J. CIGLER

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by

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A well-known theorem of H. Weyl [4] asserts the uniform distribution mod 1 of the sequence $\{p(n)\}$, where $p(t) = a_0 t^k + \cdots + a_k$ is any polynomial with at least one irrational coefficient a_i, $i \neq k$. There are many different proofs of this theorem. The simplest one uses the so-called fundamental inequality of J. G. van der Corput (Cf. e.g. [1], where all mentioned definitions and results from the theory of uniform distribution may be found). In the last years new interest in this theorem arose in connection with the study of ergodic properties of affine transformations on compact Abelian groups. In 1963 F. J. Hahn [2] gave a new proof depending on the individual ergodic theorem.

The purpose of this note is to give another purely functional-analytical proof which uses only the mean ergodic theorem.

LEMMA. *Let G be a compact Abelian group with Haar measure λ, let $S : G \to G$ be a continuous map and let μ be an S-invariant positive normed Radon measure on G. Denote by U the isometry on $L^2_\mu(G)$ defined by $(Uf)(x) = f(Sx)$. Then*

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n\leqq N} (U^n \chi, \chi) = 0$$

for every continuous character $\chi \not\equiv 1$ implies $\mu = \lambda$.

PROOF. By the mean ergodic theorem (see e.g. P. R. Halmos [3])

$$\lim_{N\to\infty} \left\| \frac{1}{N+1} \sum_{n\leqq N} U^n \chi - P\chi \right\| = 0,$$

where P denotes the projection onto the space of all S-invariant elements of $L^2_\mu(G)$. Therefore we have

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n\leqq N} (U^n \chi, \chi) = \lim_{N\to\infty} \left(\frac{1}{N+1} \sum_{n\leqq N} U^n \chi, \chi \right)$$
$$= (P\chi, \chi) = (P\chi, P\chi) = ||P\chi||^2.$$

The assumption implies $P\chi = 0$ and therefore also

$$(\chi, 1) = (\chi, P1) = (P\chi, 1) = 0.$$

But $(\chi, 1) = \int \chi(x) d\mu(x)$ and thus we have $\int \chi(x) d\mu(x) = 0$ for all $\chi \neq 1$. But this relation characterizes Haar measure on G.

Let $R^k(Z^k)$ be the group of all k-tuples of real numbers (integers) and let $G = T^k = R^k/Z^k$ be the k-dimensional torus group in its usual topology. The elements of T^k are the cosets $\tilde{x} = x + Z^k$, where $x = (x_0, \cdots, x_{k-1}) \in R^k$. In order to avoid cumbersome notation we shall indicate elements $\tilde{x} \in T^k$ by proper chosen representatives $x \in R^k$.

Theorem. *Let $p(t) = a_0 t^k + \cdots + a_k$, $k \geqq 1$, be a polynomial with real coefficients, where $a_0 = \theta$ is irrational. Then the sequence $(p(n), p(n+1), \cdots, p(n+k-1))$ is uniformly distributed in T^k.*

Proof. The immediately verified equality

$$p(n+k) = \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} p(n+k-j) + k!\,\theta$$

implies that

$$(p(n), p(n+1), \cdots, p(n+k-1)) = S^n(p(0), \cdots, p(k-1)),$$

where $S : T^k \to T^k$ is defined by

$$Sx = S(x_0, \cdots, x_{k-1}) = (x_1, \cdots, x_{k-1}, \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} x_{k-j} + k!\,\theta).$$

On the other hand for every $x \in T^k$ there exists some polynomial p with $a_0 = \theta$ such that $x = (p(0), \cdots, p(k-1))$. Therefore the theorem asserts that for every $x \in T^k$ the sequence $\{S^n x\}$ is uniformly distributed in T^k.

Denote now by $M(T^k)$ the (compact) space of all normed positive Radon measures on T^k with the usual vague topology.

For $x \in T^k$ define $V(x)$ as the set of all limit measures in $M(T^k)$ of the sequence $\{\mu_{x,N}\}$ defined by

$$\mu_{x,N}(f) = \frac{1}{N+1} \sum_{n \leqq N} f(S^n x),\ f \in C(T^k).$$

It is sufficient to prove that $V(x) = \{\lambda\}$ for all $x \in T^k$.

We prove this by induction on k. For $k = 1$ this is a trivial consequence of Weyl's criterion. Assume now that it is already true for all $k' < k$. We show then that for every continuous character $\chi \neq 1$ of T^k and all $l = 1, 2, 3, \cdots$

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{n\leqq N}\chi(S^{n+l}x)\overline{\chi(S^n x)} = \alpha_{\chi,l}(x)$$

exists. Every such character χ has the form

$$\chi(x) = e^{2\pi i\sum_{j=0}^{k-1}h_j x_j}$$

with integers h_j not all zero. Let $x = (p(0), \cdots, p(k-1))$. Then

$$\chi(S^{n+l}x)\overline{\chi(S^n x)} = e^{2\pi i\sum_{j=0}^{k-1}h_j(p(n+l+j)-p(n+j))}$$

and

$$\begin{aligned}\sum_{j=0}^{k-1}h_j(p(t+l+j)-p(t+j)) &= \sum_{j=0}^{k-1}h_j\sum_{m=0}^{k-1}\frac{j^m}{m!}(p^{(m)}(t+l)-p^{(m)}(t))\\ &= \sum_{j=0}^{k-1}h_j\sum_{m=0}^{k-1}\frac{j^m}{m!}\sum_{s=1}^{k-m}p^{(m+s)}(t)\frac{l^s}{s!}\\ &= \sum_{m=0}^{k-1}\frac{1}{m!}\sum_{j=0}^{k-1}h_j j^m\sum_{s=1}^{k-m}p^{(m+s)}(t)\frac{l^s}{s!}.\end{aligned}$$

For at least one $m = 0, 1, \cdots, k-1$ the sum $\sum_{j=0}^{k-1}h_j j^m$ must be $\neq 0$, otherwise all h_j would be zero (Vandermonde determinant).

Let m_0 be the least such number m. For $m_0 < k-1$ we have $\alpha_{\chi,l}(x) = 0$ for all l because $\sum_{j=0}^{k-1}h_j(p(t+l+j)-p(t+j))$ is then a polynomial of degree less than k with irrational leading coefficient and thus the sequence $\{\sum_{j=0}^{k-1}h_j(p(n+l+j)-p(n+j))\}$ is uniformly distributed mod 1. Let now $m_0 = k-1$. Then

$$\sum_{j=0}^{k-1}h_j(p(t+l+j)-p(t+j)) = \beta l,$$

where $\beta = k\theta\sum_{j=0}^{k-1}h_j j^{k-1}$ is irrational. Therefore $\alpha_{\chi,l}(x) = e^{2\pi i l\beta}$.

In every case we have thus

$$\lim_{N\to\infty}\frac{1}{N+1}\sum_{l\leqq N}\alpha_{\chi,l}(x) = 0.$$

Let now $\mu \in V(x)$. It follows from the definition of $V(x)$ that S is μ-invariant and that

$$\alpha_{\chi,l}(x) = \int_{T^k}\chi(S^l t)\overline{\chi(t)}d\mu(t),$$

because $\chi(S^l t)\overline{\chi(t)} \in C(T^k)$. If (f, g) denotes the inner product in $L^2_\mu(T^k)$ then clearly $\alpha_{\chi,l}(x) = (U^l\chi, \chi)$. The lemma implies now that $\mu = \lambda$. This holds for every $\mu \in V(x)$ and so $V(x) = \{\lambda\}$, q.e.d.

REMARK. It is routine matter to derive from our theorem the following seemingly stronger facts:

1) S is strictly ergodic with λ as unique invariant probability measure.

2) Is $p(t) = a_0t^k + \cdots + a_k$ with $a_0 = \theta$ irrational, then the sequence $\{(p(n), \cdots, p(n+k-1))\}$ is well-distributed in T^k.

3) Is $p(t) = a_0t^k + \cdots + a_k$ a polynomial with at least one irrational coefficient a_i, $i < k$, then the sequence $\{p(n)\}$ is well-distributed mod. 1.

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