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by

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1. Introduction

Let X and E be topological spaces. Let Y be a superspace of X. We say that X is E-embedded in Y provided that every continuous function $f : X \to E$ admits a continuous extension $f^* : Y \to E$. We shall be concerned with R-embedding, where R is the space of the reals. All spaces will be assumed to be Hausdorff completely regular. For some spaces X, the fact that X is R-embedded in Y can be decided by examining the extension of only one function $f : X \to R$ (this property of X is formulated precisely in the next section). This paper contains some partial results concerning the characterization of such spaces.

We shall now formulate a few statements of purely technical character.

A function $f : X \to R$ is said to be *absolutely extendable* provided that for every superspace Y of X, f admits a continuous extension $f^* : Y \to R$. We say that f *has vanishing oscillation outside compact subsets of* X provided that for every $\varepsilon > 0$ there exists a compact subset C of X such that

$$\omega(f, X \setminus C) = \sup \{|f(p)-f(q)| : p, q \in X \setminus C\} < \varepsilon.$$

1.1 Proposition. *A function $f : X \to R$ is absolutely extendable if and only if f can be continuously extended over every compactification of X.*

1.2 Proposition. *A function $f : X \to R$ has vanishing oscillation outside compact subsets of X if and only if f is the limit of a uniformly convergent sequence $f_1, f_2, \cdots$ of functions on X each of which is constant outside a compact subset of X.*

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PROOF. The sufficiency of the condition is obvious. To prove the necessity consider the functions $\alpha_{sn} : R \to R$, $s \in R$, $n = 1, 2, \cdots$, defined by

$$\alpha_{sn}(t) = s \qquad \text{for} \quad |t-s| \leqq \frac{1}{n};$$

$$\alpha_{sn}(t) = t - \frac{1}{n} \quad \text{for} \quad t \geqq s + \frac{1}{n};$$

$$\alpha_{sn}(t) = t + \frac{1}{n} \quad \text{for} \quad t \leqq s - \frac{1}{n}.$$

Each α_{sn} is continuous and $|\alpha_{sn}(t)-t| \leqq 1/n$ for every $t \in R$.

Assume that $f : X \to R$ has vanishing oscillation outside compact subsets of X. For every positive integer n find a compact set $C_n \subset X$ such that $\omega(f, X \setminus C_n) \leqq 1/n$. Let s be a value of f on $X \setminus C_n$ (if $X \setminus C_n$ is empty, then there is nothing to prove) and let f_n be the composition $\alpha_{sn} \circ f$. f_n is constant outside C_n and the sequence $f_1, f_2, \cdots$ is uniformly convergent to f on X.

1.3 PROPOSITION. *A function $f : X \to R$ is absolutely extendable if and only if f has vanishing oscillation outside compact subsets of X.*

PROOF. The necessity of the condition is obvious; the sufficiency follows from 1.1 and 1.2.

2. The property (P_1). R-compact spaces

2.1 DEFINITION. We say that a space X *has the property* (P_1) provided that there exists a continuous function $f : X \to R$ such that for every superspace Y of X, X is R-embedded in Y if and only if the function f admits the continuous extension $f^* : Y \to R$.

A function f with the above property will be called (for a lack of a better term) a *proper function* on X.

In this section we shall give a characterization of R-compact spaces having property (P_1). A space is R-compact iff it is homeomorphic to a closed subspace of some topological power R^m of R. Intuitively speaking, an R-compact space is a space which is either compact or admits a large number of continuous unbounded functions (precisely: X is R-compact iff for every $p_0 \in \beta X \setminus X$ there is a continuous function $f : X \to R$ which is unbounded on every neighbourhood of p_0). In the next section we shall state some partial results concerning the property (P_1) in arbitrary space.

A function $f : X \to R$ is said to be *bounded only on compact* *subsets of X* provided that for every subset A of X, if f is bounded on A, then $\bar{A}$ (= the closure of A in X) is compact.

2.2 Proposition. *A space X admits a continuous function* $f : X \to R$ *that is bounded only on compact subsets of X if and only if X is locally compact and Lindelöf.*

Proof is obvious [2].

2.3 Proposition. *Let* $f : X \to R$ *be a continuous function which is bounded only on compact subsets of X. Let Y be a superspace of X. X is R-embedded in Y if and only if the function f admits a continuous extension* $f^* : Y \to R$.

Proof. The necessity of the condition is obvious; we shall prove the sufficiency. Let f^* be the continuous extension of f with $f^* : Y \to R$. Let g be an arbitrary continuous function with $g : X \to R$. It is clear that if a continuous function $\alpha : R \to R$ tends sufficiently fast to $+\infty$ as $|t| \to +\infty$, then the function $g(p)/\alpha(f(p))$ has vanishing oscillation outside compact subsets of X. (It suffices to take a continuous function $\alpha : R \to R$ such that

$$\alpha(t) \geqq n \cdot \sup \{|g(p)| : p \in C_{n+1}\}+1$$

for $|t| \geqq n$, where $C_n = \{p \in X : |f(p)| \leqq n\}$.) By 1.3, $g(p)/\alpha(f(p))$ is absolutely extendable. Let g^* be a continuous extension of $g(p)/\alpha(f(p))$ with $g^* : Y \to R$. It is clear that $g^*(p) \cdot \alpha(f^*(p))$ is a continuous extension of g over Y.

2.4 Corollary. *A locally compact Lindelöf subspace X of Y is R-embedded in Y if and only if there exists a continuous function* $g : Y \to R$ *such that* $g|X$ *is bounded only on compact subsets of X.*

2.5 Theorem. *Let X be an R-compact space. X has property* (P_1) *if and only if X is locally compact and Lindelöf. Furthermore, a continuous function* $f : X \to R$ *is a proper function if and only if f is bounded only on compact subsets of X.*

Proof. Assume that $f : X \to R$ is a continuous function and assume that there is a set A such that f is bounded on A and $\bar{A}^X$ is not compact. Then $\bar{A}^{\beta X} \setminus X \neq \emptyset$; let $p_0 \in \bar{A}^{\beta X} \setminus X$. f can be

[2] Recall that for a locally compact space X the following conditions are equivalent.
(a) *X is Lindelöf;*
(b) *X is σ-compact* (i.e., *X is the union of countably many compact subsets*);
(c) $X = \bigcup_n C_n$, *where* C_n *are compact and* $C_n \subset \operatorname{Int} C_{n+1}$;
(d) *the ideal point* ∞ *in the one-point compactification* $\iota X = X \cup \{\infty\}$ *of X satisfies the first axion of countability.*

extended over $X \cup \{p_0\}$; in fact, one can modify f outside a neighborhood of p_0 so that it becomes bounded on the whole of X. But X is not R-embedded in $X \cup \{p_0\}$. Thus f is not a proper function on X. The rest of the theorem follows now from Propositions **2.2** and **2.3**.

3. Proper functions for arbitrary spaces

From the results of the previous section it is easy to obtain a characterization of proper functions for arbitrary spaces. We have to recall a few known facts and definitions.

An *extension* of X is any superspace εX of X such that X is dense in εX. The *canonical map* of extension $\varepsilon_1 X$ into an extension $\varepsilon_2 X$ is a continuous function $\varphi : \varepsilon_1 X \to \varepsilon_2 X$ which is the identity on X. The canonical map (if it exists) is unique. We write $\varepsilon_1 X =_{\text{ext}} \varepsilon_2 X$ provided that there exists a canonical map of $\varepsilon_1 X$ onto $\varepsilon_2 X$ which is a homeomorphism. (For further information see [4], Chi I.) βX admits a canonical map onto any compactification of X.

The *Q-closure* of a subset P of a space X is the set of all points $q \in X$ such that for every continuous function $f : X \to R$, if $f(p) > 0$ for every $p \in P$, then $f(p) > 0$. P is said to be *Q-closed* in X provided that it is equal to its Q-closure in X. The Q-closure of any subset of an R-compact space is again R-compact. If $c_1 X$ and $c_2 X$ are compactifications of X, A is a Q-closed subset of $c_2 X$ containing X, and φ is a canonical map of $c_1 X$ onto $c_2 X$, then φ maps the Q-closure of X in $c_1 X$ into A.

$\beta_R X$ is an R-compact extension of X such that X is R-embedded in $\beta_R X$. ($\beta_R X$ is also called the *Nachbin completion* of X.) $\beta_R X$ coincides with the Q-closure of X in βX. X is R-compact iff $X = \beta_R X$; equivalently, X is R-compact iff X is Q-closed in βX.

Every continuous function $f : X \to R$ can be continuously extended over βX if we allow $\pm\infty$ to be values of this extension. We shall denote this extension by f^β (note that f^β is a function into the two-point compactification of R). It is easy to see that f is bounded only on compact subsets of X iff $f^\beta(p) = \pm\infty$ for every $p \in \beta X \setminus X$.

3.1 Theorem. *A continuous function $f : X \to R$ is proper if and only if it satisfies the following two conditions*:

(a) *f^β is one-to-one on $\beta_R X \setminus X$;*
(b) *$f^\beta(p) = \pm\infty$ for every $p \in \beta X \setminus \beta_R X$.*

PROOF. Only the sufficiency requires a proof. Assume that Y is a superspace of X such that f admits a continuous extension $g : Y \to R$. We can assume that Y is R-compact (if not, replace Y by $\beta_R Y$). Let $\tilde{X}$ be the Q-closure of X in Y; $\tilde{X}$ is R-compact, hence $\tilde{X}$ is Q-closed in $\beta\tilde{X}$. Let $g_0 = g|\tilde{X}$ (= the restriction of g to $\tilde{X}$); let φ be the canonical map of βX onto $\beta\tilde{X}$. The quality

$$f^\beta(p) = g_0^\beta(\varphi(p)) \tag{1}$$

holds for every $p \in X$; hence, by continuity, (1) holds for every $p \in \beta X$. Since g_0^β is finite on $\tilde{X}$, we infer from (1) and (b) that $\varphi^{-1}[\tilde{X}] \subset \beta_R X$. Since $\tilde{X}$ is Q-closed in $\beta\tilde{X}$, the reverse inclusion also holds. Consequently, $\beta_R X = \varphi^{-1}[\tilde{X}]$. It follows that $\varphi_0 = \varphi|\beta_R X$ is a closed map (the restriction of a closed map to a full counter-image is again closed). From (1) and (a) we infer that φ_0 is one-to-one. Thus φ_0 is a homeomorphism; consequently, $\beta_R X =_{\text{ext}} \tilde{X}$. Hence X is R-embedded in $\tilde{X}$. But from (1) and (b) we infer that $g_0^\beta(q) = \pm\infty$ for every $q \in \beta\tilde{X} \setminus \tilde{X}$; hence g_0 is bounded only on compact subsets of $\tilde{X}$. Since $\tilde{X}$ is R-compact, we infer from Theorem 2.5 that g_0 is a proper function for $\tilde{X}$. Consequently, $\tilde{X}$ is R-embedded in Y. Thus X is R-embedded in Y. The theorem is shown.

We did not find any interesting characterization of arbitrary spaces with property (P_1). The following partial results follow directly from Theorem 3.1.

3.2 PROPOSITION. *If X has (P_1), then $\beta_R X$ is locally compact and Lindelöf.*

Recall that a space X is called *extremal* (in the sense of Fréchet, see [1]) provided that every continuous function $f : X \to R$ is bounded. Such spaces are also called *pseudocompact* or *quasicompact*. X is extremal iff $\beta_R X =_{\text{ext}} \beta X$.

3.3. THEOREM. *Let X be an extremal space. A continuous function $f : X \to R$ is proper if and only if f^β is one-to-one on $\beta X \setminus X$.*

3.4. THEOREM. *Let X be an extremal locally compact space. X has (P_1) if and only if the Čech outgrowth of X, $\beta X \setminus X$, is homeomorphic to a subspace of the closed interval $I = [0, 1]$.*

3.5. THEOREM. *Every space with a countable Čech outgrowth has (P_1).*

PROOF. If card $(\beta X \setminus X) < 2^{\mathfrak{c}}$, then X is extremal. On the other hand, for every countable subset of an arbitrary space there exists a continuous real function on the space which is one-to-one on this subset (see [4], Theorem 1).

On the basis of Theorem 3.5 it is easy to give examples showing that an extremal space with property (P_1) need not to be locally compact and its Čech outgrowth need not to be homeomorphic to a subspace of I.

We say that a space X has property (P_1^*) provided that there exists continuous function $f : X \to I$ (I is the closed interval $[0, 1]$) such that for every superspace Y of X, X is I-embedded in Y iff f admits a continuous extension $g : Y \to I$. A function f with this property will be called a $*$-proper function. It can be shown that a continuous function $f : X \to I$ is $*$-proper iff f^β is one-to-one on $\beta X \setminus X$. It follows that a space X does not have property (P_1^*) unless X is extremal. But for such spaces properties (P_1) and (P_1^*) coincide.

We conclude with two questions.

It follows from Theorem 3.4 that for a locally compact extremal space X property (P_1) depends only on the topological type of the Čech outgrowth of X. Is this true for arbitrary extremal spaces?

Does $\beta X \setminus X$ being homeomorphic to a subspace of I imply that X has (P_1)?

4. Generalizations of properties (P_1) and (P_1^*)

The purpose of this section is to state some questions concerning generalizations of properties (P_1) and (P_1^*) to higher cardinalities.

Let $\mathfrak{m}$ be an arbitrary cardinal; we shall say that *a space X has property* $(P_\mathfrak{m})$ provided that there exists a class $\mathfrak{F}$ of continouous real-valued function on X such that card $\mathfrak{F} \leqq \mathfrak{m}$ and for every superspace Y of X, X is R-embedded in Y iff each function in $\mathfrak{F}$ can be extended to a continuous real-valued function on Y. Such a class $\mathfrak{F}$ will be called a proper class on X. Property $(P_\mathfrak{m}^*)$ and $*$-proper classes are defined in an analogous way. It is clear that $(P_\mathfrak{m})$ implies $(P_\mathfrak{n})$ and $(P_\mathfrak{m}^*)$ implies $(P_\mathfrak{n}^*)$ for $\mathfrak{n} > \mathfrak{m}$; furthermore, every space has property $(P_\mathfrak{m})$ (as well as $(P_\mathfrak{m}^*)$) for a sufficiently large $\mathfrak{m}$. (Properties (P_0) and (P_0^*) are equivalent; each of them asserts that X is R-embedded in each of its superspaces; such spaces coincide with those having exactly one compactification.) It can be easily shown that $\mathfrak{F}$ is a $*$-proper class on X iff the continuous extensions of functions in $\mathfrak{F}$ over βX separate points of $\beta X \setminus X$ (compare with Theorem 3.1 and the remarks at the end of § 3). This, in turn, implies that X cannot have property $(P^*_{\aleph_0})$ unless X is extremal; consequently, for $\mathfrak{m} \leqq \aleph_0$, $(P^*_{\mathfrak{m}})$ implies $(P_{\mathfrak{m}})$. Clearly, there are non-extremal spaces having property $(P^*_{2^{\aleph_0}})$; I do not know if it can be shown without the continuum hypothesis that $2^{\aleph_0}$ is the first such cardinal. It can also be shown that a locally compact extremal space has $(P_{\mathfrak{m}})$ iff its Čech outgrowth is homeomorphic to a subspace of the Tihonov cube $I^{\mathfrak{m}}$ (compare with 3.4). It follows that for locally compact extremal spaces properties $(P_{\mathfrak{m}})$ and $(P_{\mathfrak{n}})$ are not equivalent for any two distinct cardinals $\mathfrak{m}$ and $\mathfrak{n}$.[3] On the other hand, for R-compact spaces, properties (P_1), (P_2), $\cdots$, (P_n), $\cdots$, $n < \aleph_0$, are equivalent; this can be demonstrated by showing that $\mathfrak{F} = \{f_1, \cdots, f_n\}$ is a proper class for X iff $f = \max \{|f_1|, \cdots, |f_n|\}$ is a proper function for X.

Property $(P_{\mathfrak{m}})$ for R-compact spaces is somewhat related to the concept of R-defect introduced in [3]. (An *R-non-extendable* class for X is a class $\mathfrak{F}$ of continuous real-valued functions on X such that for every extension εX of X with $\varepsilon X \neq X$, at least one of the functions in $\mathfrak{F}$ does not admit a continuous real-valued extension over εX. The *R-defect* of X [in symbols: $\text{def}_R X$] is the smallest cardinal $\mathfrak{m}$ such that X admits an R-non-extendable class of cardinality $\mathfrak{m}$. For further information see [3] and [5].) It can be easily shown that

4.1. *If an R-compact space has property* $(P_{\mathfrak{m}})$, *then* $\text{def}_R X \leqq \mathfrak{m}$. *In fact, a proper class on X is an R-non-extendable class for X.*

From 2.5 and from 5.9 in [5] we infer that for $\mathfrak{m} = 1$ the above implication can be reversed.

4.2. *Let X be R-compact. X has* (P_1) *if and only if* $\text{def}_R X \leqq 1$.

The converse of 4.1 fails for infinite $\mathfrak{m}$. We have the following.

4.3. *Let* $\mathfrak{m}$ *be an infinite cardinal of the form* $\mathfrak{m} = 2^{\mathfrak{n}}$ *and let X be a space with weight* $X \leqq \mathfrak{m}$. *X has* $(P_{\mathfrak{m}})$ *if and only if* card $C(X, R) \leqq \mathfrak{m}$.

Proof. The "if" part is obvious. To prove the converse it suffices to show that for every class $\mathfrak{F}$ of continuous real-valued functions on X with card $\mathfrak{F} \leqq m$ there is a superspace Y of X such that Y has only $\mathfrak{m}$ continuous real-valued functions and each function in

[3] In [2] Glicksberg proves that if $X \times Y$ is extremal, then $\beta(X \times Y =_{\text{ext}} \beta X \times \beta Y$. On the other hand, if X is compact and Y is extremal, then $X \times Y$ is also extremal. Consequently, for every compact space X we have $\beta X^* \setminus X^* =_{\text{top}} X$, where $X^* = X \times S(\Omega)$ and $S(\Omega)$ is the space of all ordinals $< \Omega$.

$\mathfrak{F}$ admits a continuous extension over Y. We can assume that $\mathfrak{F}$ is an R-separating class for X. The parametric map h of X corresponding to F (see Theorem 2.1 in [5]) is a homeomorphism of X into $R^{\mathfrak{m}}$. It suffices to take as Y a superspace of X that is homeomorphic to $R^{\mathfrak{m}}$ by an extension of the homeomorphism h. $R^{\mathfrak{m}}$ has only $\mathfrak{m}$ continuous real-valued functions; indeed, $R^{\mathfrak{m}}$ has a dense subset of cardinality $\mathfrak{n}$.

It follows from 4.2 that if $\mathfrak{m} = 2^{\mathfrak{n}}$ is infinite, then the discrete space $X_{\mathfrak{m}}$ of cardinality $\mathfrak{m}$ does not have $(P_{\mathfrak{m}})$. On the other hand, $\operatorname{def}_R X_{\mathfrak{m}} \leqq \mathfrak{m}$ for "almost all" infinite cardinals; in particular, for all $\mathfrak{m} = 2^{\mathfrak{n}}$, where $\mathfrak{n}$ is Ulam non-measurable.

Consequently, the converse of 4.1 fails for all such cardinals. I do not know if 4.3 holds for infinite cardinals that are not of the form $2^{\mathfrak{n}}$ as well as if 4.2 fails for such cardinals. It appears that the answer to this question depends upon the assumed rules of exponentation of cardinals. In particular, I do not know if 4.2 holds for the cardinal $\aleph_0$. Let Q be the space of irrational numbers; we have $\operatorname{def}_R Q = \aleph_0$; does Q have $(P_{\aleph_0})$? (Note that $\operatorname{def}_R P > \aleph_0$, where P is the space of rational numbers; therefore P does not have $(P_{\aleph_0})$.) We have $\operatorname{def}_R R^{\aleph_0} = \aleph_0$; does $R^{\aleph_0}$ have $(P_{\aleph_0})$?

One can discuss the above problems in a more general context. The property analogous to $(P_{\mathfrak{m}})$ but referring to functions with values in a space E will be denoted by $P_{\mathfrak{m}}(E)$; in the formulation of this property all spaces are assumed to be E-completely regular. F. Marin has pointed out to us that 4.1 holds true in this general context: if and E-compact space X has property $P_{\mathfrak{m}}(E)$, then $\operatorname{def}_E X \leqq \mathfrak{m}$. The study of property $P_{\mathfrak{m}}(E)$ for E-extremal spaces is more difficult. (An E-extremal space is an E-completely regular space X with the property: for every continuous function $f : X \to E$, $f[X]$ is compact.)

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