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by

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Introduction

A graph is *planar* if it can be drawn in the plane (or on a sphere) so that no two of its edges intersect. The *thickness* $t(G)$ of a graph G, introduced by Tutte [3], is the minimum number of edge-disjoint planar subgraphs whose union is G. In contrast to the thickness is the *coarseness* [2] $c(G)$ defined as the maximum number of edge-disjoint nonplanar subgraphs contained in G. Obviously, G is planar if and only if $c(G) = 0$.

We present some results here concerning the coarseness of a graph. In particular, bounds on the coarseness of complete graphs are given.

Basic results

A *subdivision* of a graph G is a graph obtained from G by replacing an edge $x = uv$ of G by a new vertex w and the two new edges uw and wv. Two graphs G_1 and G_2 are *homeomorphic* if there exists a graph G_3 which can be obtained from each of G_1 and G_2 by a sequence of subdivisions. Following Kuratowski [2], we call a graph *skew* if it is homeomorphic to either the complete graph K_5 or the complete bigraph $K_{3,3}$. These graphs are shown in Figure 1. The classic result of Kuratowski then states that a graph is planar if and only if it contains no skew subgraph.

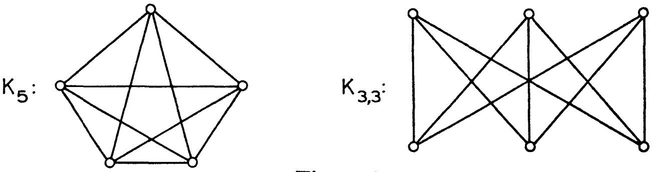

Figure 1

[1] Research supported by grants from the U. S. Air Force Office of Scientific Research and the National Institute of Mental Health, Grant MH-10834.

[2] This concept was suggested by Professor Paul Erdös.

If a graph G has coarseness n, then the n edge-disjoint nonplanar subgraphs contained in G may all be chosen to be skew, for if one were not skew, then by Kuratowski's theorem, some extraneous edges could be deleted to obtain a skew graph.

A *block of a graph* G is a maximal connected subgraph of G which contains no cut vertices. It is clear that any skew subgraph of G must be a subgraph of some block of G; thus we arrive at the following result.

THEOREM 1. The coarseness of a graph is the sum of the coarsenesses of its blocks.

This implies that it is sufficient to determine the coarseness of graphs having no cut vertices. Another observation which can be made concerns the number of edges in a graph. The smallest number of edges which a nonplanar graph can possess is 9 (and then only if the graph is $K_{3,3}$). Therefore, it immediately follows that if a graph G has q edges, then

$$c(G) \leqq \left[\frac{q}{9}\right]. \tag{1}$$

In [1] it was shown that every graph is homeomorphic to a graph having thickness 1 or 2. As we shall now see, however, coarseness is invariant under homeomorphism.

THEOREM 2. Homeomorphic graphs have the same coarseness.

PROOF. Let G_1 and G_2 be homeomorphic graphs. Then there exists a graph G_3 which can be obtained from each of G_1 and G_2 by a sequence of subdivisions. To show that $c(G_1) = c(G_2)$, it is clearly sufficient to prove that the coarseness of one of G_1 and G_2, say G_1, equals that of G_3. To prove this, however, it is sufficient to show that $c(G_1) = c(G_1')$, where G_1' is a subdivision of G_1. Thus, there is an edge $x = uv$ of G_1 which has been replaced by a new vertex w and the two new edges uw and wv to obtain G_1'. Assume $c(G_1) = n$. Consider a set of n edge-disjoint skew subgraphs of G_1. If none of these subgraphs contains x, then this is a set of subgraphs of G_1', so $c(G_1') \geqq n$. Should one of these subgraphs H contain x, then by replacing x by the vertex w and the edges uw and wv, a skew subgraph H' of G_1' is produced, which together with the other $n-1$ subgraphs constitutes a collection of n edge-disjoint skew subgraphs of G_1'. Hence, in any case, $c(G_1') \geqq n$. To show that $c(G_1') > n$ is not possible, suppose that G_1' contains $n+1$ edge-disjoint skew subgraphs. If neither of the edges uv nor wv is contained in any of these subgraphs, then these skew graphs are also subgraphs of G_1, but this cannot happen since $c(G_1) = n$. Thus, one of these edges, say uw, must belong to some skew graph S. Because no skew graph contains a vertex of degree 1, the edge wv is also necessarily in S. But then, a skew graph S' can be obtained by replacing the edges uw and wv and the vertex w by the edge x. The graph S' along with the other n skew graphs form a set of $n+1$ edge-disjoint skew subgraphs of G_1, but this is also a contradiction.

By taking a sufficiently large number of subdivisions of a graph G with high coarseness, a graph G' can be obtained whose thickness is 2. By the previous theorem, however, $c(G) = c(G')$. We state this as a corollary.

Corollary 2a. For every positive integer n, there exists a graph G such that $c(G)-t(G) = n$.

More specifically, consider the nonplanar complete bigraph $K_{3,3n}$. It is easily observed that $K_{3,3n}$ is the edge-disjoint union of $K_{1,3n}$ and $K_{2,3n}$, each of which is planar. Thus $t(K_{3,3n}) = 2$. One also sees that $K_{3,3n}$ contains n edge-disjoint copies of $K_{3,3}$ so that $c(K_{3,3n}) \geqq n$, but since this graph has $9n$ edges, by inequality (1), $c(K_{3,3n}) = n$. These facts have another, perhaps somewhat unexpected interpretation: For any positive integer n, it is possible to find two planar graphs which can be combined into a graph G, which then in turn can be decomposed into n edge-disjoint nonplanar graphs.

On the coarseness of complete graphs

The complete graph K_p with p vertices has $p(p-1)/2$ edges; therefore by (1), we have

$$c(K_p) \leqq p(p-1)/18. \tag{2}$$

In this section we also present lower bounds for $c(K_p)$. The determination of a formula for $c(K_p)$ appears to be a very difficult problem. The known values for $c(K_p)$ are given in Table 1. Thus the smallest value of p for which $c(K_p)$ is unknown is 13. Also unknown is the smallest positive integer n for which there exists no p such that $c(K_p) = n$.

For several of the entries in Table 1, a simple construction and a degree argument provides the answer. We illustrate this for $p = 9$. By dividing the 9 vertices into 3 sets of 3 vertices each, one sees that every two of these sets induces a copy of $K_{3,3}$ so that

TABLE 1

The known coarsenesses of complete graphs

p	1−4	5−7	8	9	10	11	12
$c(K_p)$	0	1	2	3	4	5	6

$c(K_9) \geqq 3$. Since K_9 has 36 edges, it follows, by (2), that $c(K_9) \leqq 4$. However, for $c(K_9)$ to have the value 4, it would necessarily have to contain 4 edge-disjoint copies of $K_{3,3}$, but each vertex of K_9 has degree 8 so at least two edges meeting at each vertex cannot be in any of these copies of $K_{3,3}$ implying that 3 is the maximum number of edge-disjoint nonplanar skew graphs contained in K_9.

For $p = 11$ and $p = 12$ an argument similar to that for $p = 9$ can be given, but for $p = 10$ a different technique is employed. We label the vertices of K_{10} as $v_1, v_2, \cdots, v_{10}$. Then $c(K_{10}) \geqq 4$ since K_{10} contains 4 edge-disjoint skew graphs, e.g., those shown in Figure 2.

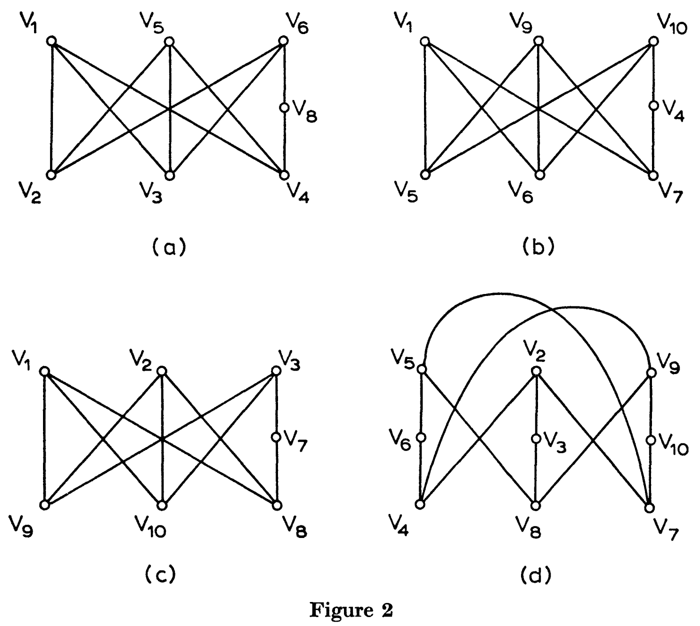

Figure 2

By (2), however, $c(K_{10}) \leqq 5$. If $c(K_{10}) = 5$, then, since K_{10} has 45 edges, K_{10} must contain 5 edge disjoint copies G_i $(1 \leqq i \leqq 5)$ of the graph $K_{3,3}$. Necessarily, each vertex appears in 3 of these copies. We now relabel the vertices of K_{10} by $u_1, u_2, \cdots, u_{10}$ so that u_1 appears in G_1, G_2, and G_3 and so that the vertices to which u_1 is adjacent are as indicated in Figure 3.

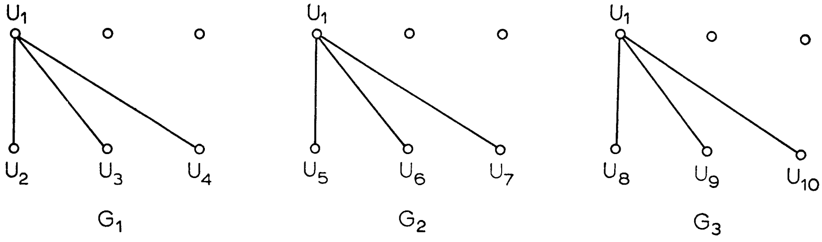

Figure 3

The two unlabeled vertices in G_1 clearly belong to the set $\{u_5, u_6, \cdots, u_{10}\}$. If both these vertices are in the set $\{u_5, u_6, u_7\}$ or in $\{u_8, u_9, u_{10}\}$, then these two vertices must appear together in some copy of $K_{3,3}$ since the edge determined by them is in some G_i. However, no such graph can exist, for these vertices must be mutually adjacent to two other vertices in this G_i. Hence, of the two unlabeled vertices in G_1, one must be in $\{u_5, u_6, u_7\}$ and the other in $\{u_8, u_9, u_{10}\}$, say u_5 and u_8. Similarly, of the two unlabeled vertices in G_2, one must belong to the set $\{u_2, u_3, u_4\}$ and the other to $\{u_8, u_9, u_{10}\}$. However, no vertex of $\{u_2, u_3, u_4\}$ can be in G_2 since each of these vertices is already adjacent to u_5 in G_1. This is a contradiction; thus K_{10} does not contain 5 edge-disjoint copies of $K_{3,3}$ so that $c(K_{10}) = 4$. This fact will be useful in obtaining a lower bound for $c(K_p)$.

THEOREM 3. For the complete graph K_p,

$$c(K_p) \geqq \begin{cases} \binom{r}{2} & \text{if } p = 3r \\ \binom{r}{2} + \left[\frac{r}{3}\right] & \text{if } p = 3r+1 \\ \binom{r}{2} + \left[\frac{3r+1}{4}\right] & \text{if } p = 3r+2. \end{cases}$$

PROOF. For $p = 3r$, the vertices of K_p can be divided into r sets of 3 vertices each. Every two of these sets clearly determine a copy of $K_{3,3}$. Since there are $\binom{r}{2}$ such pairs of sets, K_{3r} contains $\binom{r}{2}$ copies of $K_{3,3}$ and thus $c(K_{3r}) \geqq \binom{r}{2}$.

Assume $p = 3r+1$. Let v be one vertex of K_p and divide the remaining vertices into r sets of 3 vertices each. As before, one gets $\binom{r}{2}$ copies of $K_{3,3}$. However, if one takes 3 triples of vertices together with v, 4 skew graphs can be obtained as in the decomposition of K_{10}. These can replace the 3 copies of $K_{3,3}$ which had been obtained originally. There are $\left[\frac{r}{3}\right]$ disjoint collections of 3 triples, and it is possible to produce an additional skew graph for each such collection. Hence, $c(K_{3r+1}) \geqq \binom{r}{2}+\left[\frac{r}{3}\right]$.

For $p = 3r+2$, we again take r sets of 3 vertices each and denote the remaining vertices by u and v. On one of these sets together with u and v, a copy of K_5 can be formed. Let C denote the collection of $3\left[\frac{r+2}{4}\right]$ vertices in $\left[\frac{r+2}{4}\right]$ of the remaining $r-1$ sets. A skew graph homeomorphic to K_5 can then be constructed for each of the $r-1-\left[\frac{r+2}{4}\right]$ still remaining sets by using such a set with u and v and assigning a vertex of C as a vertex of degree 2 between u and v. Since

$$r-1-\left[\frac{r+2}{4}\right] = r+\left[-\frac{r+3}{4}\right] \leqq \frac{3r-3}{4} \text{ and } 3\left[\frac{r+2}{4}\right] \geqq 3\left(\frac{r-1}{4}\right),$$

there are enough vertices to accomplish this. Therefore,

$$c(K_{3r+2}) \geqq \binom{r}{2} +1+ \left(r-1-\left[\frac{r+2}{4}\right]\right) = \binom{r}{2} + \left[\frac{3r+1}{4}\right].$$

In terms of p, the bounds provided by (2) and Theorem 3 can be stated as follows:

$$\frac{p^2-3p}{18} \leqq c(K_p) \leqq \frac{p^2-p}{18} \qquad p \equiv 0 \bmod 3$$

$$(3) \qquad \frac{p^2-5p+4}{18} + \left[\frac{p-1}{9}\right] \leqq c(K_p) \leqq \frac{p^2-p}{18} \qquad p \equiv 1 \bmod 3$$

$$\frac{p^2-7p+10}{18} + \left[\frac{p-1}{4}\right] \leqq c(K_p) \leqq \frac{p^2-p}{18} \qquad p \equiv 2 \bmod 3.$$

The inequalities (3) immediately establish an estimate of $c(K_p)$ for large p.

COROLLARY 3a. The coarseness of the complete graph K_p is asymptotic to $p^2/18$.

The lower bound $a(p)$ and the upper bound $b(p)$ given by (3) are presented in Table 2 for $1 \leqq p \leqq 40$.

TABLE 2

Lower Bounds $a(p)$ and Upper Bounds $b(p)$ on $c(K_p)$, $1 \leqq p \leqq 40$

p	1	2	3	4	5	6	7	8	9	10
$a(p)$	0	0	0	0	1	1	1	2	3	4
$b(p)$	0	0	0	0	1	1	2	3	4	5

p	11	12	13	14	15	16	17	18	19	20
$a(p)$	5	6	7	9	10	11	14	15	17	19
$b(p)$	6	7	8	10	11	13	15	17	19	21

p	21	22	23	24	25	26	27	28	29	30
$a(p)$	21	23	26	28	30	34	36	39	43	45
$b(p)$	23	25	28	30	33	36	39	42	45	48

p	31	32	33	34	35	36	37	38	39	40
$a(p)$	48	52	55	58	63	66	70	75	78	82
$b(p)$	51	55	58	62	66	70	74	78	82	86

The amount by which $a(p)$ and $b(p)$ can differ is now considered.

THEOREM 4. The lower bound $a(p)$ and the upper bound $b(p)$ of $c(K_p)$, given in (3), satisfy the inequality:

$$b(p)-a(p) \leqq \left[\frac{p+2}{9}\right].$$

PROOF. We consider the 3 cases $p = 3r$, $3r+1$, and $3r+2$ individually.

First, assume $p = 3r$. Then

$$b(p)-a(p) \leqq \frac{3r(3r-1)}{18} - \frac{r(r-1)}{2} = \frac{r}{3} = \frac{p}{9}.$$

Next, suppose $p = 3r+1$. Then

$$b(p)-a(p) \leqq \frac{(3r+1)3r}{18} - \frac{r(r-1)}{2} - \frac{r-2}{2} = \frac{r+2}{3} = \frac{p+5}{9}.$$

However, for $p = 3r+1$,

$$\left[\frac{p+5}{9}\right] > \left[\frac{p+2}{9}\right] \text{ implies that } \left[\frac{r+2}{3}\right] > \left[\frac{r+1}{3}\right].$$

This inequality is satisfied only when $r \equiv 1 \pmod 3$, i.e., when $r = 3s+1$. But then $p = 9s+4$, in which case

$$b(p)-a(p) \leqq \frac{(9s+4)(9s+3)}{18} - \frac{(3s+1)3s}{2} - s = \frac{3s+2}{3} = \frac{p+2}{9}.$$

Finally, assume $p = 3r+2$. Then

$$b(p)-a(p) \leqq \frac{(3r+2)(3r+1)}{18} - \frac{r(r-1)}{2} - \frac{3r-2}{4} = \frac{9r+22}{36} = \frac{3p+16}{36},$$

and

$$\frac{3p+16}{36} \leqq \frac{p+2}{9} \qquad \text{for all } p \geqq 8.$$

Because

$$b(p)-a(p) \leqq \frac{p+2}{9} \qquad \text{for } p = 2 \text{ and } p = 5$$

also, the inequality is satisfied for all $p = 3r+2$.

Since the difference $b(p)-a(p)$ is always an integer, the theorem follows.

In conclusion, we present bounds for the number $c(K_{m,n})$. Since $K_{m,n}$ has mn edges, we have, by (1), $c(K_{m,n}) \leqq \left[\frac{mn}{9}\right]$. We now obtain a lower bound for $c(K_{m,n})$. Assume the vertex set V of $K_{m,n}$ is partitioned as $V_1 \cup V_2$, where $[V_1] = m$, $[V_2] = n$, and every edge joins a vertex of V_1 with a vertex of V_2. The set V_1 has $\left[\frac{m}{3}\right]$ distinct triples of vertices while V_2 has $\left[\frac{n}{3}\right]$ such triples. Thus, $K_{m,n}$ has at least $\left[\frac{m}{3}\right]\left[\frac{n}{3}\right]$ copies of $K_{3,3}$. These observations are summarized below.

THEOREM 5. *For the complete bigraph* $K_{m,n}$,

$$\left[\frac{m}{3}\right]\left[\frac{n}{3}\right] \leqq c(K_{m,n}) \leqq \left[\frac{mn}{9}\right].$$

COROLLARY 5a. *If* $m \equiv n \equiv 0 \bmod 3$, *then* $c(K_{m,n}) = mn/9$.

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