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For each $i > 0$, let $I_i = [-1/i, 1/i]$, ${}^0I_i = (-1/i, 1/i)$. The Hilbert cube I^∞ is the product $\prod_{i>0} I_i$, with pseudo-interior $s = \prod_{i>0} {}^0I_i$ (also denoted by ${}^0I^\infty$). Let $I^\alpha = \{x \in I^\infty : x_i = 0$ if $i \notin \alpha\}$, similarly for ${}^0I^\alpha$, and let π_i, τ^α denote the projections onto I_i, I^α, respectively. For $\alpha = \{1, \cdots, n\}$ we write $I^\alpha = I^n$, $\tau^\alpha = \tau^n$. We use the metric $d(x, y) = \max_{i>0} |x_i - y_i|$ in I^∞, and the corresponding supremum metric d^* for maps into I^∞.

DEFINITION (Anderson [1]). A closed subset K of X has *Property Z in X* if for every nonempty, homotopically trivial open set U in X, $U \setminus K$ is nonempty and homotopically trivial.

We show that the graph $G(f)$ of a continuous function $f : I^\infty \to I^\infty$ has Property Z in $I^\infty \times I^\infty$, and that a closed subset $K \subset s$ with dimension k and finite deficiency $2k+1$ has Property Z in s.

LEMMA 1. *Let K be a closed subset of s. Suppose that for every $n \geqq 0$, every map $f : I^n \to s$, and every $\varepsilon > 0$, there exists a map $g : I^n \to s$ such that $d^*(f, g) < \varepsilon$ and $g(I^n) \cap K = \emptyset$. Then K has Property Z.*

PROOF. We show that for every map $f : I^n \to s$ with $f(Bd\, I^n) \cap K = \emptyset$, and every $\varepsilon > 0$, there exists a map $g : I^n \to s$ such that $f/Bd\, I^n = g/Bd\, I^n$, $d^*(f, g) < \varepsilon$, and $g(I^n) \cap K = \emptyset$. Clearly, this implies Property Z. Choose an n-cube J^n in ${}^0I^n$ such that $d(f(I^n \setminus J^n), K) > 0$. Let $0 < \eta < \min\{d(f(I^n \setminus J^n), K), \varepsilon\}$, and let $g_0 : J^n \to s$ be a map such that $d^*(f/J^n, g_0) < \eta$ and $g_0(J^n) \cap K = \emptyset$. Extend the map $f/Bd\, I^n \cup g_0$ to a map $g : I^n \to s$ such that $d^*(f, g) < \eta$. Then $g(I^n) \cap K = \emptyset$.

The condition of the lemma is actually equivalent to Property Z. Moreover, since a closed subset K of I^∞ has Property Z in I^∞ if (and only if) $K \cap s$ has Property Z in s (Anderson [1]), and since there exist arbitrarily small maps of I^∞ into s, the lemma remains true when s is everywhere replaced by I^∞.

Lemma 2. *Let maps $f, g : I^n \to I^{3^n}$, and $\varepsilon > 0$ be given. Then there exists a map $h : I^n \to I^{3^n}$ such that $d^*(g, h) < \varepsilon$ and $f(x) \neq h(x)$ for every x.*

Proof. Assume $\varepsilon < 1$. Choose $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)), d(g(x), g(y)) < \varepsilon/3^{n+1}$. On each interval I_i, $i = 1, \cdots, n$, take a regular subdivision with mesh less than δ/n, and consider the resulting product subdivision $\{J_1^n, \cdots, J_r^n\}$ of I^n into n-cubes. Each cube has diameter less than δ, and no cube meets more than 3^n-1 other cubes. Thus to each cube J_j^n we may assign an integer i_j, $1 \leqq i_j \leqq 3^n$, such that $i_k = i_m$ only if $J_k^n \cap J_m^n = \emptyset$. For each cube J_j^n, select a point p_j in the corresponding interval I_{i_j}, such that $p_j \notin \pi_{i_j} f(J_j^n)$ and

$$d(p_j, \pi_{i_j} g(J_j^n)) < \varepsilon/3^{n+1}.$$

Let V be the collection of vertices of the cubes $\{J_j^n\}$, and define a function $h_0 : V \to I^{3^n}$ as follows: $\pi_{i_j} h_0(v) = p_j$ if $v \in J_j^n$, otherwise $\pi_i h_0(v) = \pi_i g(v)$. Extend h_0 piecewise-linearly to $h : I^n \to I^{3^n}$.

The lemma holds true with I^{3^n} replaced by I^{n+1}, but for our purposes the sharper result is unnecessary.

Theorem 1. *Let $f : I^\infty \to I^\infty$ be a map. Then $G(f)$ has Property Z in $I^\infty \times I^\infty$.*

Proof. Let $\tau_1^\infty, \tau_2^\infty, \tau_2^n$ be the projections of $I^\infty \times I^\infty$ onto $I^\infty \times 0$, $0 \times I^\infty$, $0 \times I^n$, respectively. Suppose a map $g : I^n \to I^\infty \times I^\infty$ and $\varepsilon > 0$ are given. Consider the maps $\tau_2^{3^n} \circ f \circ \tau_1^\infty \circ g, \tau_2^{3^n} \circ g : I^n \to I^{3^n}$. By Lemma 2 there exists a map $h : I^n \to I^{3^n}$ such that $d^*(\tau_2^{3^n} \circ g, h) < \varepsilon$ and $h(x) \neq (\tau_2^{3^n} \circ f \circ \tau_1^\infty \circ g)(x)$ for every x. Consider the map $\bar{g} : I^n \to I^\infty \times I^\infty$, defined by $\tau_1^\infty \circ \bar{g} = \tau_1^\infty \circ g$, $\tau_2^{3^n} \circ \bar{g} = h$, and $\pi_i \circ \tau_2^\infty \circ \bar{g} = \pi_i \circ \tau_2^\infty \circ g$ for $i > 3^n$. We have $d^*(g, \bar{g}) < \varepsilon$ and $\bar{g}(I^n) \cap G(f) = \emptyset$. By the remark following Lemma 1, $G(f)$ has Property Z.

In I^∞ (and in s), homeomorphisms between closed subsets with Property Z can be extended to space homeomorphisms (Anderson [1]). Since $I^\infty \times 0$ has Property Z in $I^\infty \times I^\infty$, the above result guarantees the existence of a homeomorphism H of $I^\infty \times I^\infty$ onto itself, such that $H/G(f) = \tau_1^\infty/G(f)$. In general, we cannot require that $\tau_1^\infty \circ H = \tau_1^\infty$. (With such an H for $f = id$, the map $h : I^\infty \to I^\infty$ defined by $h(x) = \tau_2^\infty \circ H^{-1}(x, p)$, where $0 \neq p \in I^\infty$, would have no fixed point).

A closed subset K of I^∞ or s has *finite deficiency* k if $\tau^k(K) = 0$.

COROLLARY 1. *If* $K \subset I^\infty$ *has finite deficiency* k *and can be imbedded in* I^k, *then it has Property* Z.

PROOF. Let $h : K \to {}^0I^k$ be an imbedding, and let H be a homeomorphism of I^∞ onto itself, such that $\pi_i \circ H = \pi_i$ for $i > k$ and $\tau^k \circ H/K = h$. Partition the set of positive integers into two infinite classes α, β, with $\{1, \cdots, k\} \subset \alpha$, and consider $I^\infty = I^\alpha \times I^\beta$. Clearly, $H(K) \subset G(f)$ for an appropriate map $f : I^\alpha \to I^\beta$, and since Property Z is hereditary (Lemma 1), $H(K)$, and therefore K, has Property Z.

COROLLARY 2. *A closed subset* K *in* I^∞ *with dimension* k *and finite deficiency* $2k+1$ *has Property* Z.

This result for the case $k = 0$ relates to the question, raised by Anderson, of whether a closed set with Property Z in I^∞ (or s) intersects a hyperplane of deficiency one in a set with Property Z in the hyperplane. Pełczynski suggested as a possible counterexample a wild Cantor set in the hyperplane $\{0\} \times \prod_{i>1} I_i$, and Anderson verified that such a set does have Property Z in I^∞.

Since, with K as above, $K \cap s$ has dimension less than or equal to k and finite deficiency $2k+1$, the following theorem may be regarded as a generalization of Corollary 2.

THEOREM 2. *A closed subset* K *in* s *with dimension* k *and finite deficiency* $2k+1$ *has Property* Z.

PROOF. Let a map $f : I^n \to s$ and $\varepsilon > 0$ be given. Let $\gamma = \{2k+2, \cdots, 2k+n+2\}$, $\gamma' = \{1, \cdots, 2k+1\} \cup \{2k+n+3, \cdots\}$. Choose $0 < \delta < \min \{1/2(2k+n+2), \varepsilon/12\}$. Since (s, d) is totally bounded there exists a finite open cover of K with mesh less than δ and ord $\leqq k$. Let L be a realization in

$$U = \{x \in {}^0I^{2k+1} : d(x, 0) < \varepsilon/4\}$$

of the abstract nerve of the cover. Let $\bar{m} : \prod_{i>2k+1} {}^0I_i \to U$ be an extension of the barycentric map $m : K \to L$. Then $\bar{m}$ defines a homeomorphism h of s onto itself, such that $d^*(h, id) \leqq \varepsilon/4$, $\pi_i \circ h = \pi_i$ for $i > 2k+1$, and $\tau^{2k+1} \circ h/K = m$. Let $g : I^n \to s$ be a piecewise-linear map with $d^*(g, h \circ f) < \varepsilon/4$. We construct a map $r : g(I^n) \to s$ such that $d^*(r, id) < \varepsilon/4$ and $r(g(I^n)) \cap h(K) = \emptyset$; r will change coordinates only in the directions given by γ, and will be independent of the coordinates in the directions given by $\{2k+n+3, \cdots\}$. Let $\{\sigma_i\}_1^q$ be an ordering of the simplices of L, with $\dim \sigma_i \leqq \dim \sigma_j$ if $i \leqq j$. Choose a closed cover $\{B_i\}_1^q$ of

L, such that $B_i \subset {}^0st\,\sigma_i$, and $B_i \cap B_j \neq \emptyset$ only if $\sigma_i \subset \sigma_j$ or $\sigma_j \subset \sigma_i$. Let $\eta > 0$ be such that $d(B_i, B_j) > 2\eta$ if $B_i \cap B_j = \emptyset$. For any subset D of ${}^0I^\gamma$ with diam $D < \delta$, let $[D] = \prod_{i\in\gamma} J_i$, where $J_i = [\inf \pi_i(D) - \delta, \sup \pi_i(D)+\delta] \cap I_i$. Let

$$\alpha(D) = \{i \in \gamma : \inf J_i = -1/i\},$$

and

$$\beta(D) = \{i \in \gamma : \sup J_i = 1/i\};$$

then $\alpha \cap \beta = \emptyset$. If $\alpha \cup \beta = \emptyset$, and $y \in {}^0[D] = \prod_{i\in\gamma} {}^\circ J_i$, let $p(y) : [D]\diagdown y \to Bd[D]$ be the projection from y. Otherwise, for each $0 < \xi < \delta$, define

$$[D]_\xi = \prod_{i\in\alpha} [-1/i+\xi, \sup J_i] \times \prod_{i\in\beta} [\inf J_i, 1/i-\xi] \times \prod_{i\notin\alpha\cup\beta} J_i,$$

and let $p(\xi) : [D]_\xi \to Bd[D]_\xi \cap Bd[D]$ be the projection from a point in ${}^0[D]\diagdown[D]_\xi$. In either case, there exists a homotopy $\{p_t\}_{t\geqq 0}$ of $[D]\diagdown y$ into $[D]\diagdown y$, of $[D]_\xi$ into $[D]_\xi$, with $p_0 = p(y)$, $p(\xi)$, respectively, and $p_t = id$ for $t \geqq 1$. Extend each p_t by the identity on ${}^0I^\gamma\diagdown[D]$. For each $\sigma_i \subset L$, let $D_i = \tau^\gamma(m^{-1}({}^0st\,\sigma_i))$. We successively define maps

$$r_1 : g(I^n) \to s, \cdots, r_q : (r_{q-1} \circ \cdots \circ r_1 \circ g)(I^n) \to s$$

as follows. If $\alpha(D_i) \cup \beta(D_i) = \emptyset$, choose

$$y \in {}^0[D_i]\diagdown\tau^\gamma(r_{i-1} \circ \cdots \circ r_1 \circ g)(I^n).$$

Otherwise, choose $0 < \xi < \delta$ such that

$$\tau^\gamma(r_{i-1} \circ \cdots \circ r_1 \circ g)(I^n) \cap [D_i] \subset [D_i]_\xi.$$

In either case, let $\{p_t\}_{t\geqq 0}$ be the homotopy noted above, and define $r_i : (r_{i-1} \circ \cdots \circ r_1 \circ g)(I^n) \to s$ by $\tau^{\gamma'}(r_i(x)) = \tau^{\gamma'}(x)$, and $\tau^\gamma(r_i(x)) = p_t(\tau^\gamma(x))$, where

$$t = d(\tau^{2k+1}(x), B_i)/\eta.$$

Let $r = r_q \circ \cdots \circ r_1$. Then $(h^{-1} \circ r \circ g)(I^n) \cap K = \emptyset$ and

$$d^*(h^{-1} \circ r \circ g, f) < \varepsilon.$$

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