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by

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Among many other results Andreotti and Grauert proved in [2] the following:

(1) Suppose \( n \) is a non-negative integer and \( \mathcal{F} \) is a coherent analytic sheaf on a Stein space \( X \) such that \( \text{codh} \mathcal{F} \geq n \) (where \( \text{codh} \mathcal{F} \) = homological codimension of \( \mathcal{F} \)). Then \( H^p(X, \mathcal{F}) = 0 \) for \( p < n \). (Cf. Prop. 25, [2]).

Reiffen proved in [6] the following:

(2) Suppose \( n \) is a non-negative integer and \( \mathcal{F} \) is a coherent analytic sheaf on a complex space \( X \) such that \( \dim \text{Supp} \mathcal{F} \leq n \) (where \( \text{Supp} \mathcal{F} \) = support of \( \mathcal{F} \)). Then \( H^p(X, \mathcal{F}) = 0 \) for \( p > n \). (Cf. Satz 3, [6]).

In this note we prove converses of these statements:

**Theorem 1.** Suppose \( n \) is a non-negative integer. If \( \mathcal{F} \) is a coherent analytic sheaf on an open subset \( G \) of a Stein space \( X \) and \( H^p(G, \mathcal{F}) = 0 \) for \( p < n \), then \( \text{codh} \mathcal{F}_x \geq n \) for \( x \in G \).

**Theorem 2.** Suppose \( n \) is non-negative integer, \( \mathcal{F} \) is a coherent analytic sheaf on a Stein space \( X \), and \( G \) is an open subset of \( X \). If \( H^p(G, \mathcal{F}) = 0 \) for \( p > n \), then \( \dim (G \cap \text{Supp} \mathcal{F}) \leq n \).

For the proofs of Theorems 1 and 2 we need the following Lemma:

**Lemma 1.** Suppose \( G \) is an open subset of \( \mathbb{C}^N \), \( x \in G \), and \( A \) is an at most countable subset of \( G \setminus \{x\} \). Then there exists a holomorphic function \( f \) on \( \mathbb{C}^N \) such that \( f(x) = 0 \) and \( f(y) \neq 0 \) for \( y \in A \).

**Proof.** Let \( F \) be the vector space of all holomorphic functions on \( \mathbb{C}^N \) vanishing at \( x \). \( F \) is a Fréchet space with the topology of uniform convergence on compact subsets of \( \mathbb{C}^N \). For \( y \in A \) let \( \varphi_y : F \to \mathbb{C} \) be defined by \( \varphi_y(f) = f(y) \) for \( f \in F \). Let \( K_y = \text{Ker} \varphi_y \). \( K_y \) is a nowhere dense closed subspace of \( F \). For, if we take \( g \in F \) such that \( g(y) \neq 0 \), then for any open neighborhood \( U \) in \( F \) of
h ∈ K_y we have λg + h ∈ U - K_y for λ ∈ C - {0} with |λ| sufficiently small. By Baire category theorem \( \bigcup_{\nu \in A} K_y \neq F \). \( f \in F - \bigcup_{\nu \in A} K_y \) satisfies the requirement.

**Lemma 2.** Suppose \( \mathcal{G} \) is a coherent analytic sheaf on an open subset \( G \) of \( \mathbb{C}^N \). There exist subvarieties \( X_p \) in \( G \), either empty or of pure dim \( p \), \( 0 \leq p \leq N-1 \), such that, for every \( x \in G \), if a non-identically-zero holomorphic function-germ \( f \) at \( x \) does not vanish identically on any non-empty branch-germ of \( X_p \) at \( x \) for any \( p \), then \( f \) is not a zero-divisor for the stalk \( \mathcal{G}_x \) of \( \mathcal{G} \) at \( x \).

**Proof.** For \( 0 \leq p \leq N-1 \), define a subsheaf \( \mathcal{G}_p \) of \( \mathcal{G} \) on \( G \) as follows: for \( x \in G \), \( (\mathcal{G}_p)_x = \{ s \in \mathcal{G}_x \} \) for some subvariety \( A_s \) of dimension \( \leq p \) in some open neighborhood \( U_s \) of \( x \) in \( G \) there exists \( t \in \Gamma(U_s, \mathcal{G}) \) such that \( t_x = s \) and \( t_y = 0 \) for \( y \notin A_s \). \( \mathcal{G}_p \) is a coherent analytic subsheaf of \( \mathcal{G} \) and \( \dim \text{Supp} \mathcal{G}_p \leq p \).

For, if \( \varphi : N^0 \rightarrow \mathcal{G} \) is a sheaf-epimorphism on an open subset \( D \) of \( G \) (where \( N^0 \) is the structure-sheaf of \( \mathbb{C}^N \) and \( (\text{Ker} \varphi)_p \) is the \( p \)-th step gap-sheaf of \( \text{Ker} \varphi \) in the sense of Thimm (Def. 9, [9]), then \( \mathcal{G}_p = \varphi((\text{Ker} \varphi)_p) \) on \( D \) and by Satz 3, [9] \( (\text{Ker} \varphi)_p \) is coherent and \( \dim \{ x \in D | (\text{Ker} \varphi)_p(x) \neq (\text{Ker} \varphi)_p \} \leq p \). Let \( X_p \) be the union of \( p \)-dimensional branches of \( \text{Supp} \mathcal{G}_p \). We claim that these satisfy the requirement.

Suppose \( f \) is a non-identically-zero holomorphic function-germ at a point \( x \) of \( G \) not vanishing identically on any non-empty branch-germ of \( X_p \) at \( x \) for any \( p \). We have to prove that \( f \) is not a zero-divisor for \( \mathcal{G}_x \). Suppose the contrary. Then there exist \( g \in \Gamma(U, N^0) \) and \( h \in \Gamma(U, \mathcal{G}) \) for some connected open neighborhood \( U \) of \( x \) in \( G \) such that \( g_x = f \), \( h_x \neq 0 \), and \( gh = 0 \). Let \( Z = \text{Supp} \ h \) and let \( p \) be the dimension of the germ of \( Z \) at \( x \). \( 0 \leq p \leq N-1 \). By shrinking \( U \) we can assume that \( \dim Z = p \). \( h \in \Gamma(U, \mathcal{G}_p) \) and \( Z \subset \text{Supp} \mathcal{G}_p \). Since \( \dim \text{Supp} \mathcal{G}_p \leq p \) and at \( x \) \( Z \) has dimension \( p \), \( Z \) and \( X_p \) have a branch-germ \( A \) in common at \( x \). \( gh = 0 \) implies that \( f \) vanishes identically on \( A \). Contradiction.

**Lemma 3.** Suppose \( \mathcal{S} \) is a torsion-free coherent analytic sheaf on a normal reduced irreducible complex space \( Z_0 \). Then the set \( E \) of points in \( Z_0 \) where \( \mathcal{S} \) is not locally free is a subvariety of codimension \( \geq 2 \).

**Proof.** Let \( m = \dim Z_0 \). \( D \) is a subvariety in \( Z_0 \) (Prop. 8, [1]). Suppose the Lemma is false. Then \( D \) contains an \( (m-1) \)-dimensional branch \( A \). Let \( M \) be the set of all regular points of \( Z_0 \).
Since dim \((Z_0-M) \leq m-2\), there exists \(x \in M \cap A\). There is a non-identically-zero holomorphic function \(f\) on some connected open neighborhood \(U\) of \(x\) in \(M\) such that \(f\) vanishes identically on \(A \cap U\). Since \(\mathcal{I}\) is torsion-free, for \(y \in U\) \(f_y\) is not a zero-divisor for \(\mathcal{I}_y\). Let \(\mathcal{I} = \mathcal{I}/f\mathcal{I}\) on \(U\). \(F = \{y \in U\mid \text{codh } \mathcal{I}_y \leq m-2\}\) is of dimension \(\leq m-2\) ([Satz 5, [7]]). There exists \(z \in U \cap A - F\). codh \(\mathcal{I}_z = m\). \(\mathcal{I}\) is locally free at \(z\), contradicting that \(z \in D\). q.e.d.

**Lemma 4.** Suppose \(P\) is an \(m\)-dimensional complex manifold. Suppose \(\mathcal{O}\) is the structure-sheaf of \(P\), \(\mathcal{I}\) is a locally free sheaf on \(P\), and \(\mathcal{L}\) is the sheaf of germs of holomorphic \((m, 0)\)-forms on \(P\). If \(H^m_\ast(P, \mathcal{I}) = 0\), then \(\Gamma(P, \text{Hom}_\mathcal{O}(\mathcal{I}, \mathcal{L})) = 0\).

**Proof.** Let \(B\) and \(B^*\) be respectively the holomorphic vector-bundles canonically associated with the locally free sheaves \(\mathcal{I}\) and \(\text{Hom}_\mathcal{O}(\mathcal{I}, \mathcal{L})\). For \(0 \leq p \leq m\) let \(\lambda(0, p)\) denote the vector-bundle of \((0, p)\)-forms on \(P\). Let \(\mathcal{A}^{(0, p)}(B)\) denote the sheaf of germs of infinitely differentiable sections in \(B \otimes \lambda(0, p)\) and let \(\mathcal{D}^{(0, p)}(B^*)\) denote the sheaf of germs of distribution-sections in \(B^* \otimes \lambda(0, p)\). Let \(\Gamma_\ast(P, \mathcal{A}^{(0, p)}(B))\) denote the set of all global sections in \(\mathcal{A}^{(0, p)}(B)\) with compact supports.

\[
0 \to \mathcal{I} \to \mathcal{A}^{(0, 0)}(B) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{A}^{(0, m-1)}(B) \xrightarrow{\overline{\partial}} \mathcal{A}^{(0, m)}(B) \to 0
\]

are fine-sheaf-resolutions for \(\mathcal{I}\) and \(\text{Hom}_\mathcal{O}(\mathcal{I}, \mathcal{L})\) respectively. \(H^m_\ast(P, \mathcal{I}) = 0\) means that

\[
\alpha : \Gamma_\ast(P, \mathcal{A}^{(0, m-1)}(B)) \to \Gamma_\ast(P, \mathcal{A}^{(0, m)}(B))
\]

induced by

\[
\overline{\partial} : \mathcal{A}^{(0, m-1)}(B) \to \mathcal{A}^{(0, m)}(B)
\]

is surjective. \(\Gamma_\ast(P, \mathcal{D}^{(0, 0)}(B^*))\) and \(\Gamma_\ast(P, \mathcal{D}^{(0, 1)}(B^*))\) are respectively the duals of \(\Gamma_\ast(P, \mathcal{A}^{(0, m)}(B))\) and \(\Gamma_\ast(P, \mathcal{A}^{(0, m-1)}(B))\).

\[
\beta : \Gamma_\ast(P, \mathcal{D}^{(0, 0)}(B^*)) \to \Gamma_\ast(P, \mathcal{D}^{(0, 1)}(B^*))
\]

induced by \(\overline{\partial} : \mathcal{D}^{(0, 0)}(B^*) \to \mathcal{D}^{(0, 1)}(B^*)\) is the transpose of \(\alpha\). (Cf. [8]). \(\beta\) is therefore injective. \(\Gamma(P, \text{Hom}_\mathcal{O}(\mathcal{I}, \mathcal{L})) = 0\). q.e.d.

**Proof of Theorem 1:** Since \(X\) is Stein, by imbedding \(X\) and extending \(\mathcal{F}\) trivially we can assume w.l.o.g. that \(X = \mathbb{C}^N\) and
n > 0. Fix \( x \in G \). For \( 0 \leq m \leq n \) we are going to construct by induction on \( m \) holomorphic functions \( f_0 \equiv 0, f_1, \ldots, f_m \) on \( G \) such that \( f_1(x) = \cdots = f_m(x) = 0 \), \( (f_1)_x \neq 0, \ldots, (f_m)_x \neq 0 \), and for \( 1 \leq k \leq m \)

\[
0 \to \mathcal{F}/\sum_{i=0}^{k-1} f_i \mathcal{F} \xrightarrow{\varphi_k} \mathcal{F}/\sum_{i=0}^{k-1} f_i \mathcal{F} \to \mathcal{F}/\sum_{i=0}^{k} f_i \mathcal{F} \to 0
\]

is an exact sequence on \( G \), where \( \varphi_k \) is defined by multiplication by \( f_k \).

The case \( m = 0 \) is trivial. Suppose we have constructed \( f_0 \equiv 0, f_1, \ldots, f_m \) for some \( 0 \leq m < n \). (3) implies that

\[
H^p(G, \mathcal{F}/\sum_{i=0}^{k-1} f_i \mathcal{F}) \to H^p(G, \mathcal{F}/\sum_{i=0}^{k} f_i \mathcal{F}) \to H^{p+1}(G, \mathcal{F}/\sum_{i=0}^{k} f_i \mathcal{F})
\]

is exact for \( p \geq 0 \).

Since \( H^p(G, \mathcal{F}) = 0 \) for \( p < n \), by induction on \( k \) we obtain from (4) that, for \( 0 \leq k \leq m \)

\[
H^p(G, \mathcal{F}/\sum_{i=0}^{k} f_i \mathcal{F}) = 0 \quad \text{for} \quad p < n-k.
\]

Let \( \mathcal{G} = \mathcal{F}/\sum_{i=0}^{m} f_i \mathcal{F} \). For the coherent analytic sheaf \( \mathcal{G} \) on \( G \) we have in \( G \) subvarieties \( X_p \), of pure dim \( p \) or empty, \( 0 \leq p \leq N-1 \), satisfying the requirement of Lemma 2. Since \( H^p(G, \mathcal{G}) = 0 \) by (5) \( m \), from the construction in the proof of Lemma 2 we can choose \( X_0 = \emptyset \). Let \( X_p = \bigcup_{i \in I_p} X_p^i \) be the decomposition into irreducible branches, \( 1 \leq p \leq N-1 \). For \( X_p \neq \emptyset \) take \( x_p^i \in X_p^i - \{x\} \). Let \( G - \{x\} = \bigcup_{j \in J} G_j \) be the decomposition into topological components. Take \( x_j \in G_j \). Let

\[
A = \{x_p^i \mid i \in I_p, 1 \leq p \leq N-1, X_p \neq \emptyset \} \cup \{x_j \mid j \in J\}.
\]

\( A \) is at most countable. There exists by Lemma 1 a holomorphic function \( f \) on \( G \) such that \( f(x) = 0 \) and \( f(y) \neq 0 \) for \( y \in A \). For \( z \in G \) \( f_z \) cannot vanish identically in any non-empty branch-germ of \( X_p \) at \( z \) for any \( p \). Therefore for \( z \in G \) \( f_z \) is not a zero-divisor for \( \mathcal{G}_z \). Set \( f_{m+1} = f \). The sequence \( f_0 \equiv 0, f_1, \cdots, f_m, f_{m+1} \) satisfies the construction requirement. The construction is complete. \( (f_1)_x, \cdots, (f_m)_x \) is an \( \mathcal{F}_x \)-sequence in the sense of (27.1), [5].

\( \text{c.d.d.} \)

**Proof of Theorem 2.** Again w.l.o.g. we can assume that \( X = \mathbb{C}^N \). Let \( Y = \text{Supp} \mathcal{F}, D = G \cap Y, \) and \( \dim D = m \). We have to prove that \( m \leq n \). Suppose the contrary. Then \( n < m \) and \( H^p(G, \mathcal{F}) = 0 \) for \( p \geq m \).

Let \( \mathcal{I} \) be the annihilating ideal-sheaf for \( \mathcal{F} \), i.e. for \( x \in \mathbb{C}^N \), \( \mathcal{I}_x = \{s \in \mathcal{O}_x \mid s \mathcal{F}_x = 0\} \). Let \( \mathcal{H} = \mathcal{O}_G/\mathcal{I} \). The sheaf of modules
\( \mathcal{F} \) can be regarded as over the sheaf of rings \( \mathcal{H} \). Let \( \mathcal{H} \) be the subsheaf of all nilpotent elements of \( \mathcal{H} \). The exactness of

\[ 0 \to \mathcal{H} \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{H} \mathcal{F} \to 0 \]

implies the exactness of

\[ H^p_*(G, \mathcal{F}) \to H^p_*(G, \mathcal{F}/\mathcal{H} \mathcal{F}) \to H^{p+1}_*(G, \mathcal{H} \mathcal{F}) \quad \text{for} \quad p \geq 0. \]

Since

\[ \dim G \cap (\text{Supp } \mathcal{H} \mathcal{F}) \leq m, \quad H^{p+1}_*(G, \mathcal{H} \mathcal{F}) = 0 \quad \text{for} \quad p \geq m \]

by Satz 3, [6]. Hence

\[ H^p_*(G, \mathcal{F}/\mathcal{H} \mathcal{F}) = 0 \quad \text{for} \quad p \geq m. \]

Supp \( (\mathcal{F}/\mathcal{H} \mathcal{F}) \) = Supp \( \mathcal{F} \). For, if for some \( x \in \mathbb{C}^N \), \( \mathcal{F}_x = \mathcal{H}_x \mathcal{F}_x \), then, since \( \mathcal{H}_x \) is contained in the maximal-ideal of the local ring \( \mathcal{H}_x \), we have \( \mathcal{F}_x = 0 \) by Krull-Azumaya Lemma ((4.1), [5]).

Let \( \mathcal{G} = (\mathcal{F}/\mathcal{H} \mathcal{F})|Y \) and \( \tilde{\mathcal{G}} = (\mathcal{H}/\mathcal{H})|Y \). \( \mathcal{F} \) is a coherent analytic sheaf on the reduced Stein space \( (Y, \tilde{\mathcal{G}}) \). Supp \( \mathcal{G} = Y \) and \( H^p_*(D, \mathcal{G}) = 0 \) for \( p \geq m \).

Let \( \pi : Z \to Y \) be the normalization of \( (Y, \tilde{\mathcal{G}}) \). Let \( \mathcal{G}' \) be the inverse image of \( \mathcal{G} \) under \( \pi \) (Def. 8, [3]) and let \( \mathcal{G}'' \) be the zeroth direct image of \( \mathcal{G}' \) under \( \pi \). There exists a natural sheaf-homomorphism \( \lambda : \mathcal{G} \to \mathcal{G}'' \) (Satz 7 (b), [3]). \( \lambda \) is bijective at regular points of \( Y \). Let \( \mathcal{R} = \text{Ker } \lambda \) and \( \mathcal{Z} = \lambda(\mathcal{G}) \). The exactness of

\[ 0 \to \mathcal{R} \to \mathcal{G} \to \mathcal{G}'' \to 0 \]

implies the exactness of

\[ H^p_*(D, \mathcal{R}) \to H^p_*(D, \mathcal{Z}) \to H^{p+1}_*(D, \mathcal{R}) \quad \text{for} \quad p \geq 0. \]

Since \( \dim D \cap \text{Supp } \mathcal{R} < m \), \( H^{p+1}_*(D, \mathcal{R}) = 0 \) for \( p \geq m-1 \).

\( H^p_*(D, \mathcal{Z}) = 0 \) for \( p \geq m \). The exactness of

\[ 0 \to \mathcal{Z} \to \mathcal{G}'' \to \mathcal{G}''/\mathcal{Z} \to 0 \]

implies the exactness of

\[ H^p_*(D, \mathcal{Z}) \to H^p_*(D, \mathcal{G}'') \to H^p_*(D, \mathcal{G}''/\mathcal{Z}) \quad \text{for} \quad p \geq 0. \]

Since \( \dim D \cap \text{Supp } \mathcal{G}''/\mathcal{Z} < m \), \( H^p_*(D, \mathcal{G}''/\mathcal{Z}) = 0 \) for \( p \geq m \).

\( H^p_*(D, \mathcal{G}'') = 0 \) for \( p \geq m \). Let \( L = \pi^{-1}(D) \). Since

\[ H^p_*(L, \mathcal{G}') \approx H^p_*(D, \mathcal{G}'') \quad \text{for} \quad p \geq 0, \]

\( H^p_*(L, \mathcal{G}') = 0 \) for \( p \geq m \).

Let \( \mathcal{I} \) be the torsion subsheaf of \( \mathcal{G}' \) and let \( \mathcal{I} = \mathcal{G}'/\mathcal{I} \). On \( Z \) \( \mathcal{I} \) is coherent and torsion-free (Prop. 6, [1]). Since \( \text{Supp } \mathcal{G} = Y \),
Supp \( \mathcal{S} = Z \). The exact sequence \( 0 \to \mathcal{I} \to \mathcal{G}' \to \mathcal{S} \to 0 \) gives rise to the exact sequence

\[
H^p_*(L, \mathcal{G}') \to H^p_*(L, \mathcal{S}) \to H^{p+1}_*(L, \mathcal{I}) \quad \text{for} \quad p \geq 0.
\]

Since \( \dim L \cap \text{Supp} \mathcal{I} < m \), \( H^{p+1}_*(L, \mathcal{S}) = 0 \) for \( p \geq m-1 \). \( H^p_*(L, \mathcal{S}) = 0 \) for \( p \geq m \). Let \( Z_0 \) be an \( m \)-dimensional branch of \( Z \) intersecting \( L \). \( H^p_*(L \cap Z_0, \mathcal{S}) = 0 \) for \( p \geq m \). Let \( M \) be the set of all regular points of \( Z_0 \) and let \( E \) be the set of points in \( Z_0 \) where \( \mathcal{S} \) is not locally free. By Lemma 3 \( \dim E \leq m-2 \). Since \( Z_0 \) is normal, \( \dim (Z_0 - M) \leq m-2 \). By Satz 3,\[ [6],
\[
H^p_*(L \cap (M - E), \mathcal{S}) = 0 \quad \text{for} \quad p \geq m.
\]

Let \( \mathcal{O} \) be the structure-sheaf of \( Z_0 \) and let \( \mathcal{L} \) be the sheaf of germs of holomorphic \((m, 0)\)-forms on \( M \). By Lemma 4 \( I(L \cap (M - E), \text{Hom}_\mathcal{O}(\mathcal{S}, \mathcal{L})) = 0 \). Take \( x \in L \cap (M - E) \). Since \( \mathcal{L}_x \neq 0 \) and \( Z_0 \) is Stein, there exists \( s \in I(Z_0, \text{Hom}_\mathcal{O}(\mathcal{S}, \mathcal{O})) \) such that \( s_x \neq 0 \). Since \( Z_0 \) is Stein, there exist holomorphic functions \( g_1, \ldots, g_m \) on \( Z_0 \) such that the map \( (g_1, \ldots, g_m): Z_0 \to \mathbb{C}^m \) has rank \( m \) at \( x \). \( dg_1 \wedge \cdots \wedge dg_m \) defines an element \( f \) of \( I(M, \mathcal{L}) \).

Let \( f_x \neq 0 \). Since \( \text{Hom}_\mathcal{O}(\mathcal{S}, \mathcal{L}) \approx \text{Hom}_\mathcal{O}(\mathcal{S}, \mathcal{O}) \otimes _\mathcal{O} \mathcal{L} \) on \( M \), \( s \otimes f|L \cap (M - E) \) is a nonzero element of \( I(L \cap (M - E), \text{Hom}_\mathcal{O}(\mathcal{S}, \mathcal{L})) \). Contradiction. q.e.d.

**Remark.** In Theorems 1 and 2 the assumption that \( X \) is Stein cannot be dropped altogether. Counter-examples can easily be constructed by letting \( X \) be a complex projective space and by using Theorem von Serre in [3]. However, easy modifications in the proof can show that Theorem 1 holds under the weaker assumption that holomorphic functions on \( X \) separate points.

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