

# COMPOSITIO MATHEMATICA

R. D. ANDERSON

DAVID W. HENDERSON

JAMES E. WEST

## **Negligible subsets of infinite-dimensional manifolds**

*Compositio Mathematica*, tome 21, n° 2 (1969), p. 143-150

[http://www.numdam.org/item?id=CM\\_1969\\_\\_21\\_2\\_143\\_0](http://www.numdam.org/item?id=CM_1969__21_2_143_0)

© Foundation Compositio Mathematica, 1969, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## Negligible subsets of infinite-dimensional manifolds

by

R. D. Anderson, David W. Henderson, and James E. West \*

We say that a subset  $A$  of a space  $X$  is *negligible in  $X$*  if  $X \setminus A$  is homeomorphic to  $X$ . In [1], [3], [4], [5], [6], [11], and [12], various authors have shown that certain subsets of certain infinite-dimensional topological linear spaces are negligible. We extend these results to infinite-dimensional metrizable manifolds modelled on separable Fréchet spaces. (We call such manifolds, *F-manifolds*.) In fact, we give a characterization of strongly negligible closed subsets of  $F$ -manifolds, where we say that  $A$  is *strongly negligible* in  $X$  if, for each open cover  $G$  of  $X$ , there is a homeomorphism of  $X$  onto  $X \setminus A$  which is limited by  $G$ . Theorem 5 asserts that a closed subset  $A$  is strongly negligible in an  $F$ -manifold  $X$  if and only if  $A$  has *Property Z*, i.e. (following Anderson in [4]) for each non-empty, homotopically trivial, open set  $U$  in  $X$ , it is true that  $U \setminus A$  is non-empty and homotopically trivial. In Lemma 1 Property *Z* is shown to be equivalent, in  $F$ -manifolds, to several other properties, one of which is the property shown by Eells and Kuiper (in [8]) to imply homotopy-negligibility. In the same paper, Eells and Kuiper also show how their results, together with recent results of Burghlelea, Kuiper and Moulis, can be used to imply some of our results.

The authors feel that the methods of this paper offer simple and convenient frameworks for the reduction of many point-set topological problems in  $F$ -manifolds (and thus in many Banach manifolds) to problems in the Hilbert cube  $Q$  and in  $s$  (the countably infinite product of real lines). The reduction is effected by an iterated use of star-finite coverings, the second use involving open sets homeomorphic to  $s$  and thus admitting canonical compactifications to  $Q$ . It has been shown (see [7], [9], and [2] or [5]) that all separable, infinite-dimensional, Fréchet spaces are homeomorphic to  $s$ .

\* Supported in part by National Science Foundation grant GP 7952X.

**LEMMA 1.** *The following four conditions are equivalent for a closed subset  $A$  of an  $F$ -manifold  $X$ :*

1.  $A$  has Property  $Z$  in  $X$ .
2.  $A$  has Property  $Z$  locally in  $X$  (i.e. each point of  $A$  lies in an open set  $0$  for which  $A \cap 0$  has Property  $Z$  in  $0$ .)
3. Each point of  $A$  has a fundamental system of neighborhoods  $U$  in  $X$  for which the inclusion  $U \setminus A \rightarrow U$  is a homotopy equivalence.
4. Each point of  $A$  lies in an open set  $0$  of which there is a homeomorphism onto  $s$  carrying  $A \cap 0$  into a closed linear subspace which has an infinite-dimensional closed complementary linear subspace (i.e., one for which  $s$  is the direct product (sum) of it with the first).

**PROOF.** It is trivial that 1. implies 2. implies 3. That 3. implies 1. is essentially the content of the main lemma of [8]. That 2. and 4. are equivalent is Theorems 8.4 and 9.1 of [4].

**LEMMA 2.** *If  $A$  is a subset of the  $F$ -manifold  $X$  which has Property  $Z$  in  $X$  and if  $A'$  is a closed subset of  $A$ , then  $A \setminus A'$  has Property  $Z$  in  $X \setminus A'$  and  $A'$  has Property  $Z$  in  $X$ .*

**PROOF.** The first statement follows immediately from Lemma 1; the second may be shown from the definition of  $F$ -manifold and Theorems 8.4 and 9.1 of [4] to show that  $A'$  has Property  $Z$  locally in  $X$ .

For  $G$  an open cover of a space  $X$ , a function  $f$  of  $X$  into itself is said to be *limited by  $G$*  if for each  $x$  in  $X$  there is an element of  $G$  containing both  $x$  and  $f(x)$ .

An open cover  $G$  of an open subset  $U$  of a space  $X$  is said to be a *normal cover* provided that whenever  $f$  is a homeomorphism of  $U$  into itself which is limited by  $G$ , there is an extension of  $f$  to a homeomorphism of  $X$  into itself which is the identity on  $X \setminus U$ . This extension is said to be the *normal extension* of  $f$ .

**LEMMA 3.** *For any open set  $U$  in a metric space  $X$ , there is a normal cover of  $U$ .*

**PROOF.** Let  $d$  be a metric for  $X$  and for each  $x$  in  $U$ , set  $U_x = \{y \in U \mid d(x, y) < \frac{1}{2}d(x, X \setminus U)\}$ . Let  $G = \{U_x\}_{x \in U}$ . This cover suffices, as is easily seen.

**LEMMA 4.** *Each component of an  $F$ -manifold is separable.*

**PROOF.** This is immediate from a result due to Sierpinski [14; p. 111] which asserts that a connected, locally separable, metric space is separable

**THEOREM 1.** *Let  $U$  be an open subset of  $s$ . For each open cover  $G$  of  $U$  there is a triple  $(G_1, G_2, G_3)$  of star-finite covers of  $U$  refining  $G$  such that, denoting  $G_i$  by  $\{g_j^i\}_{j \in N}$ ,*

1.  $G_1$  and  $G_2$  are open covers by sets homeomorphic to  $s$ , while  $G_3$  is a cover by closed sets of  $s$ ;
2.  $G_1$  is a normal cover of  $U$ ; and
3. for each  $j \in N$ ,  $g_j^3 \subset g_j^2 \subset \bar{g}_j^2 \subset g_j^1$ .

**PROOF.** Let  $s$  be regarded as lying in the Hilbert cube  $Q$  as the “pseudo-interior”, i.e., let  $Q = \prod_{i \in N} I_i$ , where for each  $i$ ,  $I_i = [-1, 1]$ , and let  $s = \prod_{i \in N} I_i^0$ , with  $I_i^0 = (-1, 1)$ . Let  $G_0$  be a normal cover of  $U$  refining  $G$ , and set  $V = \bigcup_{g^0 \in G_0} Q \setminus s \setminus \bar{g}^0$ . Since  $V$  is a separable, locally compact metric space, it may be written as the union of an increasing sequence  $\{X_i\}_{i \in N}$  of compacta, each lying in the interior of its successor. Let  $D$  be the set of all triples  $(A, B, C)$  of basic connected open subsets of  $V$  such that  $A \subset \bar{A} \subset B \subset \bar{B} \subset C$  and there is a  $g^0$  in  $G^0$  such that  $C \subset Q \setminus s \setminus \bar{g}^0$ , and let  $E$  be the subset of  $D$  composed of all triples  $(A, B, C)$  for which  $\bar{C} \cap X_i \neq \emptyset$  implies that  $\bar{C} \subset X_{i+1}^0$ , for each  $i$ . The collection of all initial elements of members of  $E$  is an open cover of  $V$ , and each  $X_i$  is compact; so for each  $i$ , there exists a finite subset  $H_i$  of  $E$  for which the initial elements of members of  $H_i$  cover  $X_i \setminus X_{i-1}^0$ . Set

$$G_1 = \{C \cap s \mid \text{there are an } i, \text{ a } B, \text{ and an } A \\ \text{for which } (A, B, C) \text{ is in } H_i\},$$

$$G_2 = \{B \cap s \mid \text{there are an } i, \text{ an } A, \text{ and a } C \\ \text{such that } (A, B, C) \text{ is in } H_i\}, \text{ and}$$

$$G_3 = \{\bar{A} \cap s \mid \text{there are an } i, \text{ a } B, \text{ and a } C \\ \text{for which } (A, B, C) \text{ is in } H_i\}.$$

Index  $\bigcup_{i \in N} H_i$  by  $N$  and induce the same indexing on  $G_1, G_2$ , and  $G_3$ .

**LEMMA 5.** *Each open cover of an  $F$ -manifold has a star-finite open refinement.*

**PROOF.** This is immediate from Lemma 4 and from Theorem 1 of [10], which uses the same idea as our Theorem 1 above.

**THEOREM 2.** *For any countable, star-finite, open cover  $G$  of a space  $X$ , there exists an ordering  $\{g_i\}_{i \in N}$  of the elements of  $G$  such that if  $\{f_i\}_{i \in N}$  is a sequence of homeomorphisms with  $f_1$  a homeo-*

morphism of  $X$  into itself which is the identity off  $g_1$  and, for each  $i > 1$ , with  $f_i$  a homeomorphism of  $f_{i-1} \cdots f_1(X)$  into itself which is the identity off  $g_i$ , then it follows that  $f(x) = \lim_{i \rightarrow \infty} f_i \cdots f_1(x)$  defines a homeomorphism of  $X$  into itself.

PROOF. Let  $\{G_i\}_{i \in N}$  be any ordering of the elements of  $G$ ; let  $H_1 = \{G_1\}$ , and (inductively for  $i > 1$ ) let  $H_i$  be the set of all elements of  $G \setminus \bigcup_{j < i} H_j$  which intersect members of  $H_{i-1}$  together with the least-indexed element of  $G \setminus \bigcup_{j < i} H_j$ . Since  $G$  is star-finite, each  $H_i$  is finite. Let  $\{g_i\}_{i \in N}$  be an ordering of  $G$  listing the elements of  $H_1$ , then those of  $H_2$ , followed by those of  $H_3$  and then of  $H_5$ , and so forth. Now suppose  $\{f_i\}_{i \in N}$  is a sequence of functions as in the hypothesis with respect to the cover  $\{g_i\}_{i \in N}$ . Let  $h_k = f_k \cdots f_1$ . Set  $j(k) = \max \{n | g_n \in H_k\}$ , and let  $H_k^* = \bigcup_{j \leq k} H_j$ . If  $m$  is an odd integer, then

$$j(m-2) < j(m) < j(m-1) < j(m+2) < j(m+1) < j(m+4).$$

Now for  $k \leq j(m-1)$ ,  $h_k(H_m^*) \subset H_m^*$ ,

and for  $j(m-1) < k \leq j(m+2)$ ,  $f_k|H_m^* \cap h_{k-1}(X)$

is the identity; therefore,  $h_k(H_m^*) \subset H_m^*$  for all  $k \leq j(m+2)$ . Thus, for

$$k \leq j(m+1), h_k(H_m^*) \subset H_{m+1}^*.$$

Now, since  $k > j(m+1)$  implies that  $f_k|H_{m+1}^* \cap h_{k-1}(X)$  is the identity, it follows that for

$$k > j(m+1), h_k|H_m^* = h_{j(m+1)}|H_m^*,$$

and the theorem follows.

**THEOREM 3.** *If  $U$  is open in  $s$ ,  $A$  is a relatively closed subset of  $U$  with Property  $Z$  in  $U$ , and  $G$  is an open cover of  $U$ , there is a homeomorphism of  $s$  onto  $s \setminus A$  which is the identity off  $U$  and is limited on  $U$  by  $G$ .*

PROOF. Let  $G_1, G_2$ , and  $G_3$  be as per Theorem 1, and assume that  $G_1$  is indexed in accordance with Theorem 2.

Let  $H_1$  be a normal open cover of  $g_1^2$ . Since  $g_1^2$  is homeomorphic to  $s$  and (by Lemma 2)  $A \cap g_1^3$  is a closed subset of  $g_1^2$  with Property  $Z$  in it, there exists, by Theorem 8.4 of [4]<sup>1</sup> and Lemma

<sup>1</sup> Theorem 8.4 of [4] asserts that if  $X$  is a closed subset of  $s$  with Property  $Z$ , then there is a homeomorphism of  $s$  onto itself carrying  $X$  into a closed linear subspace of  $s$  which has an infinite-dimensional complementary closed linear subspace.

9.1 of [5]<sup>2</sup>, a homeomorphism of  $g_1^2$  onto  $g_1^2 \setminus (A \cap g_1^3)$  which is limited by  $H_1$ . Let  $f_1$  be the normal extension of this homeomorphism to  $s$ .

For each  $i > 0$ ,  $A \cap (\bigcup_{j < i} g_j^3) \cap g_i^2$  is a relatively closed set in  $g_i^2$  and has Property  $Z$  in it (Lemma 2), so  $g_i^2 \setminus (A \cap (\bigcup_{j < i} g_j^3))$  is homeomorphic to  $s$  (Theorem 8.4 of [4] and Lemma 9.1 of [5], together with the choice of  $g_i^2$  as being homeomorphic to  $s$ ). Let  $a_i$  be such a homeomorphism.

Suppose, now, that  $f_1, \dots, f_{i-1}$  are homeomorphisms of  $s, \dots, s \setminus (A \cap (\bigcup_{j < i-1} g_j^3))$  onto

$$s \setminus (A \cap g_1^3), \dots, s \setminus (A \cap (\bigcup_{j < i-1} g_j^3))$$

which are the identity on  $s \setminus g_1^2, \dots, s \setminus g_{i-1}^2$ , respectively, and for which  $f_j \cdots f_1(\bar{g}_m^2 \setminus \bigcup_{n < m} g_n^2)$  lies in  $g_m^1$  for each  $j \leq i-1$  and  $m \leq j$ . Let  $H_i$  be a normal open cover of  $g_i^2 \setminus (A \cap (\bigcup_{j < i} g_j^3))$  for which no element meets both  $f_{i-1} \cdots f_1(\bar{g}_m^2 \setminus \bigcup_{n < m} g_n^2)$  and  $s \setminus g_m^1$  for any  $m < i$ . Now apply Theorem 8.4 of [4] to obtain a homeomorphism  $b_i$  of  $s$  onto itself carrying  $a_i(A \cap (g_i^3 \setminus \bigcup_{j < i} g_j^3))$  into a closed linear subspace which has an infinite-dimensional complementary closed linear subspace. Let  $c_i$  be a homeomorphism of  $s$  onto  $s \setminus a_i b_i(A \cap (g_i^3 \setminus \bigcup_{j < i} g_j^3))$  limited by the image under  $b_i a_i$  of  $H_i$ , as guaranteed by Lemma 9.1 of [5]. The homeomorphism  $a_i^{-1} b_i^{-1} c_i b_i a_i$  of

$$g_i^2 \setminus (A \cap (\bigcup_{j < i} g_j^3)) \quad \text{onto} \quad g_i^2 \setminus (A \cap (\bigcup_{j \leq i} g_j^3))$$

is now limited by  $H_i$ ; let  $f_i$  be the normal extension of it to  $s \setminus (A \cap (\bigcup_{j < i} g_j^3))$ .

By induction, there exists a sequence  $\{f_i\}_{i \in N}$  of homeomorphisms of the sort supposed in the preceding paragraph. By Theorem 2, there is a homeomorphism  $f$ , defined by the formula  $f(x) = \text{limit}_{i \rightarrow \infty} f_i \cdots f_1(x)$  for each  $x$  in  $X$ , of  $s$  onto  $s \setminus A$  which is the identity off  $U$ . This homeomorphism is limited on  $U$  by  $G$  because if  $x$  is in  $U$  and  $i$  is the integer for which  $g_i^2 \setminus \bigcup_{j < i} g_j^2$  contains  $x$ , then  $f_m \cdots f_1(x)$  is in  $g_i^1$  for each  $m$ . Thus,  $f(x)$  is in  $g_i^1$  because there is an  $l$  for which  $f_{l+k} \cdots f_1(x) = f_l \cdots f_1(x)$  for all  $k > 0$  by the nature of the ordering of  $G_1$ . Since  $G_1$  limits  $f$  and refines  $G$ ,  $G$  also limits  $f$ .

<sup>2</sup> Lemma 9.1 of [5] asserts that if  $X$  is a closed subset of  $s$  lying in a closed linear subspace with a complementary closed linear subspace of infinite dimension and if  $G$  is an open cover of  $s$ , then there is a homeomorphism of  $s$  onto  $s \setminus X$  limited by  $G$  which may be required to be the identity off any open set containing  $X$ .

**THEOREM 4.** *If  $X$  is an  $F$ -manifold and  $A$  is a closed subset of  $X$  which has Property  $Z$ , then  $X$  is homeomorphic to  $X \setminus A$ ; furthermore, the homeomorphism may be required to be limited by any open cover of  $X$  and to be the identity on the complement of any open set of  $X$  containing  $A$ .*

**PROOF.** It suffices to prove the theorem without requiring the homeomorphism to be the identity on the complement of a given open set  $U$  containing  $A$ . This is true because then if  $H$  is any open cover of  $X$ , there is a normal open cover  $H_1$  of  $U$  which refines the open cover of  $U$  composed of the intersections of elements of  $H$  with  $U$ , and the normal extension to  $X$  of a homeomorphism of  $U$  onto  $U \setminus A$  which is limited by  $H_1$  will be limited by  $H$  and will be the identity off  $U$ . Therefore, let  $H$  be any open cover of  $X$ . By Lemma 5 and the definition of  $X$ , there is a star-finite open refinement  $G$  of  $H$  by sets homeomorphic to open subsets of  $s$ . Because the homeomorphism may be defined on each component of  $X$  separately, there is no loss of generality in the supposition that  $X$  is separable and, hence, that  $G$  is countable. By Theorem 2, there is an ordering  $\{g_i\}_{i \in N}$  of  $G$  such that if  $\{f_i\}_{i \in N}$  is a sequence of homeomorphisms with  $f_1$  carrying  $X$  into itself and moving no point not in  $g_1$ , and with, for  $i > 1$ ,  $f_i$  a homeomorphism of  $f_{i-1} \cdots f_1(X)$  into itself which moves no point not in  $g_i$ , then  $f(x) = \lim_{i \rightarrow \infty} f_i \cdots f_1(x)$  defines a homeomorphism of  $X$  onto  $\bigcap_{i \in N} f_i \cdots f_1(X)$ . Assume that  $G$  is so ordered.

For each  $i \in N$ , let  $H_i$  be a normal open cover of

$$g_i \setminus (A \cap (\bigcup_{j < i} g_j)),$$

and let  $a_i$  be a homeomorphism of  $g_i$  onto an open subset of  $s$ . Let  $c_1$  be a homeomorphism of  $a_1(g_1)$  onto  $a_1(g_1 \setminus A)$  limited by the image under  $a_1$  of  $H_1$  (Theorem 3). Set  $f_1$  to be the normal extension of  $a_1^{-1}c_1a_1$  to  $X$ . For each  $i > 1$ , let  $b_i$  be a homeomorphism of  $s \setminus a_i(A \cap g_i \cap (\bigcup_{j < i} g_j))$  onto  $s$  which is the identity off  $a_i(g_i \cap (\bigcup_{j < i} g_j))$ , as guaranteed by Theorem 3. Since  $b_i a_i(g_i \setminus (A \cap (\bigcup_{j < i} g_j)))$  is open in  $s$  and  $b_i a_i(A \cap (g_i \setminus (\bigcup_{j < i} g_j)))$  is relatively closed in it and has Property  $Z$  in it, Theorem 3 yields a homeomorphism  $c_i$  of  $a_i(g_i)$  onto

$$a_i(g_i) \setminus b_i a_i(A \cap (g_i \setminus \bigcup_{j < i} g_j))$$

which is limited by the image under  $b_i a_i$  of  $H_i$ . Now,  $a_i^{-1} b_i^{-1} c_i b_i a_i$  is a homeomorphism of  $g_i \setminus (A \cap (\bigcup_{j < i} g_j))$  onto  $g_i \setminus A$  limited by

$H_i$ . Let  $f_i$  be the normal extension of it to  $X \setminus (A \cap (\bigcup_{j < i} g_j))$ , and define  $f: X \rightarrow X \setminus A$  by  $f(x) = \lim_{i \rightarrow \infty} f_i \cdots f_1(x)$ . By Theorem 2,  $f$  is the desired homeomorphism.

**THEOREM 5.** *If  $A$  is a closed subset of an  $F$ -manifold  $X$ , then  $A$  is strongly negligible in  $X$  if and only if  $A$  has Property  $Z$ .*

**PROOF.** Only the necessity remains to be proved. Let  $A$  be strongly negligible and let  $f: B^{n+1} \rightarrow U$  be a map of the  $(n+1)$ -ball into an homotopically trivial open subset  $U$  of  $X$  such that  $f(S^n)$  is contained in  $U \setminus A$ . We need to show that  $f|S^n$  can be extended to a map  $F: B^{n+1} \rightarrow U \setminus A$ . Let  $h$  be a homeomorphism of  $X$  onto  $X \setminus A$  which is limited by  $\{U, X \setminus f(B^{n+1})\}$  and which is fixed on the closed set  $f(S^n)$ . (See the proof of Theorem 4.)  $F = h \circ f$  is the desired extension.

*Added in proof.* Extensions of our results to non-separable spaces have recently been given by W. H. Cutler (*Negligible subsets of non-separable Hilbert manifolds* (to appear)). In addition, Cutler shows that our Theorem 4 may be strengthened to require the homeomorphism to be ambient isotopic to the identity by a small isotopy.

#### REFERENCES

R. D. ANDERSON

[1] On a theorem of Klee, Proc. AMS 17 (1966), 1401–4.

R. D. ANDERSON

[2] Hilbert space is homeomorphic to the countable infinite product of open intervals, Trans. AMS 72 (1966), 516–519.

R. D. ANDERSON

[3] Topological properties of the Hilbert cube and the infinite product of open intervals, Trans. AMS 126 (1967), 200–216.

R. D. ANDERSON

[4] On topological infinite deficiency, Mich. Math. J. 14 (1967), 365–383.

R. D. ANDERSON and R. H. BING

[5] A complete, elementary proof that Hilbert space is homeomorphic to the countable infinite product of lines, Bul. AMS 74 (1968), 771–792.

Cz. BESSAGA and V. L. KLEE, Jr.

[6] Two topological properties of topological linear spaces, Israel J. Math. 2 (1964), 211–220.

Cz. BESSAGA and A. PELCZYNSKI

[7] Some remarks on homeomorphisms of  $F$ -spaces, Bull. Acad. Polon. Sci. Ser. Sci Math. Astron. Phys. 10 (1962), 265–270.

J. EELLS, JR. and N. H. KUIPER

[8] Homotopy negligible subsets.



**M. I. KADEC**

- [9] On topological equivalence of all separable Banach spaces (Russian), *Dokl. Akad. Nauk. SSSR* 167 (1966), 23–25.

**S. KAPLAN**

- [10] Homology properties of arbitrary subsets of Euclidean spaces, *Trans. AMS* 62 (1947), 248–271.

**V. L. KLEE, JR.**

- [11] A note on topological properties of normed linear spaces, *Proc. AMS* 7 (1956), 673–4.

**BOR—LUH LIN**

- [12] Two topological properties concerning infinite-dimensional normed linear spaces, *Trans. AMS* 114 (1965), 156–175.

**R. S. PALAIS**

- [13] Homotopy theory of infinite-dimensional manifolds, *Topology* 5 (1966), 1–16.

**W. SIERPINSKI**

- [14] Sur les espaces métriques localement séparables, *Fund. Math.* 21 (1933), 107–113.

(Oblatum 31–5–68)

Louisiana State University,  
Cornell University,  
and  
The Institute for Advanced Study