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by

S. P. Franklin

Introduction

In recent years the category of k -spaces has come to occupy a fairly central place in topology (see for example Whitehead [20], Gale [15], Michael [18], Cohen [9], Morita [19], Arhangel'skii [1], Bagley and Yang [5], Duda [10], Whyburn [21], Arhangel'skii and Franklin [3], Gabriel and Zisman [22, p 47], and many others). R. Brown suggests this category may serve all the major purposes of topology [8]. Steenrod [23] suggests this as the most appropriate category for algebraic topology. Of more recent vintage is the interest in sequential spaces (see for example Kisyn'ski [17], Dudley [11], Aull [4], Franklin [13], [14], Boehme [7], Baron [6], Arhangel'skii and Franklin [3], Fleischer and Franklin [12], Luxemburg and Moore [24]). Several people have noted (and exploited) the similarity of the theorems which can be proved about k -spaces and sequential spaces. In this paper we offer a general theory which encompasses the common aspects of the theories of k -spaces and sequential spaces and of others also. The ideas involved can be traced from Michael's generating collections [18], thru Cohen's weak topologies [9] and Brown's natural covers [8]. We strengthen Brown's concept slightly, retaining his terminology. Section 1 is devoted to the statements of some preliminary facts and lemmas due either to Brown [8] or to Cohen [9]. The theory is then developed in sections 2 and 3.

1. The weak topology of a cover

Let \mathcal{S} be a cover for a topological space X with topology τ . The family $\Sigma(\tau)$ of those subsets of X which intersect each $S \in \mathcal{S}$ in an S -open set (i.e. open in S with the relative topology from τ) is a topology for X finer than τ . The restriction of $\Sigma(\tau)$ to S yields the same topology on S (i.e., $\Sigma(\tau)|S = \tau|S$), and repetition of the process leads to no new open sets (i.e., $\Sigma(\Sigma(\tau)) = \Sigma(\tau)$). One key fact is given in the following

1.1 LEMMA. *A function out of X is $\Sigma(\tau)$ -continuous if and only if its restriction to each $S \in \Sigma$ is continuous.*

If we let $\oplus \Sigma$ denote the disjoint topological sum of the spaces $S \in \Sigma$, the inclusion maps $S \in X$ combine to yield a natural map $\varphi : \oplus \Sigma \rightarrow X$. Suppose Σ' is a cover for a space X' with topology τ' and that $\varphi' : \oplus \Sigma' \rightarrow X'$ is the associated natural map. Then $\Sigma \times \Sigma' = \{S \times S' | S \in \Sigma, S' \in \Sigma'\}$ is a cover of $X \times X'$ and

1.2 LEMMA, *with $X \times X'$ topologized by $(\Sigma \times \Sigma') (\tau \times \tau')$, $\varphi \times \varphi' : \oplus \Sigma \times \oplus \Sigma' \rightarrow X \times X'$ is a quotient map.*

Further we have

1.3 LEMMA. $\Sigma(\tau) \times \Sigma'(\tau') \subseteq (\Sigma \times \Sigma') (\tau \times \tau')$. *If each $x \in X$ is a $\Sigma(\tau)$ -interior point of some $S \in \Sigma$, and similarly for each $x' \in X'$, then equality holds.*

The next lemma will prove central to our theory as it allows the construction of a coreflective functor.

1.4 KEY LEMMA. *If for each $S \in \Sigma$, $f : X \rightarrow Y$ satisfies i) there is an $S' \in \Sigma'$ with $f(S) \subseteq S'$ and ii) $f|S : S \rightarrow S'$ is continuous, then f is continuous with respect to $\Sigma(\tau)$ and $\Sigma'(\tau')$.*

Now suppose both Σ and Σ' are covers of X . If $\Sigma \subseteq \Sigma'$ or if Σ is a refinement of Σ' , then $\Sigma'(\tau') \subseteq \Sigma(\tau)$. Hence if each is a refinement of the other $\Sigma(\tau) = \Sigma'(\tau)$.

If $A \subseteq X$, let $\Sigma|A = \{A \cap S | S \in \Sigma\}$, and let $\tau|A$ be the relative topology on A .

1.5 LEMMA. $\Sigma(\tau)|A \subseteq (\Sigma|A)(\tau|A)$. *Further, if A and each $S \in \Sigma$ are τ -closed, then equality holds and A is $\Sigma(\tau)$ -closed.*

The hypotheses are in fact needed. For the convenience of the reader we reconstruct Example 5.1 of [14]. Let

$$M = (N \times N) \cup N \cup \{0\}$$

with each $(m, n) \in N \times N$ an isolated point, where N denotes the natural numbers. For a basis of neighborhoods at $n_0 \in N$, take all sets of the form $\{n_0\} \cup \{(m, n_0) | m \geq n_0\}$. U will be a neighborhood of 0 if and only if $0 \in U$ and U is a neighborhood of all but finitely many $n \in N$. Let Σ be the collection of all convergent sequences (with limit points) in M , and let $A = M \setminus N$. Each $S \in \Sigma$ is closed in M , but A is not. $(\Sigma|A)(\tau|A)$ is discrete and, hence, contains $\{0\}$ where $\Sigma(\tau)|A$ does not.

For another example, let I be the closed unit interval $[0, 1]$ with τ the usual topology, and let Σ be the collection of all connect-

ed subsets of I . Let A be the Cantor set. A , being compact, is closed but each S need not be. Since $I \in \Sigma$, $\tau = \Sigma(\tau)$. But $\tau|A$ is compact while $(\Sigma|A)(\tau|A)$ is discrete. Thus the conditions of lemma 1.5 are needed.

2. Natural covers and their spaces

By a *natural cover* we shall mean a function Σ which assigns to each topological space X a cover Σ_X satisfying 1) if $S \in \Sigma_X$ and S is homeomorphic to a subset T of Y , then $T \in \Sigma_Y$, and 2) if $f: X \rightarrow Y$ is continuous and $S \in \Sigma_X$ there is a $T \in \Sigma_Y$ with $f(S) \subseteq T$. For example we may choose Σ_X to be the compact subsets of X , or the connected subsets, or the countable subsets, or the convergent sequences. The first and last of these are the motivating examples which lead to the k -spaces and the sequential spaces ¹.

Now to each space X with topology τ we may associate the space σX , the same set of points topologized by $\Sigma_X(\tau)$. We may also assign to each continuous function $f: X \rightarrow Y$, the continuous (by Lemma 1.4) function $f = \sigma f: \sigma X \rightarrow \sigma Y$. Since $\sigma f = f$, σ preserves compositions and identities, i.e., σ is a functor from the category \mathcal{T} of topological spaces into itself.

Let us call a space X a Σ -space whenever $\sigma X = X$ (i.e. $\tau = \Sigma_X(\tau)$). If Σ is the natural cover which assigns to each space its compact subsets, the Σ -spaces are precisely the k -spaces. If Σ assigns the convergent sequences, the Σ -spaces are the sequential spaces. (This is almost immediate.)

2.1 LEMMA. *For each space X , σX is a Σ -space ($\sigma\sigma X = \sigma X$). Hence σ is a retraction from \mathcal{T} onto the category \mathcal{S} of Σ -spaces.*

PROOF. Clearly $\Sigma_X(\tau) \subseteq \Sigma_{\sigma X}(\Sigma_X(\tau))$. Now if $S \in \Sigma_X$, S is a subspace of σX and hence belongs to $\Sigma_{\sigma X}$, i.e., $\Sigma_X \subseteq \Sigma_{\sigma X}$. Hence $\Sigma_{\sigma X}(\Sigma_X(\tau)) \subseteq \Sigma_X(\Sigma_X(\tau)) = \Sigma_X(\tau)$ and the lemma is proved.

2.2 LEMMA. *\mathcal{S} is a coreflective subcategory ² of \mathcal{T} .*

PROOF. $1_X: X \rightarrow X$ is continuous. If $f: Y \rightarrow X$ is continuous and Y is a Σ -space the following diagram commutes.

¹ For the sequential case we must restrict ourselves to the category of Hausdorff spaces and continuous functions in order that the convergent sequences give rise to a natural cover. The compact metric subspaces also form a natural cover in this case which yields the sequential spaces.

² See [16] for the definition.

$$\begin{array}{ccc}
 Y & \xrightarrow{f} & X \\
 \uparrow 1_Y & & \uparrow 1_X \\
 \sigma Y & \xrightarrow{f = \sigma f} & \sigma X
 \end{array}$$

Hence $f : Y \rightarrow \sigma X$ is continuous and uniquely factors f thru 1_X . This completes the proof.

2.3 PROPOSITION. *The category \mathcal{S} of Σ -spaces is closed under quotients and disjoint topological sums.*

PROOF. This is a direct consequence of Lemma 2.3 and Theorem A of Kennison [16] since the disjoint topological sum is the co-product in \mathcal{S} . It is not at all difficult to give a direct proof.

2.4 COROLLARY. *Every open or closed image of a Σ -space is a Σ -space.*

2.5 COROLLARY. *When a product space is a Σ -space, so is each of its factors.*

2.6 COROLLARY. *The continuous image of a compact Σ -space in a Hausdorff space is a Σ -space.*

2.7 COROLLARY. *The inductive limit of any system of Σ -spaces is again a Σ -space.*

2.8 COROLLARY. *Any adjunction space of Σ -spaces is a Σ -space.*
 Of course Proposition 2.3 and its corollaries may be immediately asserted for k -spaces and sequential spaces.

2.9 LEMMA. *For each X , $\bigoplus \Sigma_X$ is a Σ -space.*

PROOF. If $S \in \Sigma_X$, then $S \subseteq S$ implies $S \in \Sigma_S$ and therefore $S = \sigma S$. Hence by 2.3 $\bigoplus \Sigma_X$ is a Σ -space.

This allows a characterization of Σ -spaces as follows.

2.10 PROPOSITION. *X is a Σ -space if and only if the natural mapping $\varphi : \bigoplus \Sigma_X \rightarrow X$ is a quotient mapping.*

PROOF. Since $\bigoplus \Sigma_X$ is a Σ -space and S is closed under quotients, if φ is a quotient map, X is a Σ -space. The converse has already been noted.

Hence we get the sequential spaces as quotients of zero-dimensional, locally compact metric spaces and the k -spaces as quotients of locally compact spaces.

(Lemma's 1.2 and 1.3 together with Proposition 2.10 provide a

criterion for the product of two Σ -spaces to again be a Σ -space. Although possibly useful in particular cases, it is clumsy to state and we shall omit it.)

This characterization is in fact more useful than it would appear. For example, with it we can say something about sub-spaces of Σ -spaces.

2.11 PROPOSITION. *If X and every open subset of each $S \in \Sigma_X$ is a Σ -space, then every open subset of X is a Σ -space.*

PROOF. Suppose U is open in X and $\varphi : \bigoplus \Sigma_X \rightarrow X$ is the natural mapping. Then for each $S \in \Sigma_X$, $\varphi^{-1}(U) \cap S$ is open in S and hence is a Σ -space. Thus their topological sum $\varphi^{-1}(U)$ is a Σ -space and $\psi = \varphi|_{\varphi^{-1}(U)}$ maps $\varphi^{-1}(U)$ onto U . By 2.10 it is enough to show ψ to be a quotient map. For $V \subseteq U$, if $\psi^{-1}(V)$ is open in $\varphi^{-1}(U)$, then for each $S \in \Sigma_X$, $\psi^{-1}(V) \cap S = \varphi^{-1}(V) \cap \varphi^{-1}(U) \cap S$ is open in S . Since X is a Σ -space, V is open in X and hence in U , and we are done.

From 2.11 one sees at once that open subsets of sequential spaces and k -spaces are again sequential spaces and k -spaces. It is also true that being a sequential space or a k -space is a local property. This follows from

2.12 PROPOSITION. *If each $x \in X$ has an open neighborhood which is a Σ -space, then X is a Σ -space.*

PROOF. Let $A \in \Sigma_X(\tau)$ and choose $x_0 \in A$. Let U be an open Σ -space neighborhood of x_0 . If $S \in \Sigma_U \subseteq \Sigma_X$, then

$$S \cap (U \cap A) = S \cap A \in \tau|S.$$

But since $S \subseteq U \in \tau$, $\tau|S = (\tau|U)|S$. Thus $A \cap U$ is open in U and hence in X . But A is the union of the $A \cap U$, and we are done.

Let us call a function $f : X \rightarrow Y$ Σ -continuous just in case $f|S$ is continuous for each $S \in \Sigma_X$. It follows from Lemma 1.1 that f is Σ -continuous if and only if $f : \sigma X \rightarrow Y$ is continuous. The Σ -spaces can be characterized in terms of the Σ -continuous functions.

2.13 PROPOSITION. *X is a Σ -space if and only if every Σ -continuous function out of X is continuous.*

PROOF. If each Σ -continuous function is continuous, then $1_X : X \rightarrow \sigma X$ is continuous and X is homeomorphic to σX . Conversely, f being Σ -continuous implies that $f : \sigma X \rightarrow Y$ is continuous. But $X = \sigma X$ since X is a Σ -space.

Following [3] we may assign an ordinal number to each Σ -space

in a topologically invariant manner. Let X be any space and A a subset of X . Define $A^\wedge = \bigcup \{cl_S(A \cap S) \mid S \in \Sigma_X\}$. Now let $A^0 = A$; $A^\alpha = (A^\beta)^\wedge$ if $\alpha = \beta + 1$; $A^\alpha = \bigcup \{A^\beta \mid \beta < \alpha\}$ otherwise. The Σ -characteristic of X is the least ordinal α (if it exists) such that for each subset A of X , $A^\alpha = cl_X A$. (See [3] for the existence of sequential and k characteristics.) Again we may characterize the Σ -spaces.

2.14 PROPOSITION. *X is a Σ -space if and only if it has a Σ -characteristic.*

PROOF. If X is a Σ -space, it suffices to show that for each $A \subseteq X$ there is an ordinal α such that $A^\alpha = cl_X A$. (One simply takes the sup of such ordinals.) Clearly for each α , $A \subseteq A^\alpha \subseteq cl_X A$. If A^β is not closed, there is some $S \in \Sigma_X$ with $A^\beta \cap S$ not closed in S , i.e. there is some point in $A^{\beta+1} \setminus A^\beta$. Hence by cardinality, some A^α is closed and hence equals $cl_X A$. Conversely if X has Σ -characteristic α and $A \cap S$ is closed in S for each $S \in \Sigma_X$, then $A^\wedge \subseteq A$ and hence $A^\alpha \subseteq A$ and A is closed. Thus X is a Σ -space.

For any natural cover Σ , a Σ -space has Σ -characteristic 0 if and only if it is discrete. In the case of sequential spaces and of k -spaces, those of characteristic ≤ 1 (called Fréchet spaces and k' -spaces respectively) have received special study. We formalize the common part of these theories in the next section.

3. Σ' -spaces and hereditary quotient maps

A mapping $f: X \rightarrow Y$ is an *hereditary quotient map* if and only if for each subspace Y_1 of Y , the mapping $f_1 = f|_{f^{-1}(Y_1)}$ is a quotient map. Arhangel'skii showed that *both open maps and closed maps are hereditary quotient maps* and characterized them as follows: *a mapping $f: X \rightarrow Y$ is an hereditary quotient map if and only if for each $y \in Y$ and for each open neighborhood U of $f^{-1}(y)$, y is an interior point of $f(U)$ ([2]).*

X is called a Σ' -space if its Σ -characteristic is ≤ 1 .

3.1 PROPOSITION. *If X is a Σ' -space and $f: X \rightarrow Y$ is an hereditary quotient map, then Y is a Σ' -space.*

PROOF. Suppose $A \subseteq Y$ and $y \in cl A$. We claim that

$$f^{-1}(y) \cap cl f^{-1}(A) \neq \emptyset.$$

(If it were empty, $U = X \setminus cl f^{-1}(A)$ would be an open neighborhood of $f^{-1}(y)$ and by Arhangel'skii's characterization of hered-

itary quotient maps, y would be an interior point of $f(U)$. But $X \setminus \text{cl } f^{-1}(A) \subseteq X \setminus f^{-1}(A)$ and hence $f(U) \subseteq Y \setminus A$ contradicting $y \in \text{cl } A$.) Choose $x \in f^{-1}(y) \cap \text{cl } f^{-1}(A)$. Then for some

$$S \in \Sigma_X, x \in \text{cl}_S(S \cap f^{-1}(A)).$$

Since Σ is a natural cover, there is some $S' \in \Sigma_Y$ with $f(S) \subseteq S'$. Then $y \in f[\text{cl}_S(f^{-1}(A) \cap S)] \subseteq \text{cl}_{S'}[f(f^{-1}(A) \cap S)]$, since $f : S \rightarrow S'$ is continuous. Thus $y \in \text{cl}_{S'}(A \cap f(S)) \subseteq \text{cl}_{S'}(A \cap S')$ and we are done.

The preceding proposition and the succeeding lemmas are aimed at characterizing Σ' -spaces in terms of hereditary quotient maps.

3.2 LEMMA. *Every disjoint topological sum of Σ' -spaces is again a Σ' -space.*

3.3 LEMMA. *If $X \in \Sigma_X$, then X is a Σ' -space.*

3.4 LEMMA. *Each $S \in \Sigma_X$ is a Σ' -space.*

3.5 LEMMA. *For each X , $\oplus \Sigma_X$ is a Σ' -space.*

3.6 PROPOSITION. *X is a Σ' -space if and only if $\varphi : \oplus \Sigma_X \rightarrow X$ is an hereditary quotient map.*

PROOF. Proposition 3.1 and Lemma 3.5 yield one direction immediately. For the converse we will again use Arhangel'skii's characterization. Suppose X is a Σ' -space, $x \in X$, and U is an open neighborhood of $\varphi^{-1}(x)$ in $\oplus \Sigma_X$. If $x \in \text{cl}(X \setminus \varphi(U))$, then for some $S \in \Sigma_X$, $x \in \text{cl}_S(S \cap (X \setminus \varphi(U)))$. Hence, regarding S as a subspace of $\oplus \Sigma_X$, $x \in \text{cl}(S \setminus U)$ contradicting $\varphi^{-1}(x) \subseteq U$ and U open. Hence x is an interior point of $\varphi(U)$ and we are done.

Since the Fréchet spaces are precisely the hereditary sequential spaces ([14] Proposition 7.2), one might look for a similar result for any natural cover. Unfortunately, using the connected sets to generate the natural cover, an example can be constructed of a Σ' -space with a subspace which is not a Σ' -space. However something can be said in the general case.

3.7 PROPOSITION. *If each subspace of X is a Σ' -space, then X is a Σ' -space.*

PROOF. Let $\varphi : \oplus \Sigma_X \rightarrow X$ be the natural mapping, X_1 be a subspace of X and $\varphi_0 = \varphi|_{\varphi^{-1}(X_1)}$. Let $\varphi_1 : \oplus \Sigma_{X_1} \rightarrow X_1$ be the natural mapping. If $U \subseteq X$ and $\varphi_0^{-1}(U)$ is open in $\varphi^{-1}(X_1)$, then $\varphi_1^{-1}(U) = \varphi_0^{-1}(U) \cap \oplus \Sigma_{X_1}$ is open in $\oplus \Sigma_{X_1}$. But φ_1 is a quotient

map and so U is open in X_1 . Thus φ_0 is a quotient map and φ an hereditary quotient map, and 3.6 completes the proof.

3.8 COROLLARY. *If every subspace of a Σ' -space is a Σ -space, then every subspace is a Σ' -space.*

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