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Open subsets of Hilbert space

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In this paper we prove the following

**Theorem.** If $M$ is an open subset of the separable Hilbert space $l_2$, which we shall call $H$, then $M$ is homeomorphic to an open set $N \subset H$ where ("$\cong$" denotes "is homeomorphic to" and $R \equiv$ Reals)

(a) $H - N \cong H - \text{cl}(N) \cong H$,

(b) $N \cong \text{cl}(N) \cong \text{cl}(N) - N(\equiv \text{bd}(N)) \cong H \times |K|$, where $K$ is a countable locally finite simplicial complex (clfsc), and

(c) there is an embedding $h : \text{bd}(N) \times R \to H$ such that

(i) $h(\text{bd}(N) \times R)$ is open in $H$, (ii) $h|\text{bd}(N) \times \{0\}$ is the inclusion of $\text{bd}(N)$ into $H$, and (iii) $h(\text{bd}(N) \times (-\infty, 0)) = N$.

Recent results of Eells and Elworthy [7] show that each separable $C^\infty$ Hilbert manifold is $C^\infty$-diffeomorphic to an open subset of $H$, and, since Hilbert $C^\infty$-structures are unique [7], we may assert, in this case, that $M$ is $C^\infty$-diffeomorphic to $N$. Also, each separable, infinite-dimensional, Fréchet space, $F$, is homeomorphic to $H$ (see [1]). In a later paper [9] the author will use the Theorem to show that all separable manifolds modeled on $F$ are homeomorphic to open subsets of $H$ (and thus have unique Hilbert $C^\infty$-structures).

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### 1. Special cases

It is natural to ask when a manifold $M$ modeled on $H$ has the form $P \times H$, for a finite-dimensional manifold $P$. In this section we give several answers based on the following result:

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1 The author is an Alfred P. Sloan Fellow.
If $X$ and $Y$ are manifolds modeled on $H$, then any homotopy equivalence $f : X \to Y$ is homotopic to a homeomorphism (diffeomorphism) of $X$ onto $Y$. This result is a combined effort of [7], [9], [11], and [12]. (The proof in [9] uses the Theorem of this paper but not the results of this section.) It is the first special case that motivates the proof of the Theorem; however, the proof of the Theorem is otherwise independent of this section.

**Special Case 1.** $M$ has the homotopy type of a finite-dimensional clfsc $L$, if and only if, $M$ is homeomorphic (diffeomorphic) to $P \times H$, where $P$ is an open subset of a finite-dimensional Euclidean space.

**Proof.** Embed $L$ in a sufficiently high dimensional Euclidean space $E^n$ and let $P$ be its open regular neighborhood. Then $M$ has the homotopy type of $P \times H$, and thus $M \simeq P \times H$. In fact, $M$ is homeomorphic (though not diffeomorphic) to $\text{cl}(P) \times H$.

**Special Case 2.** Let $M$ be a connected separable manifold modeled on $H$. Then $M$ is homeomorphic (diffeomorphic) to $P \times H$, where $P$ is an open subset of a Euclidean space, if and only if the following conditions are satisfied: (a) $\tilde{M}$ denotes the universal covering space of $M$, then there is an integer $n$ such that $H_i(\tilde{M}) = 0$, for $i > n$; and (b) $H^{n+1}(M, \mathcal{B}) = 0$, for all local coefficients $\mathcal{B}$ (possibly non-abelian, if $n = 1$).

**Proof.** First of all, $M$ and $\tilde{M}$ have the homotopy type of connected clfsc’s. Theorem E of Wall [14] insures that $M$ has the homotopy type of a finite-dimensional clfsc $L$ and thus we may apply the first special case.

**Special Case 3.** In the above cases $P$ is an open manifold. Conditions which insure that $M$ is homeomorphic (diffeomorphic) to $P \times H$, for $P$ a closed (compact) manifold, are much more delicate. As above, we can use Wall’s work [14] to reduce the problem to a characterization of those clfsc having the homotopy type of such $M$. Sufficient conditions are then given by the theorems of W. Browder [6] and Novikov [13]. In another direction, we can use a theorem of Berstein-Ganea [5] which asserts that the conditions (a), (b) below imply that the map $f : M \to P$ is a homotopy equivalence. From it we obtain:

There is a closed manifold $P$ such that $M$ is homeomorphic (diffeomorphic) to $P \times H$, if and only if, there is an integer $n$ such that (a) $H^n(M) \neq 0$, and (b) $M$ is dominated by a manifold $P$ (i.e., there are maps $f : M \to P$ and $g : P \to M$ such that $g \circ f$ is homotopic to the identity map on $M$.)
2. Lemmas

We need two lemmas concerning Property \( Z \) which was first introduced by R. D. Anderson in [2]. (A closed set \( Y \subset W \) is said to have Property \( Z \) in \( W \) if, for each homotopically trivial and non-void open set \( U \) in \( W \), \( U - Y \) is homotopically trivial and non-void.)

**Lemma 1.** If \( X = A \cup B \), where \( A, B, \) and \( A \cap B \) are homeomorphic to \( H \) and \( A \cap B \) is closed with Property \( Z \) in both \( A \) and \( B \), then, for any neighborhood \( N \) of \( A \cap B \), there is a homeomorphism of \( A \) onto \( X \) which is fixed on \( A - N \).

The proof (which is left to the reader) follows easily from Corollary 10.3 of [2], which asserts that any homeomorphism between two closed sets with Property \( Z \) in \( H \) can be extended to a homeomorphism of all of \( H \) onto itself.

**Lemma 2.** If \( X \) is a separable manifold modeled on \( H \) and \( A \) is a closed subset of \( X \) such that \( A = \cup \mathcal{F} \), where \( \mathcal{F} \) is a locally-finite collection of closed sets with Property \( Z \) in \( X \). Then \( A \) has Property \( Z \) in \( X \).

**Proof.** \( A \) has Property \( Z \) in \( X \) if and only if each \( x \in A \) has a neighborhood \( W \) such that \( A \cap U \) has Property \( Z \) in \( W \). (Lemma 1 of [4].) Choose \( W \) so small that it intersects only finitely many members of \( \mathcal{F} \), say \( \{F_1, F_2, \ldots, F_n\} \). Let \( U \) be a homotopically trivial and non-void open subset of \( W \). Then, since each \( F_i \) is closed and has Property \( Z \) in \( X \),

\[
U - F_1, (U - F_1) - F_2, \ldots, U - U\{F_1, F_2, \ldots, F_n\}
\]

are successively homotopically trivial and non-void open sets. Therefore, \( W \cap U\{F_1, F_2, \ldots, F_n\} = W \cap A \) has Property \( Z \) in \( W \).

3. Proof of theorem

The proof uses several basic results from the Theory of Combinatorial Topology. Most of these results can be found in any treatise in the subject. All of the needed results can be found in [15].

Let \( L \) be any countable locally-finite simplicial complex (lcfs) which has the homotopy type of \( M \). (See, for example, [8].) Consider \( |L| \) linearly embedded in \( H \) so that, if \( \{v_1, v_2, \ldots\} \) are the vertices of \( L \), then \( |v_i| \) is the point of \( H \) whose \( i \)-th coordinate is 1 and whose other coordinates are 0. Define \( E^i \) to be the set of all points of \( H \) whose \( j \)-th coordinates are 0 for each \( j > i \). Let \( T \) be any triangulation of \( U\{E^i| i = 1, 2, 3, \ldots\}\) such that \( T \cap E^i \).
is a combinatorial triangulation of $E^t$ and a subcomplex of $T$ and such that $L$ is a subcomplex of $T$. Note that $T$ (but not $T \cap E^t$) will, of necessity, fail to be locally finite. If $Q$ is a simplicial complex, then let $Q''$ denote the second barycentric subdivision of $Q$; and, if $\alpha$ is a simplex of $Q$, let $st(\alpha, Q)$ be the smallest subcomplex of $Q$ which contains all those simplices of $Q$ which have $\alpha$ as a face. For each $\alpha \in L$, let $b(\alpha)$ be the barycenter of $\alpha$ and define

$$D(\alpha) = |st(b(\alpha), (T \cap E^n(\alpha)))|,$$

where $n(\alpha)$ is the smallest integer such that $|st(\alpha, L)| \subseteq E^n(\alpha)$. If $\beta$ is a face of $\alpha$ (written "$\beta \prec \alpha"$), then $st(\beta, L) \subseteq st(\alpha, L)$; and, thus,

$$D(\beta) \cap E^n(\alpha) = |st(b(\beta), (T \cap E^n(\alpha)))|.$$

There follows a list of those properties of the $D(\alpha)$'s which we shall use. These properties may be verified by standard combinatorial arguments. (See, in particular [15], pages 196 and 197, where $C$ is a simplex and $C^* = D(C)$.)

(i) $D(\alpha)$ is an $n(\alpha)$-cell.

(ii) $D(\alpha) \cap D(\beta) \neq \emptyset$, if an only if, $\alpha \prec \beta$ or $\beta \prec \alpha$.

(iii) Let $U[D(\beta)|\beta < \alpha \text{ and } \beta \neq \alpha] = D(bd \alpha)$.

Then $(D(\alpha), D(\alpha) \cap D(bd \alpha))$ is homeomorphic to the pair $[I^{n(\alpha)} \times \dim(a), I^{n(\alpha)} \times \dim(a)]$.

Let $K$ be the subcomplex of $T''$ such that $|K| = U[D(\alpha)|\alpha \in L]$. We will show that $K$ is the clfsc whose existence is asserted in the Theorem. $K$ also corresponds to the $P$ of the first special case. Let

$$C = (|K| \times R) \cup (H \times [0, \infty)) \times H \subset (H \times R) \times H.$$

We shall finish the proof of the theorem via several propositions. In the proofs of these propositions we will several times need to use a theorem proved by Klee (Theorem III. 1.3 of [10]); Hilbert space $H$ is homeomorphic to $H \times (0, 1]$ and to $H \times [0, 1].$

**Proposition 1.** $C$ is homeomorphic to $H$.

**Proof.** Define $C_{-1} = (H \times [0, \infty)) \times H \subseteq C$. $C_{-1}$ is clearly homeomorphic to a half-space of $H$; and, therefore by Klee's Theorem, $C_{-1} \simeq H$. Inductively define

$$C_n = C_{n-1} \cup [(U[D(\alpha)|\alpha \in L \text{ and } \dim(\alpha) = n]) \times (-\infty, 0]) \times H].$$

and, for a fixed ordering $\{x^n_1, x^n_2, x^n_3, \cdots\}$ of the $n$-simplices of $L$, define

$$C_{n,0} = C_{n-1} \text{ and } C_{n,t} = C_{n,t-1} \cup (D(x^n_t) \times (-\infty, 0] \times H).$$
Let $n \geq 0$ and $i \geq 1$, and assume inductively that $C_{n, i-1} \cong H$. $(D(x^n) \times (-\infty, 0]) \times H) \equiv D_i^n$ is homeomorphic to $H$ by Klee’s Theorem and (i) above.

$$C_{n, i-1} \cap D_i^n = \left[ (D(x^n) \cap D(bd x^n) \times (-\infty, 0]) \cup (|D(x^n)| \times \{0\}) \right] \times H,$$

which by using (ii) and (iii) above can be seen (see figure) to be homeomorphic to

$$(\text{open } n\text{-cell}) \times (\text{closed } (n(x^n) - n)\text{-cell}) \times H,$$

which in turn is homeomorphic to $H$ by Klee’s Theorem.

Let $\bar{D}(x)$ be the combinatorial boundary of $D(x)$. Then, since the pair $(D(x), \bar{D}(x))$ is homeomorphic to a ball and its boundary, it is easy to see that $\bar{D}(x) \times (-\infty, 0] \times H$ has Property $Z$ in $D(x) \times (-\infty, 0] \times H$. Also $H \times \{0\} \times H$ is a closed set with Property $Z$ in $H \times [0, \infty) \times H$. Repeated applications of Lemma 2, above, and Lemma 2 of [4] lead to the conclusion that $C_{n, i-1} \cap D_i^n$ has Property $Z$ in both $C_{n, i-1}$ and in $D_i^n$. Lemma 1 now applies and we conclude that $C_{n, i} \cong H$. For each $n, i$, let $A_{n, i}$ be a neighborhood of $D_i^n$ in $\mathcal{C}_{n, i}$ such that $A_{n, i} \cap A_{m, i} \neq \emptyset$, $m \geq n$, if and only if $x^n_i < x_m^n$. We may assume by Lemma 1 that there are homeomorphisms $h_{n, i} : C_{n, i-1} \rightarrow C_{n, i}$ such that $h_{n, i} | C_{n, i-1} - A_{n, i} = \text{id}$. Then the transfinite sequence

$$\cdots \circ h_{n+1, 0} \circ \cdots \circ h_{n, i} \circ h_{n, i-1} \circ \cdots \circ h_{n, 0}$$

$$\circ \cdots \circ h_{1, 1} \circ h_{1, 0} \circ \cdots \circ h_{0, 1} \circ h_{0, 0}$$

moves some neighborhood of each point at most finitely often because $L$ is locally-finite; and thus the sequence converges to a 1–1 map $h : C_{-1} \rightarrow C$. It is easy to check that $h$ is a homeomorphism and thus $C \cong H$. [5]
Proposition 2. Let \( N = (|K| \times (-\infty, -1)) \times H \subset C \). Then \( N \cong \text{cl} (N) \cong \text{bd} (N) \cong H \times |K| \). (This is the \( N \) of the Theorem.)

Proof. This follows immediately from Klee’s Theorem and the observation that \( \text{cl}(N) = (|K| \times (-\infty, -1]) \times H \).

Proposition 3. \( M \cong N \).

Proof. \( M \) and \( N \) are both open subsets of \( H \) and they have the same homotopy type. It follows directly from recent results of Kuiper and Burghelea \cite{11} and Mouli \cite{12} that \( M \) and \( N \) are homeomorphic (in fact, diffeomorphic).

Proposition 4. \( C - N \cong C - \text{cl}(N) \cong H \).

Proof. Clearly \( C - \text{cl}(N) \) is homeomorphic to \( C \) and thus, by Proposition 1, to \( H \). The set
\[
(|K| \times \{-1\}) \times H = (C - N) - (C - \text{cl}(N))
\]
is a closed set with Property Z (because it is collared) in \( C - N \). Using Klee’s Theorem it is easy to see that \( C - N \) is a manifold; therefore, Theorem 4 of \cite{4} asserts that \( C - N \) is homeomorphic to
\[
(C - N) - ((|K| \times \{-1\}) \times H) = C - \text{cl}(N),
\]
and thus homeomorphic to \( H \).

Proposition 5. Conclusion (c) of the Theorem is satisfied.

Proof. This holds because \( \text{bd} (N) = (|K| \times \{-1\}) \times H \) and \( (|K| \times (-\infty, 0)) \times H \) is an open neighborhood of
\[
(|K| \times (-\infty, -1)) \times H = N
\]
in \( C \).

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