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Note on a theorem of J. Nagata

by

J. E. Vaughan and B. R. Wenner

In a 1963 issue of this journal, J. Nagata proved the following Theorem:

**Theorem.** A metric space $R$ has dim $\leq n$ if and only if we can introduce in $R$ a topology-preserving metric $\rho$ such that the spherical neighborhoods $S_\varepsilon(p)$, $\varepsilon > 0$ of every point $p$ of $R$ have boundaries of dim $\leq n-1$ and such that $\{S_\varepsilon(p) : p \in R\}$ is closure-preserving for every $\varepsilon > 0$. [2, Theorem 1].

Subsequently in the same article the author used the metric constructed in this Theorem to give proofs of the following two Corollaries:

**Corollary 2.** A metric space $R$ has dim $\leq n$ if and only if we can introduce a topology-preserving metric $\rho$ into $R$ such that $\dim C_\varepsilon(p) \leq n-1$ for any irrational (or for almost all) $\varepsilon > 0$ and for any point $p$ of $R$ and such that $\{C_\varepsilon(p) : p \in R\}$ is closure-preserving for any irrational (or for almost all) $\varepsilon > 0$, where $C_\varepsilon(p) = \{q : p \rho(p, q) = \varepsilon\}$.

**Corollary 3.** A metric space $R$ has dim $\leq n$ if and only if we can introduce a topology-preserving metric $\rho$ into $R$ such that for all irrational (or for almost all) positive numbers $\varepsilon$ and for any closed set $F$ of $R$, $\dim C_\varepsilon(F) \leq n-1$, where $C_\varepsilon(F) = \{p : p \rho(p, F) = \varepsilon\}$.

The purpose of this communication is two-fold: first, to show why the proofs of these two Corollaries are invalid, and second, to show that in general no such result can be obtained.

1. The first objective will be obtained by using Nagata's procedure to construct an equivalent metric on the real line $\mathbb{R}$; we shall use the notation of [2] throughout. We define the follow-
The sequence of open covers of $\mathbb{R}$: $U_0 = \{\mathbb{R}\}$, and for all $i > 0$ we let

$$U_i = \{(k \cdot 2^{-5(i-1)} - \frac{3}{4}) \cdot 2^{-5(i-1)}, k \cdot 2^{-5(i-1)} + \frac{3}{4} \cdot 2^{-5(i-1)}:\ k = 0, \pm 1, \pm 2, \cdots\}.$$

It is immediate that $\{U_i : i = 0, 1, 2, \cdots\}$ satisfies conditions (1), (2), and (3) in the proof of Theorem 1, and we define the metric $\rho$ as in that proof.

The proof of Corollary 2 now purports to show that for any metric defined in this manner, any irrational $\varepsilon > 0$, and for any point $p$, $C_\varepsilon(p) = B[S_\varepsilon(p)]$. This is done by showing that $q \notin S_\varepsilon(p)$ implies $q \notin C_\varepsilon(p)$. In our example $(\mathbb{R}, \rho)$, let us choose the irrational number $\varepsilon = 2^{-m_1} + 2^{-m_2} + \cdots$, where $m_i = \sum_{j=1}^i j$ for all $i = 1, 2, \cdots$; then choose $p = \frac{3}{4} + \sum_{i=2}^\infty 2^{-5(m_i-1)}$, and $q = 0$. A routine calculation shows that

$$S_{m_1, m_2} = \{(k-p, k+p) : k = 0, \pm 1, \pm 2, \cdots\},$$

hence $S_\varepsilon(p) = S(p, S_{m_1, m_2}, \ldots) = (1-p, 1+p)$; also, $q \notin S_\varepsilon(p)$ as

$$p < \frac{3}{4} + \sum_{i=2}^\infty 2^{-5(m_i-1)} < \frac{3}{4} + 2^{-8} < 1.$$

Now for $i = 2$ we see that $S(q, U_1) \cap S(p, S_{m_1, m_2}, \ldots) = \emptyset$ and $m_{i+1} \geq m_i + 2$, but the following statement, “Then it is easily seen that $q \notin S(p, S_{m_1, m_2}, \ldots)$” [2, p. 232, top] is false. For let $t = \frac{3}{4} + \sum_{i=3}^\infty 2^{-15}$, then $q = 0 \in (-t, t) \in S_{m_1, m_2, m_3+1}$. Moreover,

$$t-p = \frac{3}{4} + \sum_{i=3}^\infty 2^{-15} = \frac{3}{4} + \sum_{i=3}^\infty 2^{-15} = \frac{3}{4} + \sum_{i=3}^\infty 2^{-15}$$

so $p \in (-t, t)$, hence $q \in S(p, S_{m_1, m_2, m_3+1})$.

In the case of $(\mathbb{R}, \rho)$ there is no possibility of avoiding this roadblock, as it is by no means true that $C_\varepsilon(p) = B[S_\varepsilon(p)]$ for irrational $\varepsilon > 0$. This can be seen by consideration of the $\varepsilon$ and $p$ used above. For all $r \in [-1+p, 1-p]$ we see that $S_\varepsilon(r) = (-p, p)$, so $p \in B[S_\varepsilon(r)]$, which implies $\rho(p, r) = \varepsilon$; thus

$$[-1+p, 1-p] \subset C_\varepsilon(p).$$

But

$$B[S_\varepsilon(p)] = B[(1-p, 1+p)] = \{1-p, 1+p\},$$

so $C_\varepsilon(p) \neq B[S_\varepsilon(p)]$. We note finally that
hence a metric constructed as in Theorem 1 does not necessarily have the property described in Corollary 2, nor that in Corollary 3.

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Corollary 2 asserts the existence of a topology-preserving metric for any $n$-dimensional metric space $R$ which satisfies the following two properties for all irrational $\varepsilon > 0$:

(i) $\dim C_\varepsilon(p) \leq n - 1$ for all $p \in R$, and

(ii) $\{C_\varepsilon(p) : p \in R\}$ is closure-preserving.

Although the space $(\mathbb{R}, \rho)$ can be shown to satisfy (ii), we have seen that it does not fulfill (i). On the other hand, $\mathbb{R}$ with the usual metric satisfies (i) but not (ii). The following Theorem demonstrates that a connected metric space of dimension greater than zero cannot simultaneously satisfy (i) and (ii) for small $\varepsilon$:

**Theorem.** Let $(R, d)$ be a connected space with at least two points, $n > 0$, and $0 < \varepsilon < \frac{1}{2} \text{diam } R$. If (i) and (ii) are satisfied for this $n$ and $\varepsilon$, then $\dim R \leq n - 1$.

**Proof.** Let $p \in R$; the set $B = \{z : d(p, z) > \varepsilon\} \neq \emptyset$ (if not, then for all $x, y \in R$ we have $d(x, y) \leq d(x, p) + d(y, p) \leq 2\varepsilon$, so $\text{diam } R \leq 2\varepsilon$, which contradicts the hypothesis). Hence there exists a point $q \in R$ such that $d(p, q) = \varepsilon$, for otherwise the two nonempty sets $S_\varepsilon(p)$ and $B$ would yield a separation of the connected space $R$. Hence $p \in R$ implies $p \in C_\varepsilon(q)$ for some $q \in R$, so $R = \bigcup \{C_\varepsilon(q) : q \in R\}$. By a Theorem of Nagami [1, Theorem 1], conditions (i) and (ii) imply $\dim R \leq n - 1$.

This Theorem demonstrates that for the above class of spaces Corollary 2 is invalid. It remains an open question as to whether or not Corollary 3 is invalid for a similar class of spaces.

**BIBLIOGRAPHY**

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