly K is invariant under T.

If E/K is finite dimensional Leray defines the trace of T to be the trace of the induced operator on E/K. This definition is justified because:

a) if E' is a subspace of E invariant under T and T'' is the induced operator on $E'' = E/E'$, then writing $T' = T|E$,

$$\text{Tr}\,(T) = \text{Tr}\,(T') + \text{Tr}\,(T'') \qquad \text{(see [9])},$$

and b) if T is nilpotent on some finite dimensional space K then

$$\text{Tr}\,(T, K) = 0.$$

However, if the dimension of E is infinite the following simple example shows that K may not be closed:

EXAMPLE 2.1. Consider the Hilbert space

$$l^2 = \{(x_1, x_2, \cdots) : \sum |x_i|^2 < \infty\}$$

and consider the bounded linear operator $T : l^2 \to l^2$ defined by:

$$T(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, \cdots).$$

Then, for example, T^n is zero on the subspace E_n of l^2 spanned by the first n coordinates, and $K = \bigcup_{p=1}^{\infty}\{x : T^p x = 0\}$ consists of all vectors with only finitely many non-zero coordinates.

Clearly K is not the whole of l^2 but the closure of K is, that is K is dense in l^2. Therefore K is not closed.

This example is useful for indicating some of the other difficulties that occur, as the following discussion shows.

Consider the particular vector

$$y = (1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots 1/n, \cdots) \text{ in } l^2$$

and write

$$y_n = (1, \tfrac{1}{2}, \tfrac{1}{3}, \cdots 1/n, 0, 0 \cdots) \in K \subset l^2,$$

so that $y_n \to y$ in l^2.

It might be supposed that, although K for some operator S on l^2 is not closed, for any $y \in \bar{K}$ we have $S^n y \to 0$ as $n \to \infty$. However, by considering the operator $S = 2T$ in the above situation we see that we still have

$$y_n \to y \text{ and } S^m y_n = 0 \text{ for } m > n, \text{ but now } ||S^n y|| \to \infty.$$

In spite of the above difficulties it might still be hoped that some of Leray's theory of trace could be extended if the closure $\bar{K}$, of the subspace K on which the operator T is nilpotent, is considered in place of K. However, an important property of K is that $T^{-1}K = K$.

EXAMPLE 2.2. In general $T^{-1}\bar{K} \not\subset \bar{K}$.

To see this consider the space $E = l^2$ and the operator T as described in 2.1. Also consider a one dimensional space $F \simeq R$.

Consider the linear operator P on $E \oplus F$ which acts like T on E and sends the unit vector $f \in F$ onto the vector $y = \{1/n\} \in E$. With the usual norm $|| \;||_E + || \;||_F$ on $E \oplus F$, $P(0, f) = y$ is in $\bar{K}$ for P, but $(0, f) \notin \bar{K}$.

However, a probably more important reason why the closure $\bar{K}$ should not be factored out is also indicated by Example 2.1.

On each subspace of l^2 spanned by finitely many of the standard coordinates the left shift operator T is nilpotent, and so has zero spectrum and zero trace. It is, therefore perhaps reasonable to define the trace of $T|K$ to be zero.

However, $\bar{K}$ is the whole of l^2 and it is well known that the spectrum of the left shift on l^2 is the whole of the unit circle $\{\lambda : |\lambda| \leqq 1\}$, see [14] page 266.

If there is to be any relation between trace and spectrum, and if addition formulae for trace, (such as a) at the beginning of this section), are to be valid in any sense, it does not seem reasonable that operators with such a large non-zero spectrum should be ignored. Ideas such as this are behind Definition 4.3 and Definitions 8.1 and 8.2, where operators with zero spectrum, that is quasi-nilpotent operators, are defined to have zero trace.

As described in the introduction, instead of picking out the part of the space E on which T is nilpotent, we pick out the nilpotent part of T. We do this by considering Riesz operators.

3. Riesz operators

In this section we consider a bounded operator T on a complex Banach space E whose spectrum is denoted by $\sigma(T)$.

DEFINITION 3.1. $\lambda \in \sigma(T)$ is a Riesz point for T if E is the direct sum

$$E = N(\lambda) \oplus F(\lambda)$$

where

i) $N(\lambda)$ is finite dimensional,

ii) $F(\lambda)$ is closed,

iii) $TN(\lambda) \subset N(\lambda)$ and $TF(\lambda) \subset F(\lambda)$,

iv) $\lambda - T$ is nilpotent on $N(\lambda)$,

and

v) $\lambda - T$ is a homeomorphism of $F(\lambda)$.

DEFINITION 3.2. T is called a Riesz operator if every non-zero point of its spectrum is a Riesz point.

A compact operator is therefore a Riesz operator and the spectrum of a Riesz operator is like that of a compact operator: the non zero part of the spectrum consists of an, at most countable, number of eigenvalues whose only accumulation point is zero.

Riesz operators were introduced by Ruston [11] in his work on Fredholm equations in Banach space. Ruston [12] showed that Riesz operators were the largest class of operators to which his Fredholm determinant theory could be applied. Ruston also showed that the Riesz operators coincided with the class of asymptotically quasi-compact operators:

DEFINITION 3.3. K is quasi-compact if K^n is compact for some n. K is asymptotically quasi-compact if

$$\lim_{n\to\infty} \{ \inf_{C \in \text{compact}} ||K^n - C||\}^{1/n} = 0.$$

Another proof that the asymptotically quasi-compact operators are the Riesz operators is outlined in the exercises on page 323 in Dieudonné [2]. Certainly, therefore, quasi-compact operators are Riesz operators and such operators occur in potential theory. (See Yosida [17) Chapter X.)

DEFINITION 3.4. An element x in a Banach algebra is said to be quasi-nilpotent if $\lim_{n\to\infty} ||x^n||^{1/n} = 0$.

Clearly nilpotent elements are quasi-nilpotent.

If $\mathscr{B}(E)$ denotes the algebra of bounded operators on our Banach space E and $\mathscr{K}(E)$ denotes the closed ideal of compact operators write φ for the quotient map $\varphi : \mathscr{B}(E) \to \mathscr{B}(E)/\mathscr{K}(E)$. We then observe immediately:

LEMMA 3.5. *$T \in \mathscr{B}(E)$ is a Riesz operator if and only if $\varphi(T)$ is quasi-nilpotent in $\mathscr{B}(E)/\mathscr{K}(E)$.*

The study of Riesz operators was taken up in the work of Caradus [1] and West [15], [16].

West [16] in particular obtains useful results on the decomposition of Riesz operators and we quote some of these now:

DEFINITION 3.6. If K is a Riesz operator on a Banach space E, K is said to be fully decomposable if $K = C+Q$ where C is a compact operator, Q is quasi-nilpotent and $CQ = QC = 0$.

K is said to be decomposable if we just know it can be written: $K = C+Q$. We call this a 'West decomposition' of K.

Suppose the non-zero points of the spectrum of K are arranged in a sequence $\{\lambda_i\}$, so that $\lambda_i \to 0$ as $n \to \infty$ if the spectrum is infinite. Write P_i for the spectral projection associated with λ_i, so that in the notation of 3.1 the range of P_i is $N(\lambda_i)$. West then shows that K is fully decomposable if $\sum_{i=1}^{\infty} KP_i$ is convergent in the uniform topology for some arrangement of the λ_i. This is because if $C_n = \sum_{i=1}^{n} KP_i$ and $Q_n = K - C_n$ then $C_n Q_n = Q_n C_n = 0$ and by hypothesis $C_n \to C = \sum_{i=1}^{\infty} KP_i$, which is therefore the limit of the finite dimensional operators C_n, and so is compact. In this situation West also proves:

THEOREM 3.7. *Suppose $K = C+Q$ is a fully decomposable Riesz* *operator and suppose $f(z)$ is any single valued function analytic on a neighbourhood of $\sigma(K)$ satisfying $f(0) = 0$. Then $f(K)$ is a fully decomposable Riesz operator and*

$$f(K) = f(C) + f(Q).$$

This result is true because of the strong disjointness of C and Q.

However, the main achievement of West's paper is to show that if K is any Riesz operator on a Hilbert space H then K is decomposable. West's method uses a super-diagonalization process to represent the Riesz operator K as

$$Kx = \sum_j \alpha_j \langle x, e_j \rangle e_j + \sum_j \langle x, e_j \rangle f_{j-1} + Ky.$$

Here the e_j are an orthonormal set, $\langle y, e_j \rangle = 0$ for $j = 1, 2, \cdots$, and $Ke_j = \alpha_j e_j + f_{j-1}$, where f_{j-1} is in the subspace spanned by $[e_1 \cdots e_{j-1}]$. It is easily seen that each α_j is in $\sigma(K)$, and if λ is a non-zero point of $\sigma(K)$ the diagonal multiplicity of λ is the number of times λ occurs in the sequence $\{\alpha_j\}$. Finite dimensional arguments then show that the diagonal multiplicity of λ is equal to its algebraic multiplicity as an eigenvalue of K – that is the dimension of $N(\lambda)$.

The operator C is defined to be

$$C = \sum \alpha_j \langle x, e_j \rangle e_j, \quad (x \in H).$$

C is the limit in the uniform norm of the finite dimensional operators $C_n = \sum_{j=1}^{n} \alpha_j \langle x, e_j \rangle e_j$ and so C is compact.

If Q is defined to be

$$Q = K - C$$

so that $Qx = \sum_j \langle x, e_j \rangle f_{j-1} + Ky$, it can be shown that Q is quasi-nilpotent. Thus the decomposition gives an analogue of Fitting's Lemma: on any space spanned by finitely many of the $\{e_j\}$ K is a homeomorphism, and on the complement of the closure of the space spanned by all the $\{e_j\}$ it is quasi-nilpotent.

Details of the above work can be found in West [16] and the result can be summarized in

THEOREM 3.8. *If K is a Riesz operator on a Hilbert space H then $K = C + Q$ where C is a compact operator and Q is quasi-nilpotent. Furthermore, C is normal, $\sigma(C) = \sigma(K)$ and the non-zero eigenvalues of C and K have the same algebraic multiplicities.*

The eigenvalues of C and their diagonal multiplicity in the above decomposition, that is the sequence $\{\alpha_j\}$, is therefore uniquely determined up to re-arrangement.

4. Riesz trace class operators

We first recall the definition that an arbitrary bounded operator A on a Hilbert space H be in the usual trace class. For any bounded operator A the operator A^*A is positive definite and so has a positive square root

$$B = (A^*A)^{\frac{1}{2}}.$$

DEFINITION 4.1. The operator A is said to be in the trace class of operators on H if for any orthonormal basis $\{\varphi_i\}$ of H the series $\sum_{i=1}^{\infty} \langle B\varphi_i, \varphi_i \rangle$ is absolutely convergent, where $B = (A^*A)^{\frac{1}{2}}$.

It is easily checked (see [7]) that this sum is independent of the orthonormal basis $\{\varphi_i\}$, and if A is of trace class then the sum $\sum_{i=1}^{\infty} \langle A\varphi_i, \varphi_i \rangle$ is absolute convergent and independent of the basis. As in the finite dimensional situation we state:

DEFINITION 4.2. If A is of trace class then the trace of A is

$$\mathrm{Tr}\,(A) = \sum_{i=1}^{\infty} \langle A\varphi_i, \varphi_i \rangle.$$

A result of Lidskii [8] shows that $\mathrm{Tr}\,(A) = \sum_{i=1}^{\infty} \mu_i$, where $\{\mu_i\}$ is a listing of the eigenvalues of A, repeated according to multiplicity. Furthermore, $\sum_{i=1}^{\infty} \mu_i$ is absolutely convergent.

If A has finite dimensional range and, as above, $B = (A^*A)^{\frac{1}{2}}$ then we define a norm $||A||_1 = \mathrm{tr}\,(B)$.

The trace class operators are the closure of the finite dimensional operators in this norm. For a full discussion of operators of trace class see [7] or [13].

The object of this paper is to define a notion of trace for a wider class of operators, including the nilpotent ones. The trace of a finite dimensional nilpotent operator is zero: such an operator has no non-zero eigenvalues. More generally, the only point in the spectrum of a quasi-nilpotent operator is zero and in Dunford and Schwartz [3] it is shown that if a quasi-nilpotent operator N is of trace class then $\mathrm{Tr}\,(N) = 0$. Thus motivated we state:

DEFINITION 4.3. If N is a quasi-nilpotent operator on a Hilbert space H, that is $\lim_{n=\infty} ||N^n||^{1/n} = 0$, then, whether N is of trace class or not, we define the trace of N to be $\mathrm{Tr}\,(N) = 0$.

REMARKS 4.4. There are, however, some problems with definition 4.3. For example even if Q_1 and Q_2 are quasi-nilpotent we do not, in general, know that $Q_1 + Q_2$ is quasi-nilpotent. Also, if Q is quasi-nilpotent and A is any bounded operator, then it is not necessarily true that AQ is quasi-nilpotent. However, we do have:

LEMMA 4.5. *If* $r(B) = \sup_{\lambda \in \sigma(B)} |\lambda| = \lim_{n\to\infty} ||B^n||^{1/n}$ *denotes the spectral radius of a bounded operator B on H then if A and B commute*

i) $$r(A+B) \leqq r(A)+r(B)$$

ii) $$r(AB) \leqq r(A)r(B).$$

PROOF. See [3].

Therefore, if Q_1 and Q_2 are quasi-nilpotent and $Q_1 Q_2 = Q_2 Q_1$ then $Q_1 + Q_2$ is quasi-nilpotent so

$$\mathrm{Tr}\,(Q_1+Q_2) = 0.$$

Similarly, if Q is quasi-nilpotent and A commutes with Q

$$\mathrm{Tr}\,(QA) = \mathrm{Tr}\,(AQ) = 0.$$

Continuing this line of thought we give

DEFINITION 4.6. Suppose K is a Riesz operator on a Hilbert space H, so that by theorem 3.8, $K = C+Q$ is a West decomposition of K, where C is compact and Q is quasi-nilpotent. If C in the above decomposition is of trace class then we say K is of Riesz trace class and we define

$$\mathrm{Tr}\,(K) = \mathrm{Tr}\,(C).$$

REMARKS 4.7. This definition is independent of the decomposition of K into $C+Q$ because

i) $\sigma(C) = \sigma(K)$ and the non-zero eigenvalues of C and K have the same algebraic multiplicities and

ii) we have observed that $\mathrm{Tr}\,(C) = \sum \mu_i$, this sum being absolutely convergent, where $\{\mu_i\}$ is a listing of the eigenvalues of C, repeated according to multiplicity. Thus we have

PROPOSITION 4.8. *If K is an operator of Riesz trace class then* $\mathrm{Tr}\,(K) = \sum_{i=1}^{\infty} \mu_i$, *where* $\{\mu_i\}$ *is a listing of the eigenvalues of K, repeated according to multiplicity.*

PROPOSITION 4.9. *Suppose K is a bounded operator on a Hilbert space H such that the non-zero part of* $\sigma(K)$ *consists only of isolated eigenvalues each of finite algebraic multiplicity, and suppose if* $\{\mu_i\}$ *is a listing of the non-zero eigenvalues, repeated according to multiplicity, that we have* $\sum_{i=1}^{\infty} |\mu_i| < \infty$.

Then

a) K is Riesz operator,

b) in the West decomposition of K as $C+Q$, C is a trace class operator,

c) trace K is defined and $\mathrm{Tr}\,(K) = \mathrm{Tr}\,(C) = \sum \mu_i$.

That is, K is of Riesz trace class.

PROOF: a) The hypothesis implies that any non-zero eigenvalue $\lambda \in \sigma(K)$ occurs in the listing $\{\mu_i\}$ only finitely many times. Thus the spectral projection $P(\lambda; K)$ corresponding to λ has finite dimensional range $N(\lambda; K)$, of dimension equal to the number of times λ occurs in the $\{\mu_i\}$. $N(\lambda; K)$ thus has a closed complement $F(\lambda; K)$ and $\lambda - K$ is nilpotent on $N(\lambda; K)$ and a homeomorphism on $F(\lambda; K)$. Thus λ is a Riesz point and K is a Riesz operator.

b) As a Riesz operator on H by Theorem 3.8. K has a West decomposition $K = C+Q$ where, for any $x \in H$,

$$x = \sum_{j=1}^{\infty} \langle x, \theta_j \rangle e_j + y,$$

the e_j are an orthonormal set and $\langle y, e_j \rangle = 0$, and: $Kx = Cx+Qx$ where, in the present situation,

$$Cx = \sum_{j=1}^{\infty} \mu_j \langle x, e_j \rangle e_j .$$

Suppose the orthonormal set $\{e_j\}$ is extended by the set $\{e'_j\}$ so that $\{e_j\} \cup \{e'_j\}$ is an orthonormal basis for H. Then clearly $Ce'_j = 0$ for all j. For this orthonormal basis therefore: $\sum_{j=1}^{\infty} ||Ce_j|| = \sum_{j=1}^{\infty} |\mu_j| < \infty$, so (see Theorem 8 of Chapter I, § 2 of [7])

C is of trace class.

c) Furthermore, $\text{Tr}(K) = \text{Tr}(C) = \sum_{i=1}^{\infty} \mu_i$

REMARKS 4.10. We have, therefore, defined the trace of any operator on H, the sum of whose eigenvalues, repeated according to multiplicity is absolutely convergent. In view of the finite dimensional situation, where the trace of a matrix is the sum of its eigenvalues, our definition seems quite satisfactory.

EXAMPLE 4.11. We give an example to show that we have defined the trace of a larger class of operators and that the trace is not now related to sums of basis elements.

$H = l^2$ and C and Q are defined on l^2 by

$$C : (x_1, x_2, x_3 \cdots) \to \left(x_1, \frac{x_2}{2^2}, \frac{x_3}{3^2} \cdots\right)$$

$$Q : (x_1, x_2, x_3 \cdots) \to (0, x_1, 0, x_2 \cdots)$$

so that C is compact and of trace class: $\text{Tr}(C) = \sum 1/n^2$. $Q^2 = 0$ and so Q is certainly quasi-nilpotent.

We put $K = C+Q$ so that K is a Riesz operator and $\text{Tr}(K) = \text{Tr}(C)$. However, if we take the usual basis $\varphi_i = (\delta_{in})$ for l^2 then the series

$\sum \langle K^* K\varphi_i, \varphi_i \rangle$ diverges. Thus $K^* K$ is not of trace class, so certainly $(K^* K)^{\frac{1}{2}}$ is not of trace class and, therefore, (see [13]) K is not of trace class.

The trace of a Riesz trace class operator K is, though, independent of the basis of the space H, in fact:

LEMMA 4.12. *If B is a bounded map on H with a bounded inverse, then*

$$\text{Tr}(BKB^{-1}) = \text{Tr}(K).$$

PROOF. If $K = C+Q$ is a decomposition of K then

$$BKB^{-1} = BCB^{-1} + BQB^{-1}$$

and if Q is quasi-nilpotent so is

$$BQB^{-1}.$$

Therefore $\text{Tr}(BKB^{-1}) = \text{Tr}(BCB^{-1}) = \text{Tr}(C)$, by the theorem for trace class operators.

LEMMA 4.13. i) *If K is of Riesz trace class and α is any scalar then K is of Riesz trace class and*

$$\text{Tr}(\alpha K) = \alpha \, \text{Tr}(K)$$

ii) *If K is of Riesz trace class and K^* is its adjoint then K^* is of Riesz trace class and*

$$\text{Tr}(K^*) = \overline{\text{Tr}(K)}.$$

PROOF. i) Suppose $K = C+Q$ is a decomposition of K. Then $\alpha K = \alpha C + \alpha Q$. However, Q quasi-nilpotent implies αQ quasi-nilpotent so

$$\text{Tr}(\alpha K) = \text{Tr}(\alpha C) = \alpha \, \text{Tr}(C).$$

ii) Similarly, $K^* = C^* + Q^*$ and Q quasi-nilpotent implies Q^* is, so $\text{Tr}(K^*) = \text{Tr}(C^*) = \overline{\text{Tr}(C)}$.

LEMMA 4.14. i) *If K_2, K_1 are Riesz trace class operators on H and $K_1 K_2 = K_2 K_1$ then $K_1 + K_2$ is a Riesz trace class operator and*

$$\text{Tr}(K_1 + K_2) = \text{Tr}(K_1) + \text{Tr}(K_2).$$

ii) *If K is a Riesz trace class operator and A is a bounded operator such that $KA = AK$ then AK and KA are of Riesz trace class and* $\text{Tr}(KA) = \text{Tr}(AK)$.

PROOF. i) Suppose

$$K_1 = C_1 + Q_1$$

and

$$K_2 = C_2 + Q_2.$$

Then the hypothesis ensures that

$$Q_1 Q_2 = Q_2 Q_1,$$

so by Lemma 4.5, Q_1+Q_2 is quasi-nilpotent. Thus $(C_1+C_2)+(Q_1+Q_2)$ is a decomposition for K_1+K_2.

ii) If $K = C+Q$ then $AK = AC+AQ$ and the hypothesis ensures that AQ is quasi-nilpotent, so

$$\begin{aligned}\operatorname{Tr}(AK) = \operatorname{Tr}(AC) &= \operatorname{Tr}(CA)\\ &= \operatorname{Tr}(KA).\end{aligned}$$

5. A topological 'five' lemma

DEFINITION 5.1. If T is a bounded operator on a complex Banach space X then the complex number λ is said to be in the resolvent set of T, $\lambda \in \rho(T)$, if

i) $\lambda - T$ has a (left) inverse $R(\lambda; T) = (\lambda - T)^{-1}$,

ii) the range of $\lambda - T$, $\mathscr{R}(\lambda - T)$, is dense in X, and

iii) $R(\lambda; T)$ is continuous.

Under these conditions $R(\lambda - T)$ can be extended from $\mathscr{R}(\lambda - T)$ to a bounded operator on the whole of X, and $R(\lambda; T)$ is then a left and right inverse for $\lambda - T$.

The complement of $\rho(T)$ is the spectrum of T, $\sigma(T)$.

Suppose now that M is a closed subspace of X invariant under T. Write $T|M$ for the restriction of T to M and T_M for the operator induced by T on X/M.

We now quote some results from West [16]:

LEMMA 5.2. *Suppose T has a connected resolvent set and M is a closed subspace invariant under T, then M is invariant under $R(\lambda; T)$ for all $\lambda \in \rho(T)$.*

West uses this result to prove the following lemmas:

LEMMA 5.3. *K is a Riesz operator on the Banach space X and M is a closed subspace invariant under K. Then:*

i) $\rho(K) \subset \rho(K|M)$ and $R(z; K|M) = R(z, K)|M$ for $z \in \rho(K)$.

ii) *$\rho(K) \subset \rho(K_M)$ and $R(z; K)_M = R(z; K_M)$ for $z \in \rho(K)$.*

LEMMA 5.4. *Under the same hypotheses:*

i) *$K|M$ is a Riesz operator on M;*

ii) *K_M is a Riesz operator on X/M.*

Further, if for a non-zero $\lambda \in \sigma(K)$ we write $P(\lambda; K)$ for the corresponding spectral projection, $N(\lambda; K)$ for the range of $P(\lambda; K)$ and $F(\lambda; K)$ for its null space, then we have:

COROLLARY 5.5. i) *If λ is a non-zero point of $\sigma(K|M)$ then*

$$P(\lambda;\, K|M) = P(\lambda;\, K)|M,$$
$$N(\lambda;\, K|M) = N(\lambda;\, K)_{\cap} M,$$
$$F(\lambda;\, K|M) = F(\lambda;\, K)_{\cap} M.$$

ii) *If μ is a non-zero point of $\sigma(K_M)$ then*

$$P(\mu;\, K_M) = (P(\mu;\, K))_M,$$
$$N(\mu;\, K_M) = \{x+M : x \in N(\mu;\, K)\},$$
$$F(\mu;\, K_M) = \{x+M : x \in F(\mu;\, K)\}.$$

What we wish to prove now are partial converses of these results. Consider the following

HYPOTHESIS 5.6. *Suppose, as above, that X is a Banach space, T is a bounded linear operator on X and M is a closed subspace of X invariant under T. Thus we have an endomorphism of the exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & X/M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T|M} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T_M} & & \\ 0 & \longrightarrow & M & \longrightarrow & X & \longrightarrow & X/M & \longrightarrow & 0 \end{array}$$

LEMMA 5.7. *Suppose hypotheses 5.6 are satisfied and suppose λ is a complex number such that $\lambda - T|M$ is a monomorphism on M and $\lambda - T_M$ is a monomorphism on X/M. Then $\lambda - T$ has a left inverse on X.*

PROOF. The proof, by diagram chasing, is just part of the algebraic 'five' lemma (see [10]).

The following two results, Lemma 5.8 and Lemma 5.9, are special cases of results concerning diagrams of topological groups to be found in [4]. However, we include them here in their Banach space setting.

LEMMA 5.8. *Suppose hypotheses 5.6 are satisfied and suppose λ is a complex number such that*

$$\mathscr{R}((\lambda - T)|M) \text{ is dense in } M$$

and $\quad \mathscr{R}((\lambda - T)_M)$ *is dense in* X/M,

then $\quad \mathscr{R}(\lambda - T)$ *is dense in* X.

PROOF. Consider any $x \in X$ and any $\varepsilon > 0$.

In X/M there is a coset $b+M$ such that

$$\|(\lambda - T_M)(b+M) - (x+M)\| < \varepsilon/3$$

in the X/M norm.

That is $$\|(\lambda-T)b-x+M\| < \varepsilon/3$$
or $$\inf_{m\in M} \|(\lambda-T)b-x+m\| < \varepsilon/3.$$

Certainly, therefore, there is an $m \in M$ such that

$$\|(\lambda-T)b-x+m\| < 2\varepsilon/3.$$

Because $\mathscr{R}((\lambda-T)|M)$ is dense in M there is an $a \in M$ such that

$$\|(\lambda-T)a-m\| < \varepsilon/3.$$

Thus

$$\|(\lambda-T)(a+b)-x\| \leqq \|(\lambda-T)a-m\|+\|(\lambda-T)b-x+m\| < \varepsilon.$$

Therefore, $\mathscr{R}(\lambda-T)$ is dense in X.

LEMMA 5.9. *Suppose the hypotheses* 5.6 *are satisfied and suppose that* λ *is a complex number such that*:

a) $(\lambda-T)^{-1}$ *exists (but is not necessarily continuous) on* $\mathscr{R}(\lambda-T)$;

b) $\mathscr{R}(\lambda-T)_\cap M = (\lambda-T)M$ *is dense in* M *and* $(\lambda-T)^{-1}|\mathscr{R}(\lambda-T)_\cap M$ *is continuous, (so* $(\lambda-T)^{-1}$ *has a continuous extension to all* M);

c) M *is also invariant under* $(\lambda-T)^{-1}$, *and*

d) *the induced operator* $(\lambda-T)_M^{-1}$ *is continuous on* $\mathscr{R}((\lambda-T)_M)$ *and* $\mathscr{R}((\lambda-T)_M)$ *is dense in* X/M.

Then $(\lambda-T)^{-1}$ *is continuous on* $\mathscr{R}(\lambda-T)$, *(which is dense in* X *by Lemma* 5.8).

PROOF. To check the continuity of $(\lambda-T)^{-1}$ it is sufficient to do so at the origin, and because we are discussing metric spaces it is sufficient to investigate sequences. Suppose, therefore, that $\{x_n\} \subset \mathscr{R}(\lambda-T)$ is any sequence of points such that $x_n \to 0$ in X. Write z_n for the coset x_n+M in X/M. Then certainly $z_n \to 0 \in X/M$ in the X/M norm, so as $(\lambda-T)_M^{-1}$ is continuous

$$(\lambda-T)_M^{-1}z_n \to 0 \text{ in } X/M.$$

Consider now the sequence $B_m = \{x \in X : \|x\| < 1/m\}$, $m = 1, 2 \cdots$, of open balls, centre the origin in X. As the quotient map Π from X to X/M is open these B_m project onto open neighbourhoods of the origin in X/M.

Therefore, for each m the sequence $\{(\lambda-T)_M^{-1}z_n\}$ lies in $\Pi(B_m)$ from some point onwards. As Π is surjective we can therefore choose a sequence $\{y_n\}$ in X such that $y_n \to 0$ in X and

$$\Pi y_n = (\lambda-T)_M^{-1}z_n.$$

Now certainly $(\lambda - T)$ is continuous on X and so

$$(\lambda - T)y_n = w_n \to 0 \text{ in } X.$$

But also, by commutativity,

$$\Pi(w_n) = z_n = \Pi(x_n).$$

So for each $n : x_n - w_n \in M$. Furthermore, $x_n - w_n \to 0$ in M, so as $(\lambda - T)^{-1}|M$ is continuous

$$(\lambda - T)^{-1}|M \ (x_n - w_n) \to 0.$$

That is $(\lambda - T)^{-1}x_n - (\lambda - T)^{-1}w_n \to 0$ or $(\lambda - T)^{-1}x_n - y_n \to 0$. However, we know $y_n \to 0$ so $(\lambda - T)^{-1}x_n \to 0$ as required. Therefore, $(\lambda - T)^{-1}$ is continuous on X.

REMARK. Dr. T. West has given a shorter proof of 5.9, using the closed graph theorem. However, our proof is valid in incomplete spaces.

Assuming now that X is a Hilbert space, so that we may use Proposition 4.9, relating the trace and eigenvalues, the above three lemmas give us the following general result:

THEOREM 5.10. *Suppose the hypotheses* 5.6 *are satisfied then*

$$\rho(T|M)_\cap \rho(T_M) \subset \rho(T).$$

Furthermore, if T has a connected resolvent set, so that lemmas 5.3 *and* 5.4 *apply, we have that*

$$\rho(T) \subset \rho(T|M)_\cap \rho(T_M)$$

so in this case $\rho(T) = \rho(T|M)_\cap \rho(T_M)$.

Specializing to Riesz operators we have:

COROLLARY 5.11. If *K is an operator on X satisfying the hypotheses* 5.6, *and if $K|M$ and K_M are Riesz operators, then K is a Riesz operator on X and, as above,* $\sigma(K) = \sigma(K|M)_\cup \sigma(K_M)$.

PROOF. Taking complements in the first statement of Theorem 5.10, we have $\sigma(K) \subset \sigma(K|M)_\cup \sigma(K_M)$ so K has a connected resolvent set. Therefore $\rho(K) = \rho(K|M)_\cap \rho(K_M)$ or, equivalently,

$$\sigma(K) = \sigma(K|M)_\cup \sigma(K_M).$$

Finite dimensional arguments then show that for any non-zero $\lambda \in \sigma(K)$ λ is an eigenvalue of K, and its multiplicity is equal to the sum of its multiplicities as an eigenvalue of $K|M$ and K_M. (Counting its multiplicity zero if it is not in $\sigma(K|M)$ or $\sigma(K_M)$.)

We recall that we already know from Lemma 5.4 that if K is a Riesz operator then $K|M$ and K_M are Riesz operators, and we now see in this situation that

$$\sigma(K) = \sigma(K|M) \cup \sigma(K_M),$$

with the eigenvalues on each side counted according to their multiplicity.

THEOREM 5.12. *If K is a Riesz trace class operator, then $K|M$ and K_M are Riesz trace class.*

Conversely, if $K|M$ and K_M are Riesz trace class then K is.

In the above situation we have

$$\operatorname{Tr}(K) = \operatorname{Tr}(K|M) + \operatorname{Tr}(K_M).$$

PROOF. Suppose $\{\mu_i\}$, $\{\mu_i'\}$, $\{\mu_i''\}$ are listings of the non-zero eigenvalues of K, $K|M$ and K_M respectively, with repetitions according to multiplicity.

Then $\{\mu_i'\} \cup \{\mu_i''\} = \{\mu_i\}$, and if $\{\lambda_i\}$ is a listing of $\{\mu_i'\} \cup \{\mu_i''\}$ then $\{\lambda_i\}$ will be just a re-arrangement of $\{\mu_i\}$.

However, all the sums are absolutely convergent so

$$\sum \lambda_i = \sum \mu_i = \sum \mu_i' + \sum \mu_i''.$$

That is

$$\operatorname{Tr}(K) = \operatorname{Tr}(K|M) + \operatorname{Tr}(K_M).$$

The following is just a re-phrasing of the above result:

COROLLARY 5.13. *Suppose K is an endomorphism of the exact sequence of Hilbert spaces*

$$0 \to E' \to E \to E'' \to 0.$$

Suppose either

a) *K is of Riesz trace class on E or*

b) $K|E'$ and $K_{E''}$ *are of Riesz trace class.*

Then all three operators are of trace class and

$$\operatorname{Tr}(K, E) = \operatorname{Tr}(K, E') + \operatorname{Tr}(K, E'').$$

6. Lefschetz formulae

Suppose $E = (E_i, d_i)_{i \in Z}$ is a cochain complex of Hilbert spaces ($d_{i+1} \circ d_i = 0$ for all i). Put

$$Z_i = \operatorname{Ker}\{d_i : E_i \to E_{i+1}\}$$
$$B_i = \operatorname{Im}\{d_{i-1} : E_{i-1} \to E_i\}$$

and $H(E_i) = Z_i/B_i$, the i^{th} cohomology of the complex.

Suppose that $K = (K_i)$ $i \in Z$ is an endomorphism of the complex, i.e. K_i is an endomorphism of E_i and the diagram commutes:

$$\begin{array}{ccc} E_i & \xrightarrow{d_i} & E_{i+1} \\ K_i \downarrow & & \downarrow K_{i+1} \\ E_i & \xrightarrow{d_i} & E_{i+1} \end{array}$$

Clearly (K_i^p) is an endomorphism of E too.

The following formula includes the algebraic part of the Lefschetz theorem, using our extended notion of trace:

THEOREM 6.1. *Suppose* $K = (K_i)$ *is an endomorphism of the complex* $E = (E_i; d_i)$ $i \in Z$.

Assume

a) K_i *is of Riesz trace class on* E_i,

b) $\mathrm{Tr}\,(K_i|E_i)$ *and* $\mathrm{Tr}\,(K_i|B_i)$ *are non-zero for only finitely many* $i \in Z$. *Then in the following expression both sides are defined and equal*:

$$\sum (-1)^i \,\mathrm{Tr}\,(K_i, E_i) = \sum (-1)^i \,\mathrm{Tr}\,(H(K_i), H(E_i)).$$

DEFINITION 6.2. The common value is called the Lefschetz number

$$\Lambda(K) = \Lambda(K, E) \text{ of } K.$$

PROOF. Hypothesis a) ensures that

$$\mathrm{Tr}\,(K_i, E_i),\ \mathrm{Tr}\,(K_i, B_i),\ \mathrm{Tr}\,(K_i, Z_i) \text{ and } \mathrm{Tr}\,(H(K_i), H(E_i))$$

are defined and that K_i is an endomorphism of the exact sequences:

$$0 \to Z_i \to E_i \xrightarrow{d_i} B_{i+1} \to 0$$

$$0 \to B_i \to Z_i \to H(E_i) \to 0.$$

Corollary 5.13 ensures that:

$$\mathrm{Tr}\,(K_i, E_i) = \mathrm{Tr}\,(K_i, Z_i) + \mathrm{Tr}\,(K_{i+1}, B_{i+1})$$

and

$$\mathrm{Tr}\,(K_i, Z_i) = \mathrm{Tr}\,(K_i, B_i) + \mathrm{Tr}\,(H(K_i), H(E_i)).$$

The formula follows by taking alternating sums.

THEOREM 6.3. *Suppose*

$$\cdots \to E''_{i-1} \xrightarrow{\gamma_{i-1}} E'_i \xrightarrow{\alpha_i} E_i \xrightarrow{\beta_i} E''_i \xrightarrow{\gamma_i} E'_{i+1} \xrightarrow{\alpha_{i+1}} \cdots$$

is an exact sequence of Hilbert spaces and suppose $K = (K_i)$ *is an endomorphism of the sequence, for which each* K_i *is of Riesz trace class, so that the traces*

$$\operatorname{Tr}(K_i, E_i'), \operatorname{Tr}(K_i, E_i), \operatorname{Tr}(K_i, E_i'')$$

are defined. Then from the sequences:

$$\begin{aligned}
0 &\to \alpha_i E_i' &&\to E_i &&\to \beta_i E_i &&\to 0\\
0 &\to \gamma_{i-1} E_{i-1}'' &&\to E_i' &&\to \alpha_i E_i' &&\to 0\\
0 &\to \beta_i E_i &&\to E_i'' &&\to \gamma_i E_i'' &&\to 0
\end{aligned}$$

we obtain:

$$\begin{aligned}
\operatorname{Tr}(K_i, E_i) &= \operatorname{Tr}(K_i, \alpha_i E_i') + \operatorname{Tr}(K_i, \beta_i E_i)\\
\operatorname{Tr}(K_i, E_i') &= \operatorname{Tr}(K_i, \gamma_{i-1} E_{i-1}'') + \operatorname{Tr}(K_i, \alpha E_i')\\
\operatorname{Tr}(K_i, E_i'') &= \operatorname{Tr}(K_i, \beta_i E_i) + \operatorname{Tr}(K_i, \gamma_i E_i'').
\end{aligned}$$

If suitable finiteness conditions hold, for example if $\operatorname{Tr}(K_i, \alpha_i E_i')$, $\operatorname{Tr}(K_i, \beta_i E_i)$ *and* $\operatorname{Tr}(K_i, \gamma_i E_i'')$ *are all non-zero for only finitely many* $i \in Z$, *then by taking alternating sums we obtain*

$$\Lambda(K, E) = \Lambda(K, E') + \Lambda(K, E'').$$

7. Determinants

THEOREM 7.1. *Suppose* T *is a Riesz operator on a Hilbert space* H *and suppose* $f(\lambda)$ *is a function which is analytic on some open set containing* $\sigma(T)$. *If* Γ *is a positively oriented contour containing* $\sigma(T)$ *and if* $f(T)$ *is defined, as usual, by the operational calculus*

$$f(T) = \frac{1}{2\Pi i} \int_\Gamma f(\lambda) R(\lambda; T) d\lambda$$

then $f(T)$ *is a Riesz operator whenever* $f(0) = 0$.

PROOF. See Caradus [1] or West [15].

COROLLARY 7.2. *If* T *is of Riesz trace class then* $f(T)$ *is of Riesz trace class.*

PROOF. Suppose $\{\lambda_i\}$ is a listing of the eigenvalues of T repeated according to multiplicity. Then

$$\sum_{i=1}^{\infty} |\lambda_i| < \infty.$$

By the spectral mapping theorem:

$$\sigma(f(T)) = f(\sigma(T)).$$

If μ_0 is a non-zero eigenvalue of $f(T)$ write P_0 for the spectral projection associated with μ_0. If $P(\lambda_k; T)$ is the spectral projection associated with

T and $\lambda_k \neq 0$, then we know $P(\lambda_k; T)$ has finite dimensional range, of dimension equal to the multiplicity of λ_k in $\{\lambda_i\}$.

However, by the operational calculus

$$P_0 = \sum_{k \in \mathscr{S}} P(\lambda_k; T)$$

where

$$\mathscr{S} = \{k : f(\lambda_k) = \mu_0\}.$$

So the multiplicity of μ_0 is the sum of the multiplicities of $\lambda_k \in \sigma(T)$ such that $f(\lambda_k) = \mu_0$. Thus, if $\{\mu_i\}$ is a listing of the eigenvalues of $f(T)$, repeated according to multiplicity, then this listing is the same, up to rearrangement, as the listing $\{f(\lambda_i)\}$ of f acting on the eigenvalues of T. Now $f(\lambda)$ is analytic on a neighbourhood of $\sigma(T)$ and $f(0) = 0$, so $f(\lambda) = \lambda\varphi(\lambda)$, where φ is analytic in the same region. Thus $\varphi(\lambda)$ is bounded in a bounded neighbourhood of the origin, and certainly there is an M such that

$$|\varphi(\lambda_i)| \leqq M, \; i = 1, 2, \cdots.$$

Therefore

$$\sum_{i=1}^{\infty} |\mu_i| = \sum_{i=1}^{\infty} |f(\lambda_i)| = \sum_{i=1}^{\infty} |\lambda_i||\varphi(\lambda_i)| \leqq M \sum_{i=1}^{\infty} |\lambda_i| < \infty.$$

So $f(T)$ is of Riesz trace class and, in fact,

$$\operatorname{Tr}(f(T)) = \sum f(\lambda_i).$$

DEFINITION 7.3. If $-z^{-1} \notin \sigma(T)$ the function $f(\lambda) = \log(1+z\lambda)$ satisfies the above conditions.

Thus $\det(1+zT) = \exp.(\operatorname{tr.}\log(1+zT))$ is defined if T is of Riesz trace class and $-z^{-1}$ is in $\rho(T)$. Approximation arguments using the definition of the integral show it depends continuously on T and is an analytic function of z for $-z^{-1} \in \rho(T)$.

The motivation for the above definition of determinant is the following observation in finite dimensions:

If T is an endomorphism of a finite dimensional space over the complex numbers then:

$$\xi(z) = \det(I - zT) = \prod_{j=1}^{m} (1 - zx_j)$$

where the x_j are the eigenvalues of T. Thus

$$\xi'(z)/\xi(z) = -\sum_{j=1}^{m} x_j/(1-zx_j)$$

and as $\xi(0) = 1$, working formally we have:

$$\log \xi(z) = -\sum_{j=1}^{m} \sum_{k=1}^{\infty} z^k x_j^k / k,$$

that is:

$$\xi(z) = \det(I - zT) = \exp\{-\sum_{k=1}^{\infty} z^k \operatorname{Tr} T^k / k\}.$$

We are indebted to Professor Eells for the following remarks: Suppose as in Theorem 6.1. $K = (K_i)$ is an endomorphism of a complex of Hilbert spaces and suppose each K_i is of Riesz trace class. Then $\operatorname{Tr} K_i^p$ is defined for all p, as is

$$\xi_i(z) = \det(I - zK_i).$$

Working completely formally, under the finiteness hypothesis of Theorem 6.1, so that only finitely many $\xi_i(z)$ are non-zero, we have for each $i \in Z$:

$$\log \xi_i(z) = -\sum_{k=1}^{\infty} (z^k \operatorname{Tr} K_i^k)/k$$

so

$$\sum_i (-1)^i \log \xi_i(z) = -\sum_{k=1}^{\infty} (z^k \Lambda(K^k))/k$$

Therefore, for such a complex and such an endomorphism (K_i) by Riesz trace class operators one can introduce a zeta function

$$\equiv (z) = \frac{\prod (\xi_{2j}(z))}{\prod (\xi_{2j+1}(z))} = \exp\{-\sum s^k \Lambda(K^k))/k\}$$

which looks similar to the zeta function of A. Weil's conjectures. (See the Proceedings of the International Congress of Mathematics, Amsterdam, (1954), or the paper by J. L. Kelley and E. H. Spanier, Euler Characteristics, Pacific Jour. Math. 26 (1968), 317–339.

8. Extensions and concluding remarks

Let us return to our discussion of a Riesz trace class operator K in a Hilbert space H. Such an operator had a West decomposition:

$$Kx = Cx + Qx \text{ for } x \in H,$$

where

$$Cx = \sum_j \alpha_j \langle x, e_j \rangle e_j$$

and

$$Qx = \sum_j \langle x, e_j \rangle f_{j-1} + Ky.$$

Here the α_j are the non-zero eigenvalues of K repeated according to multiplicity, $x = \sum_j \langle x, e_j \rangle e_j + y$ and f_{j-1} is in the subspace spanned by $e_1 \cdots e_{j-1}$. Write M for the closed subspace spanned by all the e_j, $j = 1, 2 \cdots$ and write N for its orthogonal complement, so that $H = M \oplus N$, and in the above decomposition of a vector $x \in H$, $\sum_j \langle x, e_j \rangle e_j \in M$ and $y \in N$.

Because by definition

$$Ke_j = \alpha_j e_j + f_{j-1}$$

we see that $KM \subset M$. However, even restricted to MK is not in general of trace class, in the usual sense of Schatten. For $x \in M$, of course,

$$\begin{aligned} Kx &= \sum \alpha_j \langle x, e_j \rangle e_j + \sum \langle x, e_j \rangle f_{j-1} \\ &= Cx + (Q|M)x, \end{aligned}$$

and $Q|M$ is certainly quasi-nilpotent.

Because M is invariant under K there is an induced operator K_M on $H/M \simeq N$. By Lemma 5.4 K_M is a Riesz operator and by Lemma 5.3 (ii) $\sigma(K_M) \subset \sigma(K)$. Being a Riesz operator $\sigma(K_M)$ consists of a countable collection of non-zero eigenvalues whose only accumulation point is zero. However, a non-zero eigenvalue λ of K_M would be a non-zero eigenvalue of K and the range of the spectral projection corresponding to λ is included, by construction, in M. Thus $\sigma(K_M) = \{0\}$ so K_M is quasi-nilpotent.

The West decomposition, therefore, gives us a subspace M such that on H/M K_M is quasi-nilpotent, and an orthogonal basis $\{e^j\}$ of M such that, because of the super-diagonalisation procedure, $K|M$ is very clearly the sum of a trace class operator and a quasi-nilpotent operator.

Since for every Riesz trace class operator in a Hilbert space there is a subspace M satisfying the above conditions, the situation of the previous sentence could be taken as a definition of Riesz trace class operators. However, a less general but simpler definition is:

Definition 8.1. A bounded linear operator K on a Hilbert space H is said to be of 'generalized trace class' if there is a subspace M of H invariant under K such that: a) $K|M$ is of trace class in the usual sense, (Definition 4.1), and b) the induced operator K_M on H/M is quasi-nilpotent.

The trace of K is then defined to be the trace of $K|M$.

Note that such a K is certainly of Riesz trace class, and to obtain the West decomposition one need only superdiagonalize $K|M$.

If an operator is of trace class or Riesz trace class then so is its adjoint, so considering adjoints and annihilator subspaces we see Definition 8.1 is equivalent to:

DEFINITION 8.2. A bounded linear operator K is of generalized trace class if there is an invariant subspace N of H such that

a) $K|N$ is quasi-nilpotent, and

b) the induced operator K_N on H/N is of trace class in the usual sense.

The trace of K is then defined to be the trace of K_N.

Definition 8.2 may be thought preferable to 8.1 because it is the nilpotent part of the operator that is factored out.

REMARK 8.3. We mention here that following the above lines of development we could study operators in Hilbert space whose non-zero spectrum consisted of a sequence of eigenvalues belonging to l^p for any p, $0 < p < \infty$. Riesz operators in general have a spectrum belonging to c_0 and our Riesz trace class corresponds to the case $p = 1$. In particular, for $p = 2$ we should have a notion of Riesz Hilbert-Schmidt operators. The West decomposition would give us such an operator expressed as the sum of a quasi-nilpotent operator and a compact operator with eigenvalues in l^p. Definition 8.2 could be generalized in a similar manner. These ideas will be dealt with in a future publication.

Using definition 8.2 it is possible to develop a little more detail for the theory of the determinant $\det(1+zT)$ (see Definition 7.3), for a generalized trace class operator T. Because having factored out the nilpotent part, we are considering a trace class operator on $H/N \simeq N^{\perp}$ some of the determinant theory using exterior products can be introduced, as in the work of Grothendieck [6].

We remark that our definition of determinant is more closely related to the algebra, of our more specialized situation, than the definition of Ruston [11], where an almost arbitrary entire function, having zeros of the appropriate order at the eigenvalues of T, plays the role of the determinant of $(1+zT)$.

Finally, Definition 8.2 could be given for Banach spaces, and it might be thought that some of the preceeding theory could be developed in this more general setting. However, this does not seem too promising because, as is discussed by Grothendieck in [5], § 5 no. 1, page 170, unless one knows that the Banach space under consideration satisfies the 'condition d'approximation' one does not know, for example, that if T is a trace class operator such that $T^2 = 0$ then $\mathrm{Tr}\,(T) = 0$. We should not feel justified, therefore, in defining the trace of a quasi-nilpotent operator to be zero, as is implicitly done in Definition 8.2.

Also, not having in the Banach space situation the result that the trace is the sum of the eigenvalues, difficulties occur in proving the analogue of Theorem 5.12.

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REFERENCES

S. R. Caradus
[1] Operators of Riesz type. Pacific Jour. Math. 18 (1966), 61–72.

J. Dieudonné
[2] Foundations of Modern Analysis. Academic Press. New York, 1960.

N. Dunford and J. T. Schwartz
[3] Linear Operators, Part II. Interscience. New York, 1963.

R. J. Elliott
[4] Diagrams of topological groups. To appear.

A. Grothendieck
[5] Produits Tensoriels Topologiques et Espaces Nucléaires. American Math. Soc. Memoirs no. 16. Providence R.1. (1955).

A. Grothendieck
[6] La Théorie de Fredholm. Bull. Soc. Math., France. 84 (1956), 319–384.

I. M. Gelfand and N. Ya. Vilenkin.
[7] Generalized Functions. Volume 4. Academic Press, New York (1964).

V. B. Lidskii
[8] Non self-adjoint operators which have a trace. (Russian). Doklady Acad. Nauk. S.S.S.R. 125 (1959). 485–487.

J. Leray
[9] Théorie des points fixes: indice total et nombre de Lefschetz. Bull. Soc. Math., France, 87 (1959) 221–223.

S. MacLane
[10] Homology. Springer-Verlag. Berlin.

A. F. Ruston
[11) Formulae of Fredholm type for compact linear operators on a general Banach space. Proc. London Math. Soc. (3), 3, (1953), 368–377.

A. F. Ruston
[12] Operators with a Fredholm theory. Jour. London Math. Soc. 29 (1954) 318–326.

R. Schatten
[13] A theory of cross spaces. Annals of Mathematics Studies no. 26. Princeton 1950.

A. E. Taylor
[14] Introduction to Functional Analysis. Wiley. New York 1958.

T. T. West
[15] Riesz operators in Banach spaces. Proc. London Math. Soc. 16 (1966) 131–140.

T. T. West
[16] The decomposition of Riesz operators. Proc. London Math. Soc. 16 (1966) 737–752.

K. Yoshida
[17] Functional Analysis Springer-Verlag. Berlin 1965.

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