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Some remarks on Poincaré series

by

Clifford J. Earle [1]

For Professor H. W. Brinkmann on his 70th birthday

1. Introduction

In [4], Bers proved that the Poincaré series in one variable defines a surjective map between certain function spaces in the unit disk (or in any bounded homogeneous domain in $\mathbb{C}^1$). Ahlfors [1] indicated a short proof of Bers' theorem, in the special case of quadratic differentials, relying on an integral reproducing formula. The author [7] made Ahlfors' argument more explicit and extended it to forms of higher weight. Using similar methods, Bell [3] proved Bers' theorem for any product of classical domains.

A closer examination of [7] revealed two facts: First, the crucial integral formula can be easily verified with the help of Godement's theorem [8] on the boundedness of Poincaré series. Second, the reasoning in [7] conceals a very simple proof of Godement's theorem. That proof is offered here in a quite general setting: the argument holds in any bounded homogeneous domain $B \subset \mathbb{C}^n$ whose Bergman kernel function satisfies a certain uniform growth condition (2.1).

After devoting § 2 to some preliminary matters, patterned on Selberg [10], we offer our proof of Godement's theorem in § 3. The integral reproducing formula is obtained in § 5. That formula and a projection operator introduced in § 4 enable us to prove in § 6 that certain spaces of holomorphic automorphic forms are conjugate. Finally, in § 7 we prove an extension of the theorems of Bers and Bell on the surjectivity of the Poincaré series map.

It would be interesting to know whether there are bounded homogeneous domains $B \subset \mathbb{C}^n$ in which (2.1) fails. It holds in homogeneous tube domains (oral communication from O. S. Rothaus) and in all bounded symmetric domains, as H. L. Resnikoff has shown in the supplement to this paper [11].

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Much of this paper, including the crucial condition (2.1), took shape as a result of a series of discussions with H. L. Resnikoff. The author wishes to express his gratitude.

2. Preliminaries

Consider a bounded homogeneous domain B in $\mathbf{C}^n$ with Bergman kernel function $k(z, \zeta)$. (As usual, z and ζ represent n-tuples $(z_1, \cdots, z_n)$ and $(\zeta_1, \cdots, \zeta_n)$.) *We assume that* $k(z, \zeta)$ *satisfies the following condition*:

For each $\zeta \in B$ there exist an open set $U \subset B$ and a positive number M such that

$$M^{-1}|k(z, \zeta)| \leqq |k(z, w)| \leqq M|k(z, \zeta)|, \qquad z \in B,\ w \in U. \tag{2.1}$$

We call $\gamma : B \to B$ an automorphism if γ and γ^{-1} are holomorphic maps of B onto itself. If $\gamma'(z)$ is the complex Jacobean of the automorphism γ at z, then

$$k(z, \zeta) = k(\gamma z, \gamma\zeta)\gamma'(z)\gamma'(\zeta)^*. \tag{2.2}$$

Hence the volume element $dm(z) = k(z, z)dz$ is invariant under automorphisms of B, where dz is the euclidean volume element in B.

Let Γ be a discrete group of automorphisms. Choose a fundamental domain D for Γ so that $\partial D \cap B$ has zero volume. Choose $t > 0$ such that the functions $\gamma'(z)^t$ are well defined in Γ. (For example, t may be a positive integer.) For $1 \leqq p \leqq \infty$, we define $L^p(t, \Gamma)$ as the Banach space of complex-valued measurable functions $f(z)$ in B such that

$$f(\gamma z)\gamma'(z)^t = f(z) \quad \text{for all } \gamma \text{ in } \Gamma \tag{2.3}$$

and

$$f(z)k(z, z)^{-t/2} \in L^p(D, dm). \tag{2.4}$$

The $L^p(t, \Gamma)$ norm of f, written $\|f\|^{\Gamma}_{p,t}$, is defined as the $L^p(D, dm)$ norm of $f(z)k(z, z)^{-t/2}$. The subspace $H^p(t, \Gamma)$ is defined as the set of holomorphic functions $f \in L^p(t, \Gamma)$. If Γ is the trivial group we use the notations $\|f\|_{p,t}$, $L^p(t)$, and $H^p(t)$.

Remark. The function $|f(z)|k(z, z)^{-t/2}$ is Γ — automorphic for all f which satisfy (2.3). Hence the norm $\|f\|^{\Gamma}_{p,t}$ is independent of the choice of D. Moreover, $\|f\|^{\Gamma}_{\infty,t} = \|f\|_{\infty,t}$ for all f in $L^\infty(t, \Gamma)$, so that $L^\infty(t, \Gamma)$ is a closed subspace of $L^\infty(t)$.

If $1 \leqq p < \infty$ and $1/p+1/q = 1$, then $L^q(t, \Gamma)$ is identified with the conjugate space of $L^p(t, \Gamma)$ by the *Petersson inner product*

$$(2.5)\quad (f, g)_t^\Gamma = \int_D f(z)g(z)^* k(z, z)^{-t} dm(z),\ f \in L^p(t, \Gamma),\ g \in L^q(t, \Gamma).$$

If Γ is the trivial group we write $(f, g)_t$ for $(f, g)_t^\Gamma$. The space $H^2(t)$ is a Hilbert space with the inner product $(f, g)_t$. Since the point evaluations are bounded linear functionals, $H^2(t)$ has a kernel function $k_t(z, \zeta)$. The homogeneity of B implies (see [10]) that $k_t(z, \zeta) = c(t)k(z, \zeta)^t$. $H^2(t)$ is non-trivial if and only if $c(t) \neq 0$. In fact

$$(2.6)\qquad f(z) = c(t) \int_B f(\zeta)k(z, \zeta)^t k(\zeta, \zeta)^{-t} dm(\zeta),\ f \in H^2(t).$$

Following Selberg [10], we shall assume t so large that

$$(2.7)\qquad \int_B k(z, z)^{-t/2} dm(z) < \infty.$$

Then $c(t)$ and $c(t/2)$ are non-zero, $k(\cdot, z)^{t/2} \in H^2(t/2)$ for each fixed $z \in B$, and

$$(2.8)\qquad k(z, z)^{t/2} = c(t/2) \int_B |k(z, \zeta)|^t k(\zeta, \zeta)^{-t/2} dm(\zeta).$$

Since $c(1) = 1$, all our conditions on t are satisfied by the integers $\geqq 2$.

3. Godement's theorem

If $f(z)$ is a complex-valued function on B, the *Poincaré series* $P_t f$ is defined by

$$(3.1)\qquad P_t f(z) = \sum_{\gamma \in \Gamma} f(\gamma z)\gamma'(z)^t.$$

Godement's theorem asserts that for suitable functions f, $P_t f \in H^\infty(t, \Gamma)$. We give this formulation:

THEOREM 3.1. (Godement). *For each $\zeta \in B$ there is a number K, depending on ζ, t, and the discrete group Γ, such that*

$$\sum_{\gamma \in \Gamma} |k(\gamma z, \zeta)|^t k(\gamma z, \gamma z)^{-t/2} < K \quad \text{for all } z \text{ in } B.$$

PROOF. Let U be an open set satisfying (2.1). Choose a fundamental domain D so that $U \cap D$ has non-empty interior. Define

$$f(z) = c(t/2) \int_D |k(z, w)|^t k(w, w)^{-t/2} dm(w).$$

Then, for any $\gamma \in \Gamma$,

$$\begin{aligned} f(\gamma z)|\gamma'(z)|^t &= c(t/2)\int_D |k(\gamma z, w)\gamma'(z)|^t k(w, w)^{-t/2} dm(w) \\ &= c(t/2)\int_{\gamma^{-1}(D)} |k(\gamma z, \gamma w)\gamma'(z)|^t k(\gamma w, \gamma w)^{-t/2} dm(w) \\ &= c(t/2)\int_{\gamma^{-1}(D)} |k(z, w)|^t k(w, w)^{-t/2} dm(w) \end{aligned}$$

by (2.2). Hence, by (2.8)

$$\begin{aligned} &\sum_{\gamma \in \Gamma} f(\gamma z) k(\gamma z, \gamma z)^{-t/2} \\ &\qquad = k(z, z)^{-t/2} c(t/2) \int_B |k(z, w)|^t k(w, w)^{-t/2} dm(w) = 1. \end{aligned}$$

To complete the proof we need to bound $f(z)|k(z, \zeta)|^{-t}$ away from zero. (2.1) yields

$$\begin{aligned} f(z)|k(z, \zeta)|^{-t} &\geqq c(t/2)\int_{D \cap U} |k(z, w)k(z, \zeta)^{-1}|^t k(w, w)^{-t/2} dm(w) \\ &\geqq c(t/2) M^{-t} \int_{D \cap U} k(w, w)^{-t/2} dm(w) > 0, \end{aligned}$$

since $D \cap U$ has positive volume. The theorem is proved.

Corollary 3.2. *For $\zeta \in B$ and $f \in L^p(t, \Gamma)$ let*

$$A_t f(z) = f(z) k(z, \zeta)^t.$$

Then $A_t : L^p(t, \Gamma) \to L^1(2t)$ is continuous, $1 \leqq p \leqq \infty$.

Proof. For $p = \infty$ the continuity of A_t is immediate from (2.8). For $p = 1$ we compute

$$\begin{aligned} ||A_t f||_{1, 2t} &= \int_B |f(z)|\,|k(z, \zeta)|^t k(z, z)^{-t} dm(z) \\ &= \sum_{\gamma \in \Gamma} \int_{\gamma(D)} |f(z)|\,|k(z, \zeta)|^t k(z, z)^{-t} dm(z) \\ &= \sum_{\gamma \in \Gamma} \int_D |f(z) k(z, z)^{-t/2}|\,|k(\gamma z, \zeta)|^t k(\gamma z, \gamma z)^{-t/2} dm(z) \\ &\leqq K ||f||_{1, t}^{\Gamma} \end{aligned}$$

by Theorem 3.1. The Riesz convexity theorem shows that A_t is continuous for all p.

Remark. Special cases of Corollary 3.2 were proved by Earle [7] and Bell [3]. Other statements of Godement's theorem can be found in [8] and [2, § 5].

4. A projection operator

Corollary 3.2 implies that for any $f \in L^p(t, \Gamma)$, $1 \leqq p \leqq \infty$, the function

$$T_t f(z) = c(t) \int_B f(\zeta) k(z, \zeta)^t k(\zeta, \zeta)^{-t} dm(\zeta) \tag{4.1}$$

is defined as an absolutely convergent integral. Of course T_t is the orthogonal projection of $L^2(t)$ onto $H^2(t)$. More generally we have

LEMMA 4. *$T_t : L^p(t, \Gamma) \to L^p(t, \Gamma)$ is a bounded linear map, $1 \leqq p \leqq \infty$. Moreover, $T_t f \in H^p(t, \Gamma)$, $T_t^2 = T_t$, and*

$$(T_t f, g)_t^\Gamma = (f, T_t g)_t^\Gamma,\ f \in L^p(t, \Gamma),\ g \in L^q(t, \Gamma). \tag{4.2}$$

PROOF. If $f \in C_0^2(B)$, then $T_t f$ is holomorphic (hence measurable!) in B. It follows readily from (2.8) and (4.2) that

$$||T_t f||_{\infty, t} \leqq M ||f||_{\infty, t} \quad \text{for} \quad f \in C_0^2(B).$$

For any $f \in L^\infty(t)$ there is a sequence (f_n) in $C_0^2(B)$ such that $||f_n||_{\infty, t} \leqq ||f||_{\infty, t}$ and $f_n \to f$ in the weak* sense. Then of course, $T_t f_n \to T_t f$ pointwise. Since the sequence (Tf_n) is norm bounded in $H^\infty(t)$, a subsequence converges uniformly on compact sets in B; the limit is of course $T_t f$. We conclude that $T_t : L^\infty(t) \to L^\infty(t)$ is a bounded map into $H^\infty(t)$.

Next we verify that T_t maps $L^\infty(t, \Gamma)$ into itself. For $f \in L^\infty(t, \Gamma)$ we have, by (2.2) and (2.3),

$$\begin{aligned} &T_t f(\gamma z) \gamma'(z)^t \\ &\quad = c(t) \int_B f(\zeta) \big(k(\gamma z, \zeta) \gamma'(z)\big)^t k(\zeta, \zeta)^{-t} dm(\zeta) \\ &\quad = c(t) \int_B f(\gamma\zeta) \big(k(\gamma z, \gamma\zeta) \gamma'(z)\big)^t k(\gamma\zeta, \gamma\zeta)^{-t} dm(\zeta) = T_t f(z) \end{aligned}$$

for all $\gamma \in \Gamma$, as required. If in addition $f \in L^1(t, \Gamma)$ we find

$$\begin{aligned} ||T_t f||_{1,t}^\Gamma &\leqq c(t) \int_D k(z, z)^{-t/2} dm(z) \int_B |f(\zeta)|\, |k(z, \zeta)|^t k(\zeta, \zeta)^{-t} dm(\zeta) \\ &= c(t) \sum_{\gamma \in \Gamma} \int_D k(z, z)^{-t/2} dm(z) \int_{\gamma(D)} |f(\zeta)|\, |k(z, \zeta)|^t k(\zeta, \zeta)^{-t} dm(\zeta) \\ &= c(t) \sum_{\gamma \in \Gamma} \int_{\gamma^{-1}(D)} k(\gamma z, \gamma z)^{-t/2} dm(z) \\ &\qquad \cdot \int_D |f(\gamma\zeta)|\, |k(\gamma z, \gamma\zeta)|^t k(\gamma\zeta, \gamma\zeta)^{-t} dm(\zeta) \end{aligned}$$

$$= c(t) \sum_{\gamma \in \Gamma} \int_{\gamma^{-1}(D)} k(z, z)^{-t/2} dm(z) \int_D |f(\zeta)|\, |k(z, \zeta)|^t k(\zeta, \zeta)^{-t} dm(\zeta)$$
$$= c(t) \int_B k(z, z)^{-t/2} dm(z) \int_D |f(\zeta)|\, |k(z, \zeta)|^t k(\zeta, \zeta)^{-t} dm(\zeta)$$
$$= c(t) \int_D |f(\zeta)| k(\zeta, \zeta)^{-t} dm(\zeta) \int_B |k(z, \zeta)|^t k(z, z)^{-t/2} dm(z)$$
$$= \frac{c(t)}{c(t/2)} \, ||f||_{1,t}^{\Gamma}.$$

Once more the Riesz convexity theorem implies that

$$T_t : L^p(t, \Gamma) \to L^p(t, \Gamma)$$

is continuous for $1 \leqq p \leqq \infty$. $T_t f$ is holomorphic for all f because $L^\infty(t, \Gamma) \cap L^p(t, \Gamma)$ is dense in $L^p(t, \Gamma)$ for all p. That $T_t^2 = T_t$ is a simple consequence of the Fubini theorem and the identity

$$k(\zeta, \zeta')^t = c(t) \int_B k(\zeta, z)^t k(z, \zeta')^t k(z, z)^{-t} dm(z),$$

a special case of (2.6). Finally, (4.2) amounts to the identity

$$\int_D g(z)^* k(z, z)^{-t} dm(z) \int_B f(\zeta) k(z, \zeta)^t k(\zeta, \zeta)^{-t} dm(\zeta)$$
$$= \int_B g(t)^* k(z, z)^{-t} dm(z) \int_D f(\zeta) k(z, \zeta)^t k(\zeta, \zeta)^{-t} dm(\zeta),$$

which is proved by writing the integral over B as a sum of integrals over $\gamma(D)$ and changing variables (compare Ahlfors [1]). We omit the details.

Remarks. Irwin Kra has pointed out to us that our proof of Lemma 4 (which he found independently) makes no essential use of Corollary 3.2. The proof that T_t is bounded on $L^1(t, \Gamma)$ implies that the integrəl (4.1) converges absolutely for almost all $\zeta \in B$. Thus Lemma 4 holds in any bounded homogeneous domain B, whether or not (2.1) holds.

Lemma 4 has the immediate consequence that the Petersson inner product identifies $T_t L^q(t, \Gamma)$ with the conjugate space of $T_t L^p(t, \Gamma)$ for all finite $p \geqq 1$. In the next section we shall prove that $T_t f = f$ for all $f \in H^p(t, \Gamma)$, so that $T_t L^p(t, \Gamma) = H^p(t, \Gamma)$.

5. Reproducing formulas

Lemma 5.1. *$H^p(t)$, $1 \leqq p < \infty$, is contained in $H^\infty(t)$ with a* continuous inclusion map.

Proof. Fix z_0 in B. Let K be a ball centered at z_0 and contained in a compact subset of B. Let $m(K)$ be the euclidean volume of K. Since $|f|^p$ is plurisubharmonic in B for all f in $H^p(t)$,

$$|f(z_0)|^p \leqq \frac{1}{m(K)} \int_K |f(z)|^p dz.$$

But $k(z, z)$ is bounded away from zero and infinity in K, so

$$\int_K |f(z)|^p dz \leqq M \int_K |f(z)k(z, z)^{-t/2}|^p dm(z) \leqq M||f||^p_{p,t},$$

and $|f(z_0)k(z_0, z_0)^{-t/2}| \leqq C||f||_{p,t}$ for all f in $H^p(t)$.

Now let z_1 in B be any point, and let γ be an automorphism of B such that $\gamma z_0 = z_1$. If $f \in H^p(t)$, then $f^\gamma = (f \circ \gamma)(\gamma')^t \in H^p(t)$ also, and $||f||_{p,t} = ||f^\gamma||_{p,t}$. Hence

$$|f(z_1)k(z_1, z_1)^{-t/2}| = |f^\gamma(z_0)k(z_0, z_0)^{-t/2}| \leqq C||f^\gamma||_{p,t} = C||f||_{p,t}.$$

That proves the lemma.

Corollary 5.2. *The functions $k(\cdot, z)^t$, $z \in U \subset B$, span a dense subspace of $H^1(t)$ if U is open in B.*

Proof. By Lemma 5.1, $H^1(t) \subset H^2(t)$. Thus (2.6) holds in $H^1(t)$, and $T_t L^1(t) = H^1(t)$. Now suppose $g \in L^\infty(t)$ is such that $(f_z, g)_t = 0$ for all $f_z = k(\cdot, z)^t$, $z \in U$. It follows that the holomorphic function $T_t g$ vanishes in U and hence in B; thus $g \in \ker T_t = H^1(t)^\perp$.

Lemma 5.3. *For each $f \in H^1(t)$ and each $\zeta \in B$,*

$$f(\zeta)k(\zeta, \zeta)^{-t/2} = c(t/2) \int_B f(z)k(z, \zeta)^{-t/2}k(\zeta, z)^{t/2}k(z, z)^{-t/2}dm(z).$$

Proof. The integral defines a bounded linear functional l on $H^1(t)$. Choose an open set $U \subset B$ satisfying (2.1). By Corollary 5.2, it suffices to prove that $l(f_w) = f_w(\zeta)k(\zeta, \zeta)^{-t/2}$ for the functions $f_w(z) = k(z, w)^t$, $w \in U$. For such w, (2.1) implies that

$$k(z, w)^t k(z, \zeta)^{-t/2} \in H^2(t/2),$$

and (2.6) gives

$$\begin{aligned} k(\zeta, w)^t k(\zeta, \zeta)^{-t/2} \\ &= c(t/2) \int_B (k(z, w)^t k(z, \zeta)^{-t/2}) k(\zeta, z)^{t/2} k(z, z)^{-t/2} dm(z) \\ &= l(f_w). \end{aligned}$$

The lemma is proved.

Proposition 5.4. *The formula (2.6) holds whenever f is holo-*

morphic and the integral converges absolutely. Hence, in particular, (2.6) *holds for* $f \in H^p(t, \Gamma)$ *and* $\zeta \in B$.

PROOF. The absolute convergence means that

$$g(z) = f(z)k(z, \zeta)^t \in H^1(2t).$$

Hence, by Lemma 5.3

$$f(\zeta) = g(\zeta)k(\zeta, \zeta)^{-t} = c(t) \int_B f(z)k(\zeta, z)^t k(z, z)^{-t} dm(z).$$

The remaining assertion is a consequence of Corollary 3.2.

REMARK. Proposition 5.4 extends theorems of Innis [9] and Bell [3].

As another application of (2.6), we shall find the norm of the inclusion map from $H^p(t)$ to $H^\infty(t)$.

PROPOSITION 5.5. *The inclusion map of* $H^p(t)$ *in* $H^\infty(t)$ *has norm* $c(pt/2)^{1/p}$, $1 \leqq p < \infty$. *The norm is attained at the functions* $k(z, \zeta)^t$.

PROOF. Let $f_\zeta = k(\cdot, \zeta)^t \in H^p(t)$. Using (2.8) one verifies easily that $||f_\zeta||_{\infty, t} = c(pt/2)^{1/p}||f_\zeta||_{p,t}$. To see that

$$||f||_{\infty, t} \leqq c(pt/2)^{1/p}||f||_{p,t} \quad \text{for all} \quad f \in H^p(t),$$

apply Holder's inequality to

$$f(\zeta)k(\zeta, \zeta)^{(p-2)t/2} = c(pt/2) \int_B f(z)k(z, \zeta)^{(p-2)t/2}k(\zeta, z)^{pt/2}k(z, z)^{-pt/2}dm(z).$$

That identity is proved by applying (2.6) to

$$g(z) = f(z)k(z, \zeta)^{(p-2)t/2}, \quad f \in H^p(t).$$

REMARK. For each g on the unit sphere of $H^p(t)$, $1 \leqq p < \infty$, the *unique* linear functional l on $H^p(t)$ of norm one with $l(g) = 1$ is

$$l(f) = \int_B f(z)|g(z)|^p g(z)^{-1}k(z, z)^{-pt/2}dm(z), \quad f \in H^p(t).$$

Thus Lemma 5.3 is *equivalent* to the assertion that the functional $l(f) = f(\zeta)$ on $H^1(t)$ attains its norm at $k(z, \zeta)^t k(\zeta, \zeta)^{-t/2}$.

6. Conjugate spaces of forms

THEOREM 6. $H^q(t, \Gamma)$ *is identified with the conjugate space of* $H^p(t, \Gamma)$ *through the Petersson inner product* (2.5) *whenever* $1 \leqq p < \infty$.

Theorem 6 is an immediate consequence of Lemma 4 and Proposition 5.4. It extends earlier theorems of Godement [8], Bers [4], and Bell [3].

7. An application to Poincaré series

It is well known that the Poincaré series (3.1) is a continuous map from $H^1(t)$ into $H^1(t, \Gamma)$.

THEOREM 7. *$P_t : H^1(t) \to H^1(t, \Gamma)$ is surjective.*

REMARK. That was proved in less generality by Bers [4] and Bell [3].

PROOF. It suffices to prove that the adjoint map

$$P_t^* : H^\infty(t, \Gamma) \to H^\infty(t)$$

is one-to-one with closed range [6, p. 488]. But for $f \in H^1(t)$ and $g \in H^\infty(t, \Gamma)$

$$\begin{aligned}
(P_t f, g)_t^\Gamma &= \int_D P_t f(z) g(z)^* k(z, z)^{-t} dm(z) \\
&= \sum_{\gamma \in \Gamma} \int_D f(\gamma z) g(z)^* \left(k(z, z)^{-1} \gamma'(z)\right)^t dm(z) \\
&= \sum_{\gamma \in \Gamma} \int_D f(\gamma z) g(\gamma z)^* \left(k(z, z)^{-1} |\gamma'(z)|^2\right)^t dm(z) \\
&= \sum_{\gamma \in \Gamma} \int_{\gamma(D)} f(z) g(z)^* k(z, z)^{-t} dm(z) = (f, g)_t.
\end{aligned}$$

In other words P_t^* is the inclusion map of $H^\infty(t, \Gamma)$ in $H^\infty(t)$. Since $H^\infty(t, \Gamma)$ is a closed subspace, the theorem is proved.

REMARK. For another proof of Theorem 7 one can show that

$$S_t f(z) = c(t) \int_D f(\zeta) k(z, \zeta)^t k(\zeta, \zeta)^{-t} dm(\zeta),\ f \in H^1(t, \Gamma),$$

provides a bounded right inverse of P_t. Its adjoint S_t^*, of course, is a bounded projection of $H^\infty(t)$ on $H^\infty(t, \Gamma)$. Explicitly

$$S_t^* f(z) = \int_D f(\zeta) k_t^\Gamma(z, \zeta) k(\zeta, \zeta)^{-t} dm(\zeta),\ f \in H^\infty(t),$$

where

$$k_t^\Gamma(z, \zeta) = P_t k_t(z, \zeta) = c(t) \sum_{\gamma \in \Gamma} k(\gamma z, \zeta)^t \gamma'(z)^t.$$

The functions $k_t^\Gamma(\cdot, \zeta)$, $\zeta \in B$, span a dense subspace of $H^1(t, \Gamma)$, by virtue of Corollary 5.2 and Theorem 7. Theorem 3.1 implies that $H^1(t, \Gamma) \cap H^\infty(t, \Gamma)$ is dense in $H^1(t, \Gamma)$. We don't know whether $H^1(t, \Gamma)$ is contained in $H^\infty(t, \Gamma)$. For a partial answer in the one variable case see [5].

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