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On distribution of arithmetical functions on the set prime plus one

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On distribution of arithmetical functions
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1. Introduction

P. Erdősf proved the following theorem [1].

Let \( f(n) \) be a real valued additive number-theoretical function, and put

\[
f^*(n) = \begin{cases} f(n) & \text{for } |f(n)| \leq 1, \\ 0 & \text{for } |f(n)| > 1. \end{cases}
\]

Put

\[
F_N(x) = \frac{1}{N} \sum_{n \leq x \leq N} \frac{1}{f(n)}.
\]

Then the distribution-functions \( F_N(x) \) tend for \( N \to +\infty \) to a limiting distribution function \( F(x) \) at all points of continuity of \( F(x) \), if the following three conditions are satisfied:

1. \( \sum_{p} \frac{f^*(p)}{p} \) is convergent,
2. \( \sum_{p} \frac{(f^*(p))^2}{p} < +\infty \),
3. \( \sum_{|f(p)| > 1} \frac{1}{p} < +\infty \).

It has been shown also by P. Erdős that \( F(x) \) is continuous if and only if the series \( \sum_{f(p) \neq 0} 1/p \) diverges.

New proof of this theorem has been given by H. Delange [2] and by A. Rényi [3].

A multiplicative function \( g(n) \) is called strongly multiplicative, if for all primes \( p \) and all positive integers \( k \) it satisfies the condition

\[ g(p^k) = g(p). \]

H. Delange proved the following theorem [4].
If \( g(n) \) is a strongly multiplicative number-theoretical function such that \(|g(n)| \leq 1\) for \( n = 1, 2, \ldots \), and such that the series 
\[
\sum_p \frac{g(p) - 1}{p}
\]
converges, then 
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(n) = M(g)
\]
exists and 
\[
M(g) = \prod_p \left(1 + \frac{g(p) - 1}{p}\right).
\]
A new proof of this theorem has been given by A. Rényi [5].

Throughout the paper \( p, q \) denote primes, and \( \sum_p \) and \( \prod_p \) denote a sum and a product, respectively, taken over all primes. Let further \( \text{li} x = \int_2^x \frac{du}{\log u} \).

The aim of this paper is to prove the following statement.

**Theorem 1.** Let \( g(n) \) be a complex-valued multiplicative function such that \(|g(n)| \leq 1\) for \( n = 1, 2, \ldots \), and such that the series 
\[
(1.1) \quad \sum_p \frac{g(p) - 1}{p}
\]
converges. Let \( N(g) \) denote the product 
\[
(1.2) \quad N(g) = \prod_p \left(1 + \sum_{k=1}^{\infty} \frac{g(p^k) - g(p^{k-1})}{p^{k-1}(p-1)}\right).
\]
Then 
\[
(1.3) \quad \lim_{x \to \infty} \frac{1}{\text{li} x} \sum_{p \leq x} g(p+1) = N(g).
\]

From this theorem easily follows the

**Theorem 2.** Let \( f(n) \) be a real valued additive number-theoretical function which satisfies the conditions 1, 2, 3, of the theorem of Erdös.

Put 
\[
F_N(y) = \frac{1}{\text{li} N} \sum_{f(p+1) \leq y} 1.
\]

Then the distribution-functions \( F_N(y) \) tend for \( N \to \infty \) to a limiting distribution-function \( F(y) \) at all points of continuity of \( F(y) \).

Further \( F(y) \) is a continuous function if and only if 
\[
\sum_{f(p) \neq 0} \frac{1}{p} = \infty.
\]
2. Deduction of Theorem 2 from Theorem 1

In what follows, \( c, c_1, c_2, \ldots \) denote constants not always the same in different places.

For the proof of Theorem 2 we need to prove only that the sequence of characteristic functions

\[
\varphi_N(u) = \frac{1}{\ln N} \sum_{p \leq N} e^{iu \ln(p)}
\]

converges to a function \( \varphi(u) \), which is a continuous one on the real axis.

It is easy to verify that from the conditions 1/2/3 it follows that

\[
\sum_p \frac{e^{iu \ln(p)} - 1}{p}
\]

converges for every real \( u \). Using now Theorem 1 with \( g(n) = e^{iu \ln(n)} \) we obtain that \( \varphi_N(u) \rightarrow \varphi(u) \), where

\[
\varphi(u) = \prod_p \left( 1 + \sum_{k=1}^{\infty} \frac{e^{iu \ln(p^k)} - e^{iu \ln(p^{k-1})}}{p^{k-1}(p-1)} \right).
\]

The continuity of (2.3) is guaranteed by the continuity of (2.2), which follows from the conditions 1/2/3 evidently.

For the proof of the continuity of \( F(x) \) in the case

\[
\sum_{f(p) \neq 0} \frac{1}{p} = \infty
\]

we remark the following.

P. Levy proved the following theorem [8].

Let \( X_1, X_2, \ldots, X_n, \ldots \) be a sequence of independent random variables with discrete distribution and suppose that there exists the sum

\[
\sum_{k=1}^{\infty} X_k = X
\]

with probability 1. Let

\[
d_k = \sup_x P(X_k = x).
\]

Then the distribution function of \( X \) is continuous if and only if

\[
\prod_{k=1}^{\infty} d_k = 0.
\]
Let now the $X_p$'s be independent random variables with characteristic functions
\[ 1 + \sum_{k=1}^{\infty} \frac{e^{it\mu(p^k)} - e^{it\mu(p^{k-1})}}{p^{k-1}(p-1)} \]
and let
\[ X = \sum_{p} X_p. \]

It is evident from (2.3) that $X$ has the distribution function $F(x)$, and so this is continuous if and only if
\[ \prod_{f(p) \neq 0} \left( 1 - \frac{1}{p} \right) = 0, \quad \text{i.e.} \quad \sum_{f(p) \neq 0} \frac{1}{p} = \infty. \]

3. The proof of Theorem 1

We need the following Lemmas.

**Lemma 1.** Let $g(p)$ be a complex-valued function defined on the primes, for which $|g(p)| \leq 1$ and
\[ \sum \frac{g(p)-1}{p} \]
converges. Then
\[ \sum \frac{1}{|\arg g(p)| > \epsilon} \frac{1}{p} < + \infty \]
for every positive constant $\epsilon$, further
\[ \sum_{p} \frac{|g(p)-1|^2}{p} < + \infty \]
and
\[ \sum_{a^{1/2} < p < x} \frac{|g(p)-1|}{p} \rightarrow 0 \quad \text{for} \quad x \rightarrow + \infty. \]

**Proof.** The assertion in (3.4) is an immediate consequence of (3.3) since
\[ \sum_{a^{1/2} < p < x} \frac{|g(p)-1|}{p} \leq \left( \sum_{a^{1/2} < p < x} \frac{1}{p} \right)^{1/2} \left( \sum_{a^{1/2} < p < x} \frac{|g(p)-1|^2}{p} \right)^{1/2}, \]
further $\sum_{a^{1/2} < p < x} 1/p$ is bounded, and the second sum on the right hand side tends to zero because of (3.3).

Let us put
\[ |g(p)| = r(p) \quad \text{and} \quad \arg g(p) = \theta(p), \]
where $-\pi < \theta(p) \leq +\pi$, i.e. we suppose that $g(p) = r(p)e^{i\theta(p)}$. 


From the convergence of (3.1) it follows that
\[ \sum_p \frac{1 - \text{Re } g(p)}{p} (\leq +\infty) \]
converges too. This sum has positive terms and the inequality
\[ |\arg g(p)| > c \]
involves that \( 1 - \text{Re } g(p) > c_1 (> 0) \). Hence (3.2) follows.

From the inequality \( |a + bi|^2 \leq 2(|a|^2 + |b|^2) \) it follows that
\[ \sum_p \frac{|g(p) - 1|^2}{p} \leq 2 \sum_p \frac{|\text{Re } (1 - g(p))|^2}{p} + 2 \sum_p \frac{|\text{Im } g(p)|^2}{p}. \]
The first sum on the right hand side evidently converges since
\[ \sum_p \frac{|\text{Re } (1 - g(p))|^2}{p} < \sum_p \frac{1 - \text{Re } g(p)}{p} + 4 \sum_{|\arg(p)| > \frac{1}{3}} \frac{1}{p}. \]
It is sufficient to prove that
\[ \sum_{|\arg(p)| \leq \frac{1}{3}} \frac{r^2(p) \sin^2 \vartheta(p)}{p} < \infty. \]

Using the inequality
\[ r^2(p) \sin^2 \vartheta(p) \leq c \vartheta^2(p) \leq 2c \sin^2 \frac{\vartheta(p)}{2} \leq 1 - \cos \vartheta(p) \]
we have (3.5).

**LEMMA 2.** Let \( N_k(x) \) denote the number of solutions of the equation
\[ p + 1 = kq, \ p \leq X \]
in primes \( p, q \). Then
\[ N_k(X) < c \frac{x}{\varphi(k) \log^2 X/k} \]
for \( k < x \), where \( c \) is an absolute constant.

For the proof see Prachar’s book [6], Theorem 4.6, p. 51.

Let \( \pi(x, k, l) \) denote the number of primes in the arithmetical progression \( \equiv l \pmod{k} \) not exceeding \( x \).

**LEMMA 3.** (Brun–Titchmarsh). For all \( k \leq x^{1-\delta}, \delta > 0 \)
\[ \pi(x, k, l) < c_2 \frac{x}{\varphi(k) \log x}, \]
where \( c_2 \) is a constant depending on \( \delta \) only.

For the proof see [6].
LEMMA 4. (E. Bombieri [7]).

\[ \sum_{D \leq Y} \max_{l \equiv 1 \pmod{D}} \left| \pi(x, D, l) - \frac{\text{li} x}{\phi(D)} \right| < \frac{cx}{(\log x)^A} \]

where \( Y = x^{\delta} (\log x)^{-B} \); \( B \geq 2A + 23 \), \( A \) arbitrary constant.

Let \( \tau(n) \) be the number of divisors of \( n \).

LEMMA 5.

\[ \sum_{n \leq y} \frac{\tau^2(n)}{\varphi(n)} < c (\log y)^A, \]

where \( c \) is a constant.

The proof is very simple and so can be omitted.

Let us define the multiplicative function \( g_K(n) \) by putting

\[ g_K(p^a) = \begin{cases} g(p^a), & \text{if } p^a \leq K, \\ 1, & \text{if } p^a > K. \end{cases} \]

By other words we put for any natural number \( n \)

\[ g_K(n) = \prod_{p^a \mid n, p^a \leq K} g(p^a). \]

Let us put further

\[ h_K(n) = \sum_{d \mid n} \mu \left( \frac{n}{d} \right) g_N(d), \]

where \( d \) runs over all (positive) divisors of \( n \) and \( \mu(n) \) is the Möbius function. Then \( h_K(n) \) is also a multiplicative function, \( h_K(p^a) = g_K(p^a) - g_K(p^{a-1}) \); \( h_K(p) = 0 \) for \( p > K \); \( h_K(p) = 0 \) for \( p^{a-1} > K \), \( a \geq 2 \).

Let further \( h(n) \) be defined by

\[ h(n) = \sum_{d \mid n} \mu \left( \frac{n}{d} \right) g(d). \]

Let us introduce the notation

\[ I_K(x) = \sum_{p \leq x} g_K(p+1); \quad I(x) = \sum_{p \leq x} g(p+1). \]

Choose now \( K_1 = (1 - \varepsilon) \log x, K_2 = x^{\delta}, K_3 = x^{1-\varepsilon} \), where \( \varepsilon \) and \( \delta \) are suitable small positive numbers.

We shall prove the following relations:

(3.6) \[ I_{K_1}(x) = (1 + o(1)) \text{ li } x N(g) \quad \text{for } x \to \infty, \]

(3.7) \[ I_{K_2}(x) - I_{K_1}(x) = o(\text{ li } x) \quad \text{for } x \to \infty, \]
Theorem 1 follows if we choose \( \delta = \delta(x) \) tending to zero so slowly that the right hand side of (3.8) is \( o(\text{li } x) \).

First we prove (3.6). We have

\[
I_{K_1}(x) = \sum_{p \leq x} g_{K_1}(p+1) = \sum_{p \leq x} \sum_{d \mid p+1} h_{K_1}(a) = \sum_{d} h_{K_1}(d) \tau(x, d, -1)
\]

\[
= \text{li } x \sum_{d} \frac{h_{K_1}(d)}{\varphi(d)} + R,
\]

where

\[
|R| \leq \sum_{d} |h_{K_1}(d)| \left| \tau(x, d, -1) - \frac{\text{li } x}{\varphi(d)} \right| = R_1.
\]

Using the prime number theorem, we obtain that \( h_{K_1}(d) = O \) for \( d \geq x^{1-\varepsilon/2} \) because

\[
\prod_{p \leq K_1} p^n < e^{\left(\frac{1-\varepsilon}{2}\right) \log x},
\]

if \( x \) is sufficiently large.

Since \( |g(n)| \leq 1 \), so \( |h_{K_1}(p^n)| \leq 2 \) and \( |h_{K_1}(d)| \leq \tau(d) \).

For the estimation of \( R_1 \) we split all of the \( d \)'s, \( d \leq x^{1-\varepsilon/2} \) into two classes \( \mathcal{U}_1, \mathcal{U}_2 \) as follows:

\( d \) belongs to \( \mathcal{U}_1 \) or \( \mathcal{U}_2 \) according to that \( \tau(d) \leq (\log x)^5 \) or \( \tau(d) > (\log x)^5 \), respectively.

Using Lemma 3 and Lemma 5 we have

\[
\sum_{d \in \mathcal{U}_1} |h_{K_1}(d)| \left| \tau(x, d, -1) - \frac{\text{li } x}{\varphi(d)} \right| \leq c \text{ li } x \sum_{d \in \mathcal{U}_1} \frac{\tau(d)}{\varphi(d)}
\]

\[
\leq c \text{ li } x (\log x)^{-5} \sum_{d \leq x} \frac{\tau^2(d)}{\varphi(d)} < c \frac{x}{\log^2 x}.
\]

Otherwise, using the Bombieri’s result (Lemma 4), we have that the sum

\[
\sum_{d \in \mathcal{U}_1} |h_{K_1}(d)| \left| \tau(x, d, -1) - \frac{\text{li } x}{\varphi(d)} \right|
\]

not exceed

\[
(\log x)^5 \sum_{d \leq x^{1/4}} \left| \tau(x, d, -1) - \frac{\text{li } x}{\varphi(d)} \right| = O \left( \frac{x}{(\log x)^{A-5}} \right) = O \left( \frac{x}{\log^2 x} \right),
\]

if \( A \geq 7 \).

Hence

\[
R = O \left( \frac{x}{\log^2 x} \right).
\]
Further we have
\[ \sum_d \frac{h_{K_1}(d)}{\varphi(d)} = \prod_{p < K_1} \left( 1 + \sum_{x=1}^{\infty} \frac{g_{K_1}(p^x) - g_{K_1}(p^{x-1})}{p^x - 1} \right). \]

From the convergence of the series \( \sum (g(p) - 1)/p \) it follows that the product on the right hand side tends to \( N(g) \) for \( x \to +\infty \). So (3.6) is proved.

Let now \( \tilde{g}(n) \) be a multiplicative function defined by
\[ \tilde{g}(p^x) = \begin{cases} g(p^x), & \text{if } p \leq K_1, \\ g(p), & \text{if } p > K_1. \end{cases} \]

It is evident that \( \tilde{g}(n) = g(n) \) except eventually those \( n \)'s for which there exists a \( q, q > K_1 \), such that \( q^2 \mid n \). So

\begin{equation}
\sum_{p \leq x} |g(p+1) - \tilde{g}(p+1)| \leq 2 \sum_{q > K_1} \sum_{p+1 \equiv q^3} 1 < 2c \log x \sum_{K_1 < q < x^{1/3}} \frac{1}{q(q-1)}
+ x \sum_{q > x^{1/3}} \frac{1}{q^2} = o(\log x).
\end{equation}

From (3.10)
\[ |I_{K_1}(x) - I_{K_1}(x)| \leq \sum_{p \leq x} |\tilde{g}_{K_1}(p+1) - \tilde{g}_{K_1}(p+1)| + o(\log x) \]
\[ \leq \sum_{q \leq x} \prod_{q \mid p+1} \sum_{K_1 < q \leq K_2} g(q) - 1 + o(\log x) = V + o(\log x) \]
follows. Using the formulas
\[ \log (1+z) = z + O(|z|^2); \quad \exp (z + O(|z|^2)) = 1 + z + O(|z|^2) \]
for \(|z| \leq 1, |\arg z| \leq \pi/2\), we have that
\begin{equation}
\prod_{K_1 < q \leq K_2} g(q) - 1 = \sum_{K_1 < q \leq K_2} h_{K_1}(q) + O\left( \sum_{K_1 < q \leq K_2} |h^2(q)| \right),
\end{equation}
if all primdivisor \( q \) of \( p+1 \) in the interval \( K_1 < q \leq K_2 \) satisfies the relation \(|\arg g(q)| \leq \pi/2\). Let \( \mathbb{U}_3 \) denote the set of the \( p \)'s possessing this property, and \( \mathbb{U}_4 \) the other \( p \)'s.

We can easily estimate the sum
\[ V_1 = \sum_{p \in \mathbb{U}_4} |\sum_{q \mid p+1} g(q) - 1|,
\]
since
and by (3.2)

\[ V_1 = o \left( \log x \right). \]

Let

\[ V_2 = \sum_{p \in \mathbb{P}} \left| \prod_{q \neq 1}^{q \leq K_2} g(q) - 1 \right|. \]

From (3.11) we have that

\[ V_2 \leq \sum_{p} \left| \sum_{q \leq K_2} h(q) \right| + O \left( \sum_{p} \sum_{q \leq K_2} |h^2(q)| \right) = V_3 + O(V_4). \]

Using (3.3) in Lemma 1 and Lemma 3 we have

\[ V_4 < \sum_{K_1 < q \leq K_2} |h^2(q)| \pi(x, q, -1) < c \sum_{q > K_1} \frac{|h^2(q)|}{q - 1} = o \left( \log x \right). \]

Further, from the Cauchy’s inequality

\[ V_3 < c \left( \log x \right)^{3/2} \left\{ \sum_{K_1 < q_1, q_2 < K_2, q_1 \neq q_2} h(q_1) h(q_2) \pi(x, q_1 q_2, -1) + \sum_{K_1 < q \leq K_2} |h(q)|^2 \pi(x, q, -1) \right\}^{1/2}. \]

Using Bombieri’s result we have that

\[ V_3 < c \left( \log x \right)^{3/2} \left| \sum_{K_1 < q \leq K_2} \frac{h(q)}{q - 1} \right| (\log x)^{1/2} + O \left( \frac{x}{\log^2 x} \right) = o \left( \log x \right), \]

since

\[ \sum_{K_1 < q \leq K_2} \frac{h(q)}{q - 1} = \sum_{K_1 < q \leq K_2} \frac{h(q)}{q} + O \left( \sum_{K_1 < q \leq K_2} \frac{1}{q^2} \right) = o(1). \]

So we proved that

\[ V_2 = V_3 + O(V_4) = o(\log x); \quad V_1 = o(\log x); \quad V = V_1 + V_2 = o(\log x), \]

whence (3.7) follows.

Similarly we have

\[ |I_{K_3}(x) - I_{K_2}(x)| \leq \sum_{p \leq x} \left| \sum_{q | p+1}^{q \leq K_3} h(q) \right| + c \sum_{p \leq x} \sum_{q | p+1}^{q \leq K_3} |h(q)|^2 \\
+ c \sum_{K_3 < q \leq K_3} \pi(x, q, -1) = V_5 + cV_6 + cV_7. \]
Using Lemma 3 and (3.4) in Lemma 1 we have that

\[
V_6 \leq \sum_{K_1 < q \leq K_2} |h(q)| \pi(x, q, -1) < c_3 \text{ li } x \sum_{K_1 < q \leq K_2} \frac{|h(q)|}{q} = o(c_3 \text{ li } x).
\]

Further using (3.3) and (3.2) we obtain that

\[
V_6 \leq \sum_{K_1 < q \leq K_2} |h^2(q)| \pi(x, q, -1) < c_3 \text{ li } x \sum_{K_1 < q \leq K_2} \frac{|h^2(q)|}{q} = o(c_3 \text{ li } x),
\]

\[
V_7 \leq c_4 \text{ li } x \sum_{q > K_1} \frac{1}{q} = o(c_4 \text{ li } x).
\]

Hence (3.8) follows.

Finally using Lemma 2 we have

\[
|I(x) - I_{K_1}(x)| \leq 2 \sum_{K_1 < q < x} \sum_{j \leq x} \frac{x}{\varphi(q) \log^2 x/j} < c \frac{x}{\log^2 x} \sum_{j \leq y} \frac{1}{\varphi(j)} < c\delta \frac{x}{\log x},
\]

because

\[
\sum_{j \leq y} \frac{1}{\varphi(j)} < c \log y.
\]

So the inequality (3.9) is proved, and from (3.6)—(3.9) our theorem follows.

4. Some remarks

1. From our Theorem 2 it follows evidently that if \( g(n) \) is a positive valued multiplicative number-theoretical function such that

\[
\sum_p \frac{((\log p)*)^*}{p} \text{ is convergent},
\]

\[
\sum_p \frac{(\log g(p))^{*2}}{p} < +\infty,
\]

\[
\sum_{|\log g(p)| > 1} \frac{1}{p} < +\infty,
\]

then putting
the distribution functions $F_N(y)$ tend for $N \to +\infty$ to a limiting distribution function $F(y)$ at all points of continuity of $F(y)$.

Hence it follows especially that the functions

$$\frac{\varphi(p+1)}{p+1}, \quad \frac{\sigma(p+1)}{p+1}$$

$(\sigma(n)$ denotes the sum of the divisors of $n$) have limiting distribution functions.

2. Recently M. B. Barban, A. I. Vinogradov, B. V. Levin proved that all results of J. P. Kubilius theory (see [9]) are valid for strongly additive arithmetic functions belonging to the class $H$, when the argument runs through “shifted” primes $\{p-l\}$, (see [10], [11]).

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