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Asymptotic expansions in renewal theory

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by

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1. Introduction and statement of the results

Let \( \mu \) be a probability measure defined on the Borel sets of \((-\infty, \infty)\) with \( \int |x|\mu(dx) < \infty \) and \( \mu_1 = \int x\mu(dx) > 0 \). Then the renewal measure \( \nu \) belonging to \( \mu \), defined by \( \nu = \sum_{n=0}^{\infty} \mu^* \), assigns finite measure to bounded Borel sets.

In this paper our aim is to get approximations of \( \nu\{x+E\} \), \( E \) some Borel set, for \( x \to -\infty \) if \( \mu\{(-\infty, x)\} \) decreases exponentially, and for \( x \to \infty \) if \( \mu\{(x, \infty)\} \) has this property. Work on this has been done in Stone [1] and [2]. Results are obtained for \( \mu \) lattice and for the case that some \( \mu^* \) is non-singular (we call \( \mu \) lattice with span \( d \) if \( \mu \) is concentrated on \( \{nd : -\infty < n < \infty\} \) but not on \( \{nd' : -\infty < n < \infty\} \) for any \( d' > d \), and we call \( \mu^* \) non-singular if it contains an absolutely continuous component).

Let \( g(s) \) be the moment generating function of \( \mu \), defined by \( g(s) = \int e^{sx} \mu(dx) \), the domain being all complex numbers for which the integral exists absolutely. As far as defined let \( A(s_0) \) denote the residue of \( 1/(1-g(s)) \) at \( s = s_0 \).

**Theorem 1.** Let \( \mu \) be lattice with span 1.

a) If \( g(s) \) exists for some \( s \) with \( \text{Re } s = -R \leq 0 \), then for any \( r \in (0, R] \) with \( g(s) \neq 1 \) on \( \text{Re } s = -r \), the set

\[ S = \{s_0 : g(s_0) = 1, -r < \text{Re } s_0 < 0, -\pi < \text{Im } s_0 \leq \pi\} \]

is finite, \( A(s_0) \) exists for \( s_0 \in S \) and for integer \( k \to -\infty \)

\[
\nu\{k\} = \sum_{s_0 \in S} A(s_0)e^{-s_0k} + o(e^{rk})
\]

\[
\nu\{(-\infty, k]\} = \sum_{s_0 \in S} (1-e^{s_0})^{-1} A(s_0)e^{-s_0k} + o(e^{rk}).
\]

b) If \( g(s) \) exists for some \( s \) with \( \text{Re } s = R > 0 \), then for any \( r \in (0, R] \) with \( g(s) \neq 1 \) on \( \text{Re } s = r \), the set

\[ S' = \{s_0 : g(s_0) = 1, 0 < \text{Re } s_0 < r, -\pi < \text{Im } s_0 \leq \pi\} \]
is finite, $A(s_0)$ exists for $s_0 \in S'$ and for integer $k \to \infty$
\begin{equation}
(1.3) \quad \nu\{k\} = \mu_1^{-1} \sum_{s_0 \in S} A(s_0) e^{-s_0 k} + o(e^{-rk}).
\end{equation}
Moreover, if $\mu_2 = \int x^2 \mu(dx) < \infty$ then
\begin{equation}
(1.4) \quad \nu\{(-\infty, k]\} = k/\mu_1 + \frac{1}{2}\mu_2/\mu_1 + \sum_{s_0 \in S'} (1-e^{-s_0})^{-1} A(s_0) e^{-s_0 k} + o(e^{-rk}).
\end{equation}

Under mild conditions $S$ is not empty and contains even one real point which provides the leading term. This does not hold for the set $S'$.

**Theorem 2.** Let $\mu$ be lattice with span 1, $\mu\{(-\infty, 0)\} > 0$ and let $I$ be the interior of the interval $I$ of real points $s < 0$ for which $g(s)$ exists. Suppose $I$ is not empty.

a) If $I = I$ or if there exists some $s$ with $g(s) = 1$ and $\text{Re } s \in I$, or even $\text{Re } s \in I$ and $\text{Im } s \neq 2\pi k$, $k = 0, \pm 1, \cdots$, then there exists exactly one real $s_0 \in I$ with $g(s_0) = 1$. Moreover, $g'(s_0) < 0$ and for some $r > -s_0$
\begin{equation}
(1.5) \quad \nu\{k\} = -e^{-s_0 k}/g'(s_0) + o(e^{rk}), \quad k \to -\infty
\end{equation}
\begin{equation}
(1.6) \quad \nu\{(-\infty, k]\} = -e^{-s_0 k}/[g'(s_0)(1-e^{s_0}) + o(e^{rk}), \quad k \to -\infty.
\end{equation}

b) If $I \neq I$ and there does not exist such an $s_0 \in I$ then for any $-r \in I$
\begin{equation}
\nu\{k\} = o(e^{rk}), \quad k \to -\infty
\end{equation}
\begin{equation}
\nu\{(-\infty, k]\} = o(e^{rk}), \quad k \to -\infty.
\end{equation}
Moreover, if even there does not exist such an $s_0 \in I$ then these order relations hold for $r = R$, where $-R$ is the (finite) left boundary of $I$.

The corresponding theorems for $\mu$ non-lattice are:

**Theorem 3.** Let $\mu^{m*}$ be non-singular.

a) If $g(s)$ exists for some $s$ with $\text{Re } s = -R < 0$, then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\text{Re } s = -r$, for which the singular part $\zeta$ of $\mu^{m*}$ satisfies
\begin{equation}
(1.7) \quad \int_{-\infty}^{0} e^{-rx} \zeta(dx) + \int_{0}^{\infty} (1+x) \zeta(dx) < 1,
\end{equation}
the set
\[ S = \{s_0 : g(s_0) = 1, -r < \text{Re } s_0 < 0\} \]
is finite, $A(s_0)$ exists for $s_0 \in S$ and for $x \to -\infty$
for every Borel set $E$ bounded from above. In particular, for $x \to -\infty$

(1.9) \[ v\{(-\infty, x]\} = - \sum_{s_0 \in S} s_0^{-1} A(s_0) e^{-s_0 x} + o(e^{rx}). \]

b) If $g(s)$ exists for some $s$ with $\text{Re} s = R > 0$, then for any $r \in (0, R]$ with $g(s) \neq 1$ on $\text{Re} s = r$, for which the singular part $\zeta$ of $\mu^{m*}$ satisfies

\[ \int_{-\infty}^{\infty} (1-x) \zeta(dx) + \int_{0}^{\infty} e^{rx} \zeta(dx) < 1, \]

the set

\[ S' = \{s_0 : g(s_0) = 1, 0 < \text{Re} s_0 < r\} \]

is finite, $A(s_0)$ exists for $s_0 \in S'$ and for $x \to \infty$

(1.10) \[ v\{x+E\} = |E|/\mu_1 - \sum_{s_0 \in S'} A(s_0) e^{-s_0 x} \int_{E} e^{-s_0 t} dt + o(e^{rx}), \]

for every Borel set $E$ bounded from below of finite length $|E|$. Moreover, if $\mu_2 = \int x^2 \mu(dx) < \infty$ then

(1.11) \[ v\{(-\infty, x]\} = x/\mu_1 + \frac{1}{2}(\mu_2 / \mu_1)^2 + \sum_{s_0 \in S'} s_0^{-1} A(s_0) e^{-s_0 x} + o(e^{rx}). \]

**Theorem 4.** Let $\mu^{m*}$ be non-singular, $\mu\{(-\infty, 0]\} > 0$, let the singular part of $\mu^{m*}$ be restricted to $(-\infty, 0]$, let $I$ be the interior of the interval $I$ of real points $s < 0$ for which $g(s)$ exists and let $E$ be a Borel set bounded from above. Suppose $I$ is not empty.

a) If $I = I$ or if there exists some $s$ with $g(s) = 1$ and $\text{Re} s \in I$, or even $\text{Re} s \in I$ and $\text{Im} s \neq 0$, then there exists exactly one real $s_0 \in I$ with $g(s_0) = 1$. Moreover, $g'(s_0) < 0$ and for some $r > -s_0$

(1.13) \[ v\{x+E\} = -e^{-s_0 x} \int_{E} e^{-s_0 t} dt |g'(s_0)| + o(e^{rx}), \quad x \to -\infty. \]

In particular

(1.14) \[ v\{(-\infty, x]\} = e^{-s_0 x} |g'(s_0)| + o(e^{rx}), \quad x \to -\infty \]

b) If $I \neq I$ and there does not exist such an $s_0 \in I$ then for any $-r \in I$

\[ v\{x+E\} = o(e^{rx}), \quad x \to -\infty \]
\[ v\{(-\infty, x]\} = o(e^{rx}), \quad x \to -\infty. \]

Moreover, if even there does not exist such an $s_0 \in I$ then these order relations hold for $r = R$, where $-R$ is the (finite) left boundary of $I$. 
2. Proof of the theorems

**Proof of Theorem 1a.** \( g(s) \) is analytic for \( \Re s \in (-R, 0) \), continuous for \( \Re s \in [-R, 0] \), \( g(i\theta) \neq 1 \) for \( |\theta| \in (0, 2\pi) \) and

\[
(2.1) \quad g(s) = 1 + \mu_1 s + o(|s|), \quad \text{for } |s| \to 0 \text{ and } \Re s \leq 0.
\]

Therefore, for any \( r \in (0, R] \) with \( g(s) \neq 1 \) on \( \Re s = -r \) and \( \varepsilon > 0 \) sufficiently small the function \( 1/(1-g(s)) \) is continuous on \( \Gamma \), and analytic within \( \Gamma \) with the exception of a finite number of poles. Here \( \Gamma \) is the contour in the complex \( s \)-plane shown in fig. 1. If for one or more \( s_0 \) with \( \Re s_0 \in (-r, 0) \) it occurs that

\[
g(s_0) = 1 \text{ with } \Im s_0 = \pi \text{ and also } g(\tilde{s}_0) = 1 \text{ with } \Im \tilde{s}_0 = -\pi
\]

then the parts \( \Gamma_1 \) and \( \Gamma_2 \) of \( \Gamma \) are slightly deformed as indicated.

Setting

\[
\psi(s) = (1-g(s))^{-1} + \mu_1 (1-e^s)^{-1}
\]

we get with the Cauchy residue theorem

\[
(2.2) \quad \frac{1}{2\pi i} \int_{\Gamma} e^{-sk\psi}(s) ds = \sum_{s_0 \in S} A(s_0) e^{-s_0 k}.
\]

According to Stone [3], (20), for \( k < 0 \) we have

\[
(2.3) \quad v\{k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Re \{e^{-i\theta \psi}(i\theta)\} d\theta = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\varepsilon \leq |\Im s| \leq \pi \atop \Re s = 0} e^{-sk\psi}(s) ds.
\]
With the Riemann-Lebesgue lemma

\[
(2.4) \quad \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{-sk} \Psi(s) ds = \frac{1}{2\pi} e^{\epsilon k} \int_{-\pi}^{\pi} e^{-i\epsilon k} \Psi(i\theta - \epsilon) d\theta = o(e^{\epsilon k}), \quad \epsilon \to -\infty.
\]

Since \( g(s+2\pi i) = g(s) \), the contributions of \( \Gamma_1 \) and \( \Gamma_2 \) to the integral in (2.2) cancel out. With (2.1) we see that the contribution of \( C \) to the integral in (2.2) tends to zero for \( \epsilon \to 0 \). So (1.1) follows from (2.2)–(2.4) and (1.2) follows from (1.1).

b) The proof of (1.3) is similar to that of (1.1). Use Stone [1], (20), for \( k \geq 0 \). With (1.3)

\[
\nu(k, N) = \frac{N-k}{\mu_1} + \sum_{s_0 \in S'} (1-e^{-s_0})^{-1} A(s_0) e^{-s_0 k} + o(e^{-k}) + o(e^{-\epsilon N}),
\]

\( k \to \infty, N \to \infty \)

and, as is well-known,

\[
\lim_{N \to \infty} \left[ \nu\left((-\infty, N)\right) - \frac{N}{\mu_1} \right] = \frac{1}{2}(\mu_2/\mu_1)^2
\]

we get (1.4).

**Lemma A.** Let \( I \) and \( I' \) be defined as in theorem 2. Suppose \( I \) is not empty and \( \mu\{(-\infty, 0)\} > 0 \). If \( I = I' \) or if \( g(s_1) = 1 \) for some \( s_1 \) with \( \text{Re} \ s_1 \in I \) then there exists exactly one real \( s_0 \in I \) with \( g(s_0) = 1 \). We have \( g'(s_0) < 0 \).

**Proof.** Let

\[
(2.5) \quad g_1(s) = \int_{[0, \infty)} (e^{sx} - 1) \mu\{dx\}, \quad -s \in I
\]

\[
\quad g_2(s) = \int_{[0, \infty)} (1-e^{-sx}) \mu\{dx\}, \quad -s \in I.
\]

Since \( g_1(0) = g_2(0) = 0, 0 < g_1'(0^+) < g_2'(0^+) \) and \( g_1 \) is convex and \( g_2 \) concave, there is at most one \( s_0 \) with

\[
0 = g_1(-s_0) - g_2(-s_0) = g(s_0) - 1,
\]

and then \( g'(s_0) < 0 \). If \( g(s_1) = 1 \) with \( \text{Re} \ s_1 \in I \) then \( g(\text{Re} \ s_1) \geq 1 \). But \( g'(0^-) > 0 \) and so there exists \( s_0 \in I \) with \( g(s_0) = 1 \). Finally, if \( I = I' \) i.e. \( I \) is open to the left, then \( g_1(-s) \to \infty \) if \( s \) tends to the left boundary of \( I \). This also assures that there is \( s_0 \in I \) with \( g(s_0) = 1 \).
Proof of Theorem 2.

a) According to Lemma A the set $S$ in Theorem 1 contains exactly one real $s_0 \in I$ with $g'(s_0) < 0$ and $s_0 \geq \text{Re } s_1$ for any $s_1 \in S$. But $\mu$ has span 1 and so $s_0 > \text{Re } s_1$ and $s_0 \in I$. With $A(s_0) = -1/g'(s_0)$ and Theorem 1 we see that (1.5) holds for some $r > -s_0$. (1.6) follows from (1.5).

b) This part follows immediately from Theorem 1.

In the following for any signed measure $\psi$ let $|\psi|$ denote its variation. We call $\psi$ finite if the measure $|\psi|$ is finite.

Lemma B. Let $\mu^m*$ be non-singular, $\zeta$ the singular part of $\mu^m*$, and let $K(x)$ and $L_s(x)$, $s \in T$ with $T$ an arbitrary index-set, be non-negative Borel functions in $x$ so that

\[ (2.6) \quad \text{for every fixed finite interval } I, \quad \int_{\{x+I\}} K(y-x)\mu\{dy\} \text{ is bounded in } x, -\infty < x < \infty \]

\[ \lim_{\varepsilon \to 0} \int_{\{x+I\}} |K(y+\varepsilon) - K(y)|((\mu^m* - \zeta)\{dy\} = 0, -\infty > x > \infty \]

\[ (2.7) \quad \int K(x)\mu\{dx\} < \infty \]

\[ (2.8) \quad L_s(x) \leq K(x), \quad -\infty < x < \infty, \quad s \in T \]

\[ (2.9) \quad L_s(x+y) \leq L_s(x)L_s(y), \quad -\infty < x, y < \infty, \quad s \in T \]

\[ (2.10) \quad \sup_{s \in T} \int L_s(x)\zeta\{dx\} < 1. \]

Then for any $\varepsilon > 0$ there exist an integer $n_0 \geq 1$, a measure $\varphi$ with infinitely often differentiable density with compact support, and a signed measure $\varphi'$ such that

\[ (2.11) \quad \mu^{n_0*} = \varphi + \varphi', \]

\[ (2.12) \quad |\varphi'|\{(-\infty, \infty)\} < \varepsilon, \]

\[ (2.13) \quad 1 - \varepsilon \leq \varphi\{(-\infty, \infty)\} \leq 1, \]

\[ (2.14) \quad \sup_{s \in T} \int L_s(x)\varphi\{dx\} < \infty, \]

\[ (2.15) \quad \sup_{s \in T} \int L_s(x)|\varphi'|\{dx\} < \varepsilon. \]

Moreover, for $\varepsilon < 1$ the renewal measure

\[ \nu = \sum_{k=0}^{\infty} \mu^k* \]
can be written as
\begin{equation}
\nu = \nu' + \nu''
\end{equation}
with
\begin{align*}
\nu'' &= (\mu^{0*} + \cdots + \mu^{(n_0-1)*}) \ast \sum_0^\infty \varphi^{k*} \\
\nu' &= \varphi \ast \nu'' \ast \sum_0^\infty \mu^{k_{n_0}*}.
\end{align*}

Here \(\nu''\) is a finite signed measure with
\begin{equation}
\sup_{s \in \mathcal{T}} \int L_s(x)|\nu''|\{dx\} < \infty.
\end{equation}

**Proof.** With \(\xi((\infty, \infty)) < 1\), (2.9) and (2.10) it follows that for \(n\) sufficiently large
\begin{equation}
\zeta^{n*}\{(-\infty, \infty)\} < \frac{\varepsilon}{4},
\end{equation}
\begin{equation}
\sup_{s \in \mathcal{T}} \int L_s(x)\zeta^{n*}\{dx\} < \frac{\varepsilon}{4}.
\end{equation}
Setting \(\xi = \mu^{m*} - \zeta\) and \(n_0 = nm\) we get
\begin{equation}
\mu^{n_0*} = \zeta^{n*} + \sum_{k=1}^n \binom{n}{k} \cdot \xi^{k*} \ast \zeta^{(n-k)*}.
\end{equation}

The second term on the right hand side of (2.20) is absolutely continuous. Let \(h(x)\) be its density. With (2.7), (2.8) and (2.9) for \(A > 0\)
\begin{equation}
\sup_{s \in \mathcal{T}} \int_{|x| \geq A} L_s(x)\mu^{n_0*}\{dx\} \leq n_0 \left[ \int K(x)\mu\{dx\} \right]^{n_0-1} \cdot \int_{|x| \geq A}|n_0 K(x)\mu\{dx\}
\end{equation}
and so with (2.7) and (2.20) for \(A\) sufficiently large
\begin{equation}
\int_{|x| \geq A} h(x)dx < \frac{\varepsilon}{4}
\end{equation}
\begin{equation}
\sup_{s \in \mathcal{T}} \int_{|x| \geq A} L_s(x)h(x)dx < \frac{\varepsilon}{4}.
\end{equation}

Set
\[ q_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}\sigma^{-2}x^2\}, \quad \sigma > 0 \]
\[ h_\sigma(x) = \int q_\sigma(x-t)h(t)dt \]
and let, for \( \delta > 0 \), \( \theta(x) \) be some infinitely often differentiable function with

\[
\begin{align*}
\theta(x) &= 1, & |x| &\leq A - \delta \\
0 &\leq \theta(x) \leq 1, & A - \delta &\leq |x| \leq A \\
\theta(x) &= 0, & |x| &> A.
\end{align*}
\]

With (2.6),

\[K(x)\mu^\nu \{dx\} < \infty.\]  

So with (2.20)

\[\int_{|x| \leq A} K(x)h(x)dx < \infty\]

and therefore for \( \delta \) sufficiently small, again with (2.6)

\[\int_{-A}^A |h(x) - h_\sigma(x)|dx < \frac{\varepsilon}{4}\]

\[\int_{-A}^A K(x)|h(x) - h_\sigma(x)|dx < \frac{\varepsilon}{4}.\]

Finally, for \( \delta \) sufficiently small

\[\int_{A-\delta \leq |x| \leq A} (1 - \theta(x))h_\sigma(x)dx < \frac{\varepsilon}{4}\]

\[\int_{A-\delta \leq |x| \leq A} K(x)(1 - \theta(x))h_\sigma(x)dx < \frac{\varepsilon}{4}.
\]

Let \( \varphi \) be the measure with density

\[
p_\varphi(x) = \begin{cases} 
\theta(x)h_\sigma(x) & |x| \leq A \\
0 & |x| > A
\end{cases}
\]

and \( \varphi' \) the sum of the measure \( \zeta^\nu \) and the signed measure with density \( h(x) - p_\varphi(x) \). Then (2.11) holds, \( \varphi \) and \( \varphi' \) are finite with \( \varphi((-\infty, \infty)) \leq 1 \), and \( p_\varphi \) is infinitely often differentiable with compact support \([-A, A]\).

With (2.18), (2.21), (2.24), (2.26)

\[
|\varphi'|((-\infty, \infty)) \leq \zeta^\nu((-\infty, \infty)) + \int_{|x| \geq A} h(x)dx
\]

\[+ \int_{-A}^A |h(x) - h_\sigma(x)|dx + \int_{A - \delta \leq |x| \leq A} (1 - \theta(x))h_\sigma(x)dx < \varepsilon,
\]

which proves (2.12). With (2.11) this gives (2.13). From (2.8),
(2.20) and (2.23) we get (2.14). With (2.19), (2.8), (2.22), (2.25), (2.27)
\[ \sup_{s \in T} \int L_s(|\varphi'| \{dx\}) \leq \sup_{s \in T} \int L_s(x) z^n \{dx\} + \sup_{s \in T} \int L_s(x) h(x) dx \]
\[ + \int_{-A}^A K(x) |h(x) - h(x)| dx + \int_{A-\theta \leq |x| \leq A} K(x)(1 - \theta(x)) h(x) dx < \epsilon \]
which proves (2.15).

Moreover, if \( \varepsilon < 1 \) then from (2.12) it follows that \( \nu'' \) is a finite signed measure. So \( \nu - \nu'' \) is defined, and with (2.11),
\[ v - v'' = (\mu_0^* + \cdots + \mu^{(m-1)}_n) \sum_{k=1}^\infty \left( \mu_k n_k^* - \nu' k^* \right) \]
\[ = (\mu_0^* + \cdots + \mu^{(m-1)}_n) \sum_{k=1}^\infty \sum_{j=0}^{k-1} \nu^* \mu_j n_j^* \nu'(k-1-j)^* \]
\[ = (\mu_0^* + \cdots + \mu^{(m-1)}_n) \sum_{j=0}^\infty \sum_{k=j+1}^\infty \nu^* \mu_j n_j^* \nu'(k-1-j)^* \]
which proves (2.16). Note that the summations with respect to \( j \) and \( k \) may be interchanged since \( \nu'' \) is finite.

Finally, (2.17) follows with (2.7), (2.9) and (2.15).

**Proof of Theorem 3.**

a) Let \( r \in (0, R] \) with \( g(s) \neq 1 \) on \( \Re s = -r \). We apply Lemma B for \( T = [-r, 0] \),
\[ L_s(x) = e^{sx}, \quad x < 0 \]
\[ = 1 + x, \quad x \geq 0 \]
\[ K(x) = L_{-r}(x), \quad -\infty < x < \infty \]
and keep the same notations. Denoting the moment generating function of any finite measure or finite signed measure \( \psi \) different from \( \mu \) by \( \psi_1 \), we get that \( \int |x|\varphi''(dx) \), \( \varphi_1(-r) \) and \( |\varphi''|_1(-r) \) are finite.

In (1.8) and (1.9) we may replace \( \nu \) by \( \nu' \) since
\[ e^{-rx} \varphi''((\infty, x]) \leq \int_{(\infty, x]} e^{-ry} \varphi''(dy) \rightarrow 0 \text{ if } x \rightarrow -\infty. \]
Note that \( g(s) \) and \( g^n(s) \) are analytic for \( \Re s \in (-R, 0) \), continuous for \( \Re s \in [-R, 0] \), that \( g^n(i\theta) \neq 1 \) for \( \theta \neq 0 \) and that
\[ g^n(s) = 1 + n \mu_1 s + o(|s|), \text{ for } |s| \rightarrow 0 \text{ and } \Re s \leq 0. \]
Since ϕ_1(s) tends to zero if |Im s| → ∞, uniformly in Re s ∈ [−R, 0] and |ϕ'|_1(−r) can be made arbitrary small

(2.29) |g_{n_0}(s)| ≤ C < 1, Re s ∈ [−R, 0], for |Im s| sufficiently large.

Therefore, for N sufficiently large and ε sufficiently small the function 1/(1−g_{n_0}(s)) is continuous on Γ and analytic within Γ with the exception of a finite number of poles. Here Γ is the contour in the proof of theorem with π replaced by N.

If χ = n_0^{−1}·ϕ ∗ ν'', then χ is a finite signed measure with χ(−∞, ∞) = 1, and χ_1(s) is continuous on Γ and analytic within Γ. Setting

Ψ(s) = χ_1(s)[(1−g_{n_0}(s))−1+(n_0μ_1s)^−1]

we get with the Cauchy residue theorem

(2.30) \frac{1}{2\pi i} \int_{Γ} e^{−sx}Ψ(s)ds = \sum_{s_0 \in Z} B(s_0)χ_1(s_0)e^{−s_0x}.

Here B(s_0) is the residue of 1/(1−g_{n_0}(s)) at s = s_0 and Z is defined by

Z = \{s_0 : g_{n_0}(s) = 1, −r < Re s_0 < 0\}.

But χ_1(s_0) = 1 if g(s_0) = 1 and χ_1(s_0) = 0 if g_{n_0}(s_0) = 1 and g(s_0) ≠ 1. If s_0 ∈ S ⊂ Z then B(s_0) = n_0^{−1}A(s_0). So we get

(2.31) \sum_{s_0 \in Z} B(s_0)χ_1(s_0)e^{−s_0x} = n_0^{−1}\sum_{s_0 \in S} A(s_0)e^{−s_0x}

Let p(x) be the density of ν'. In the same way as in the proof of Stone [2], Theorem, it follows that

\int |χ(iθ + s)|dθ < ∞, s ∈ [−R, 0].

and

(2.32) p(x)−μ_1^{−1}·χ((-∞, x))

\frac{n_0}{2\pi} \text{Re} \{e^{−ixθ}Ψ(iθ)\}dθ = \lim_{ε→0} \frac{n_0}{2\pi i} \int_{|Im s| ≥ ε, \ Re s = 0} e^{−sx}Ψ(s)ds.

It follows easily that

(2.33) χ((-∞, x)) = o(e^{rx}), \quad x → −∞.

With (2.29) and the Riemann-Lebesgue lemma
The contributions of \( I_1 \) and \( I_2 \) to the integral of (2.30) tend to zero for \( N \to \infty \). This follows with (2.29) and the fact that \( \chi_1(s) \) tends to zero for \( |\text{Im} \, s| \to \infty \), uniformly in \( \text{Re} \, s \in [-R, 0] \). With (2.28) we see that the contribution of \( C \) to the integral in (2.30) tends to zero for \( \epsilon \to 0 \). Therefore, from (2.30)–(2.34)

\[
(2.35) \quad p(x) = \sum_{s_0 \in S} A(s_0) e^{-s_0 x} + o(e^{\varepsilon x}), \quad x \to -\infty
\]

and (1.8), (1.9) follow from (2.35).

b) Compare the corresponding part of the proof of theorem 1.

**Proof of Theorem 4.** Compare the proof of theorem 2. Use Lemma A and theorem 3. Since \( g(s) \to 1 \) for real \( s \in (s_0, 0) \) and \( g(s_0) = 1 \) the condition (1.7) is fulfilled for some \( r > -s_0 \).

3. Final remarks

**Remark 1.** Let \( \mu \) be lattice or some \( \mu^{m*} \) be non-singular. Suppose \( \mu\{(-\infty, 0)\} > 0 \) and let \( g(s) \) exist for some \( s < 0 \). Then there exists always a finite real number \( r < 0 \) such that \( \int e^{\varepsilon x} v\{dx\} \) converges for \( s \in (r, 0) \) and diverges for \( s \in (-\infty, r) \).

This follows from theorem 2 and \( v\{k\} \) bounded, and from theorem 4, (2.35) and \( p(x) \) bounded.

**Remark 2.**

a) Suppose \( g(s) \) exists for \( \text{Re} \, s \leq 0 \). If

\[
(3.1) \quad \liminf_{r \to \infty} e^{r k_0} \int_0^{\pi} |g(i \theta - r) - 1|^{-1} d\theta = 0
\]

then the sum in (1.1) converges for \( r \to \infty \) and equals \( v\{k\} \), \( k \leq k_0 < 0 \). This follows from the fact that the left side of (2.4) tends to zero for \( r \to \infty \), uniformly in \( k \leq k_0 \). Note that the sum remains a finite one and (3.1) holds if the number of lattice-points of \( \mu \) in \(( -\infty, 0)\) is finite.

b) Suppose \( g(s) \) exists for \( \text{Re} \, s \geq 0 \). Similarly, if

\[
(3.2) \quad \liminf_{r \to \infty} e^{-r k_0} \int_0^{\pi} |g(i \theta + r) - 1|^{-1} d\theta = 0
\]
then the sum in (1.3) converges for \( r \to \infty \) and equals \( r \{ k \} - \mu_1^{-1}, k \geq k_0 \geq 0 \). Note that the sum remains a finite one and (3.2) holds if the number of lattice-points of \( \mu \) in \((0, \infty)\) is finite.

Postscript. Further investigations have led to the stronger result that theorem 3 continues to hold if (1.7) and (1.10) are replaced by \( \zeta_1(-r) < 1 \) and \( \zeta_1(r) < 1 \). The condition in theorem 4 that \( \zeta((0, \infty)) = 0 \) can be dropped. We refer to van der Genugten [4].

REFERENCES

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